Conserved contact process in one to five dimensions

Munir M. S. Sabag and Mário J. de Oliveira
Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970 São Paulo, São Paulo, Brazil
(Received 20 May 2002; published 18 September 2002)

We analyze the conserved contact process in hypercubic lattices with dimensions ranging from one to five. In this process particles jump around, falling down only on empty sites beside an existing particle. The model is a version of the ordinary contact process with a strictly conserved particle number and can be seen as the contact process in an ensemble of fixed particle number. By means of numerical simulations we determine the critical point, the critical exponent β, and the fractal dimension d_F at the critical point. In the case of just two particles, the stationary state is obtained exactly in any dimension.

DOI: 10.1103/PhysRevE.66.036115 PACS number(s): 02.50.−r, 05.40.−a, 05.70.Ln

I. INTRODUCTION

Recently, Tomé and de Oliveira [1] introduced the conserved contact process (CCP), a version of the ordinary contact process with a strictly conserved particle number. In the CCP, particles jump around over the sites of a regular lattice falling down only on empty sites that have at least one neighboring site occupied by a particle. In contrast with the ordinary contact process, the CCP does not have an absorbing state. Despite of lacking this important feature, it displays properties that, in the thermodynamic limit, are identical to those of the ordinary contact process, including universal as well as nonuniversal quantities. The CCP has been then identified as the contact process in an ensemble of constant particle number and this has been indeed confirmed by numerical simulation in one dimension [1]. Later, Hilhorst and Wijland [2] have provided a proof of the equivalence between the two stationary state ensembles: the ordinary ensemble (i.e., the constant rate ensemble) and the conserved ensemble (i.e., the constant particle number ensemble).

The use of distinct ensembles to calculate the thermodynamic properties of the system in equilibrium is well established and there exists a standard procedure for passing from one to another ensemble [3–5]. For nonequilibrium systems no such general procedure exists. However, the possibility of using distinct ensembles in nonequilibrium models was put forward by Ziff and Brosilow [6] when they employed a constant coverage ensemble to analyze an irreversible surface-reaction model originally defined in a constant rate ensemble.

The ordinary contact process, proposed by Harris [7], is the simplest nonequilibrium model displaying a phase transition and critical behavior [8–17]. It exhibits a continuous phase transition from an active state, with nonzero density of particles, to an absorbing state, with zero density of particles, even in one dimension and belongs to the universality class of directed percolation [18–20]. The contact process in d dimension has the same critical exponents of directed percolation in $D=d+1$ dimensions. The upper critical dimension of the directed percolation was established to be $D_c=5$ [21] so that the critical dimension for the contact process is $d_c=4$. Therefore, for $d\geqslant 4$ the critical exponents are the classical ones with possible logarithmic corrections at the critical dimension.

Here, we perform numerical simulations of the CCP defined on hypercubic lattices with dimensions d ranging from $d=1$ up to $d=5$. The conservation of particles allowed us to perform numerical simulations that avoided the accidental fall into the absorbing state. We determine the critical point as well as the critical exponent β related to the order parameter and the fractal dimension d_F at the critical point. The use of the constant particle number ensemble permitted us the determination of the critical point with a very good precision. The results for β and d_F are in good agreement with the values of the ordinary contact process. In the special case of just two particles the stationary state is obtained exactly in any dimension by the use of the lattice Green function.

The CCP is related to the plant population model introduced by Bröker and Grassberger [22] in the sense that these two models are conserved versions of models belonging to the directed percolation universality class, namely, the contact process and the directed percolation model, respectively. In the Bröker and Grassberger model, however, the conservation of particles is achieved in a distinct way. In their model the conservation of particles is imposed in a global way by removing the excess of particles from the system.

II. THE CONSERVED CONTACT PROCESS

The ordinary contact process comprises two subprocesses: a catalytic creation and a spontaneous annihilation of particles. In the basic ordinary contact process [10,17], particles are created on the empty sites of a regular lattice with a rate λ/z times the number of occupied nearest neighbors, where z is the lattice coordination number. Particles are annihilated spontaneously with rate 1. Here, we use a definition in which the creation rate is $1/z$ times the number of occupied nearest neighbors, and the annihilation rate is $k=1/\lambda$.

Let us denote by η_i the occupation variable attached to the site i, with $\eta_i=0$ or 1 according to whether the site i is empty or occupied. The time evolution of the probability distribution $P(\eta,t)$, where $\eta=(\eta_1,\eta_2,\ldots,\eta_N)$ is the vector that represents the collection of occupation variables, is governed by the master equation

$$\frac{d}{dt}P(\eta,t)=\sum_i \{w_i(\eta^i)P(\eta^i,t)-w_i(\eta)P(\eta,t)\}, \quad (1)$$

where η^i is the vector $\eta^i=(\eta_1,\eta_2,\ldots,1-\eta_i,\ldots,\eta_N)$ and $w_i(\eta)$ is the transition rate from state η to state η^i. For the basic ordinary contact process the transtion rate is given by

$$w_i(\eta)=\frac{1}{z}(1-\eta_i)\sum_{\delta}\ \eta_{i+\delta}+k\eta_i, \tag{2}$$

where the first term accounts for the catalytic creation process and the second accounts for the spontaneous annihilation process. The summation in δ is over the z nearest neighbor sites.

Empty sites with one or more occupied neighbors, which we call active empty sites, play an important role in the contact process since particles are created only on those sites. A quantity that measures the number of such sites is the effective number of active empty sites $n_{\rm ac}$ defined as

$$n_{\rm ac}=\frac{1}{z}\sum_i\ (1-\eta_i)\sum_{\delta}\ \eta_{i+\delta}. \tag{3}$$

The number of particles n is given by

$$n=\sum_i\ \eta_i. \tag{4}$$

From the master equation (1) of the ordinary contact process it follows that the time evolution of the mean number of particles $\langle n\rangle$ is given by

$$\frac{d}{dt}\langle n\rangle=\langle n_{\rm ac}\rangle-k\langle n\rangle. \tag{5}$$

Therefore, the stationary condition gives

$$\langle n_{\rm ac}\rangle=k\langle n\rangle. \tag{6}$$

The contact process in an ensemble of constant particle number is defined as follows. An empty site becomes occupied in a way similar to the catalytic creation. But contrary to the ordinary contact process no particle is created; a randomly chosen particle of the system leaves its place and jumps to the empty site. Thus, both the processes of creation and annihilation of particles of the ordinary contact process are replaced by a jumping process. However, this is not an unrestricted jumping because particles are not allowed to jump to a vacant site surrounded by empty sites; at least one neighbor site must be occupied. The CCP is a two-site process governed by the following master equation:

$$\frac{d}{dt}P(\eta,t)=\frac{1}{n}\sum_i\ \sum_j\ \{w_{ij}(\eta^{ij})P(\eta^{ij},t)-w_{ij}(\eta)P(\eta,t)\}, \tag{7}$$

where η^{ij} is the vector $\eta^{ij}=(\eta_1,\eta_2,\ldots,1-\eta_i,\ldots,1-\eta_j,\ldots,\eta_N)$ and $w_{ij}(\eta)$ is the jumping transition rate, that is, the transition rate from state η to state η^{ij} given by

$$w_{ij}(\eta)=\eta_i\frac{1}{z}(1-\eta_j)\sum_{\delta}\ \eta_{j+\delta}. \tag{8}$$

The model strictly conserves the number of particles n.

The mean effective number of active sites per particle, denoted by α, is given by

$$\alpha=\frac{\langle n_{\rm ac}\rangle}{n}, \tag{9}$$

where the average is to be taken over the ensemble with a constant particle number. In the thermodynamic limit $\alpha=k$.

III. TWO PARTICLES

We consider in this section the case of two particles, $n=2$, in an infinite lattice. In this case it is possible to solve the master equation exactly. Invoking the translational invariance property it suffices to use the relative position of the particles to define a given configuration. Accordingly, we fix a particle at the origin of the lattice so that the position $\mathbf{r}$ of the other particle completely defines the configuration of the system. We look for the probability $P_{\mathbf{r}}(t)$ that the free particle is at position $\mathbf{r}=(\ell_1,\ell_2,\ldots,\ell_d)$ where ℓ_i takes integer values. From the master equation it follows that

$$\frac{d}{dt}P_{\mathbf{r}}(t)=\frac{1}{2z}\sum_{\boldsymbol{\delta}}\ P_{\mathbf{r}+\boldsymbol{\delta}}(t)-P_{\mathbf{r}}(t), \tag{10}$$

valid for $|\mathbf{r}|\neq 1,0$, where $\boldsymbol{\delta}$ represents any one of the $z=2d$ vectors of unit lenght. When $\mathbf{r}=\boldsymbol{\delta}$ the equation reads

$$\frac{d}{dt}P_{\boldsymbol{\delta}}(t)=\frac{1}{2z}\sum_{\boldsymbol{\delta}'(\neq-\boldsymbol{\delta})}P_{\boldsymbol{\delta}+\boldsymbol{\delta}'}(t)-\left(1-\frac{1}{z}\right)P_{\boldsymbol{\delta}}(t)+\frac{1}{2z}\sum_{\mathbf{r}'(\neq\boldsymbol{\delta},0)}P_{\mathbf{r}'}(t). \tag{11}$$

These equations should be solved for $P_{\mathbf{r}}(t)$ with $\mathbf{r}\neq 0$. Notice that $P_0(t)$ is not present above because two particles are not allowed to occupy the same site.

We are interested here only in the stationary solution $P_{\mathbf{r}}$. It is possible to show that the stationary solution is given by [23]

$$P_{\mathbf{r}}=\frac{G_{\mathbf{r}}}{1-G_0}, \tag{12}$$

where $G_{\mathbf{r}}$ is the lattice Green function

$$G_{\mathbf{r}}=\int_{Bz}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{1+d^{-1}\sum_{j=1}^{d}(1-\cos k_j)}\frac{d^dk}{(2\pi)^d}, \tag{13}$$

where $\mathbf{k}=(k_1,k_2,\ldots,k_d)$ and the integral is over the Brillouin zone, $-\pi\leqslant k_j\leqslant\pi$. In one dimension the integral (13) can be carried out explicitly which with the help of Eq. (12) gives

$$P_\ell=\frac{1}{\sqrt{3}-1}(2-\sqrt{3})^{|\ell|}, \tag{14}$$

TABLE I. Mean effective number of active sites per particle α of the CCP in an infinite hypercubic lattice of dimensions ranging from $d=1$ to $d=5$, in the subcritical regime. The last row gives the order of the magnitude of the statistical errors of the results.

n	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
2	0.633974	0.842079	0.901897	0.9292383	0.9447486
3	0.52357	0.78149	0.86573	0.903985	0.925421
4	0.46964	0.74570	0.84516	0.890173	0.915127
6	0.41611	0.70650	0.82219	0.875190	0.904183
8	0.38909	0.68508	0.80942	0.867049	0.898388
12	0.36174	0.66202	0.79539	0.858281	0.892299
16	0.34790	0.64969	0.78777	0.853580	0.889080
20	0.33944	0.64200	0.78291	0.850627	0.887106
28	0.32956	0.63281	0.77705	0.847076	0.884772
40	0.32206	0.62561	0.77237	0.844271	0.882960
56	0.31698	0.62060	0.76906	0.842302	0.881708
80	0.31299	0.61667	0.76646	0.840768	0.880748
112	0.31034	0.61396	0.76464	0.839701	0.880095
160	0.30825	0.61187	0.76320	0.838882	0.879592
224	0.30689	0.61045	0.76220	0.838308	0.879252
320	0.30585	0.60933	0.76143	0.837867	0.878993
450	0.30508	0.60856	0.76089	0.837561	0.878819
640	0.30456	0.60799	0.76048	0.837333	0.878686
900	0.30420	0.60758	0.76019	0.837170	0.878597
1300	0.30388	0.60728	0.75996	0.837042	0.878527
1800	0.30373	0.60708	0.75982	0.836966	0.878487
2600	0.30355	0.60692	0.75970	0.836900	0.878448
	0.00002	0.00001	0.00001	0.000005	0.000005

where $\ell=\pm1,\pm2,\ldots$.

The mean number of active sites, $\alpha=\langle n_{ac}\rangle/2$, is calculated by using the expression $\alpha=1-P_{\boldsymbol{\delta}}$. Using Eq. (12) and taking into account that $G_{\boldsymbol{\delta}}$ is related to G_0 by $G_{\boldsymbol{\delta}}=2G_0-1$, we obtain

$$\alpha=3-\frac{1}{1-G_0}. \tag{15}$$

The mean distance between particles R is given by $R=\sqrt{\langle|\mathbf{r}|^2\rangle}$ where

$$\langle|\mathbf{r}|^2\rangle=\sum_{\mathbf{r}(\neq0)}|\mathbf{r}|^2P_{\mathbf{r}}=\frac{1}{1-G_0}. \tag{16}$$

In one dimension $G_0=1/\sqrt{3}$ from which follows $\alpha=(3-\sqrt{3})/2$ and $R=[(3+\sqrt{3})/2]^{1/2}$. In other dimensions the integral (13) for G_0 can be performed numerically. Using this method we have obtained the numerical results shown in the first row $(n=2)$ of Table I.

IV. SUBCRITICAL REGIME

Due to the fact that the dynamics conserves the number of particles, the CCP does not have an absorbing state. This conservation law allows us to carry out numerical simulations without the danger of falling into the absorbing state as happens in the ordinary contact process. Thus, the quasistationary states [17] observed in the ordinary contact process, in the subcritical regime, become genuine stationary states in the CCP.

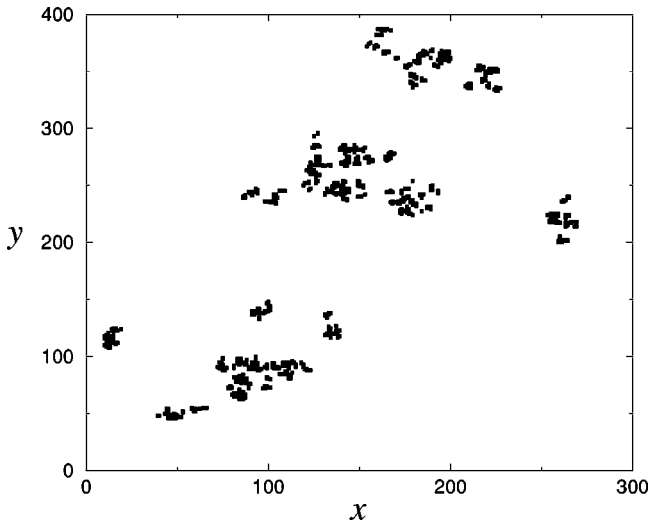

FIG. 1. Snapshot of a configuration of $n=900$ particles in an infinite square lattice, in the subcritical regime. The x and y axes give the coordinates of the occupied sites, represented by small black squares. The origin of the coordinate axes is arbitrary.

We have simulated the CCP on an infinite d-dimensional hypercubic lattice with $d=1$, 2, 3, 4, and 5 for several values of the number of particles n from $n=2$ up to $n=2600$. In an infinite lattice the particles do not scatter to infinity, as one could expect, but remain close together forming a fractal cluster as can be seen in Fig. 1. The reason is that the particles are not allowed to jump to any site of the lattice but only to those sites that are adjacent to an existing particle.

The simulation of a system with n particles is performed as follows. At each time step one of the n occupied sites is picked up at random and one of its z neighboring sites is chosen at random. If this neighboring site is empty then it will be occupied by one of the n particles, chosen randomly. If, otherwise, the neighboring site is already occupied the state remains the same. A Monte Carlo step is defined as n such time steps. Each simulation was performed by starting from a configuration were all particles are close together. The quantities of interest, such as $\langle n_{\mathrm{ac}}\rangle$, were estimated by using a number of Monte Carlo steps ranging from 10^7 to 10^8.

Since the lattice is infinite and the number of particles n is finite, the density of particles is zero and the system is naturally constrained to be into the subcritical regime. Table I shows the numerical values of $\alpha=\langle n_{\mathrm{ac}}\rangle/n$ obtained from simulations. The results shown for the case $n=2$ were exactly calculated in the preceding section. As $n\to\infty$ the effective number of active sites per particle α approaches its critical value α_c. The critical value α_c was obtained by the following extrapolation scheme applied to the data of Table I corresponding to $n\geqslant28$. To each set of three consecutive points (α_{j-1},x_{j-1}), (α_j,x_j), and (α_{j+1},x_{j+1}), where $x=1/n$, we fitted a straight line from which we withdrew a trial extrapolated value α_{cj}. The final extrapolated value α_c was then obtained by a quadratic regression over the points

TABLE II. Values of α_c, $\lambda_c = 1/\alpha_c$, the exponent β and the fractal dimension d_F for dimensions ranging from $d=1$ to $d=5$, obtained for the CCP. The values α_c are extrapolations from the results given in Table I by the method explained in the text.

d	α_c	λ_c	β	d_F
1	0.30323(2)	3.2978(2)	0.277(1)	0.747(4)
2	0.60653(1)	1.64872(3)	0.585(4)	1.20(1)
3	0.75940(1)	1.31683(2)	0.78(1)	1.56(3)
4	0.83674(1)	1.19511(1)	1	2
5	0.87837(1)	1.13847(1)	1	2

(α_{cj}, x_j). The extrapolated value of α_c is shown in Table II together with $\lambda_c = 1/\alpha_c$.

A measure of the size of the cluster is given by the quantity $R = \sqrt{\langle r_{\max}^2 \rangle}$ where $r_{\max}(\eta)$ is the maximum distance between two particles of the cluster η. As long as n is finite the mean linear size R of the cluster is also finite but diverges when $n \to \infty$. We assume the asymptotic behavior [22]

$$R \sim n^{1/d_F}, \tag{17}$$

where d_F is the fractal dimension. Figure 2 shows R as a function of n in a double-log plot for dimensions ranging from one to five. The slopes of the straight lines fitted to the data points corresponding to $d=1, 2$, and 3 are 1.338(6), 0.832(8), and 0.64(1), respectively. For $d=4$ and 5, the slopes are consistent with the classical value 1/2. The inverse of these values give the fractal dimension d_F shown in Table II.

V. SUPERCRITICAL REGIME

In the supercritical regime the density of particles ρ is nonzero. Therefore, in an infinite lattice the number of particles must be infinite. If the ratio $n/N = \rho$ is kept fixed and the limit $N \to \infty$ is taken, then the system is confined into the supercritical regime if ρ is nonzero. We have also simulated the CCP model on a finite hypercubic lattice with N sites, with n particles and periodic boundary conditions. The largest values of N used in simulations were $N = 10\,000$, $N = 100^2$, $N = 25^3$, $N = 10^4$, and $N = 6^5$, for $d = 1, 2, 3, 4$, and $d = 5$, respectively. To estimate the average $\langle n_{\rm ac} \rangle$ we used a number of Monte Carlo steps ranging from 10^6 to 10^7. A Monte Carlo step is defined here as N time steps defined in the preceding section. The quantity α was then obtained by Eq. (9).

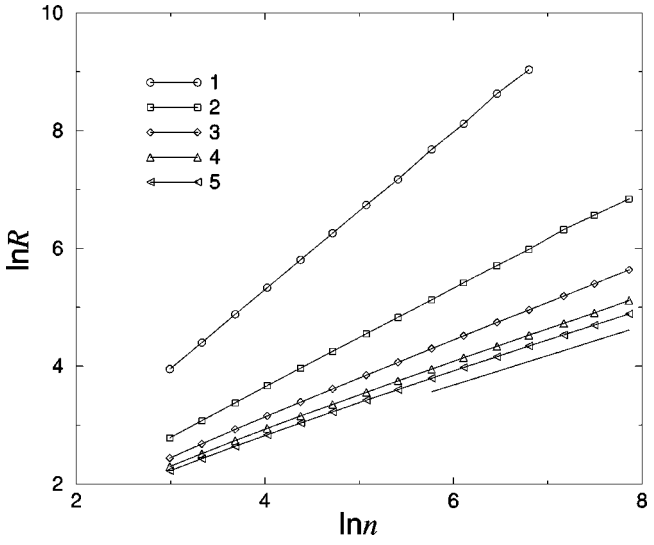

FIG. 2. Mean size of the system R, in the subcritical regime, as a function of the number of particles n for dimensions ranging from one to five on a double-log plot. For comparison we show a straight line with a slope equal to 1/2.

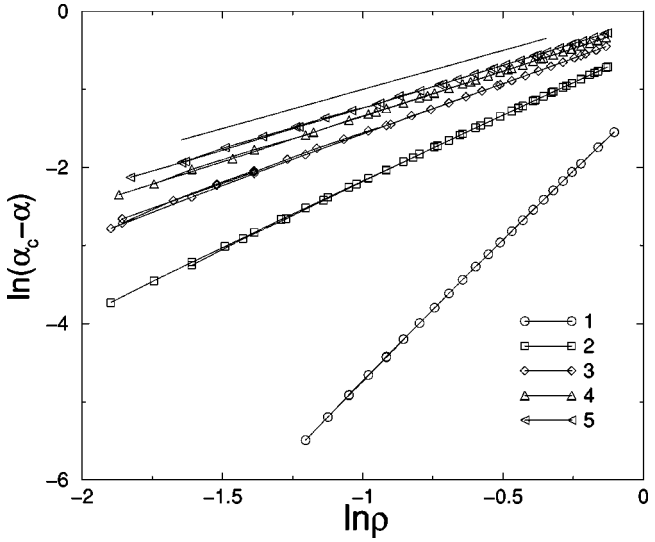

FIG. 3. Double-log plot of $\alpha_c - \alpha$ versus the density ρ, in the supercritical regime. For comparison we show a straight line with slope equal to 1.

For sufficient large N one expects the following behavior:

$$\alpha_c - \alpha \sim \rho^{1/\beta}, \tag{18}$$

where β is the order parameter exponent. Figure 3 shows a double-log plot of $\alpha_c - \alpha$ versus the density of particles ρ, where we used the values of α_c obtained in the preceding section and shown in Table II. The slope of the straight line fitted to the data points gives the values 3.61(1), 1.71(2), and 1.26(2) for $d=1, 2$, and 3, respectively. For $d=4$ and 5, the slopes are consistent with the classical value 1. The inverse of these values give the exponent β shown in Table II.

VI. CONCLUSIONS

We have analyzed the CCP in hypercubic lattices with dimensions ranging from one to five. The critical exponent β and the fractal dimension d_F obtained by numerical simulations are in good agreement with the respective values of the ordinary contact process [17]. The critical parameters λ_c obtained here are in excelent agreement with those of the ordinary contact process [17]. In fact, the results presented here

for λ_c are the best estimates, with the exception of the case $d=1$. In the subcritical regime the CCP has a remarkable property. As one increases the number of particles in an infinite system it approaches criticality, and α approaches its critical value α_c, as can be seen in Table I. The CCP has, therefore, similarities with self-organized criticality, in the sense that no adjustable parameter is necessary to drive the system to criticality.

ACKNOWLEDGMENT

We wish to acknowledge Ronald Dickman for his critical reading of the manuscript.

[1] T. Tomé and M.J. de Oliveira, Phys. Rev. Lett. **86**, 5643 (2001).
[2] H.J. Hilhorst and F. van Wijland, Phys. Rev. E **65**, 035103 (2002).
[3] L.D. Landau and E.M. Lifshitz, *Statistical Mechanics* (Pergamon Press, New York, 1958).
[4] D. Ruelle, *Statistical Mechanics: Rigorous Results* (Benjamin, Reading, MA, 1969).
[5] R.B. Griffiths, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M.S. Green (Academic Press, London, 1972), Vol. 1, p. 7.
[6] R.M. Ziff and B.J. Brosilow, Phys. Rev. A **46**, 4630 (1992).
[7] T.E. Harris, Ann. Prob. **2**, 969 (1974).
[8] R.C. Brower, M.A. Furman and M. Moshe, Phys. Lett. **76B**, 213 (1978).
[9] P. Grassberger and A. de la Torre, Ann. Phys. (N.Y.) **122**, 373 (1979).
[10] T.M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
[11] R. Dickman and M. Burschka, Phys. Lett. A **127**, 132 (1988).
[12] P. Grassberger, J. Phys. A **22**, 3673 (1989).
[13] R. Dickman, J. Stat. Phys. **55**, 997 (1989).
[14] R. Dickman and I. Jensen, Phys. Rev. Lett. **67**, 2391 (1991).
[15] I. Jensen and R. Dickman, J. Stat. Phys. **71**, 89 (1993).
[16] I. Jensen, J. Phys. A **29**, 7013 (1996).
[17] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 1999).
[18] H.K. Janssen, Z. Phys. B: Condens. Matter **42**, 151 (1981).
[19] P. Grassberger, Z. Phys. B: Condens. Matter **47**, 465 (1982).
[20] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[21] S.P. Obukhov, Physica A **101**, 145 (1980).
[22] H.-M. Bröker and P. Grassberger, Physica A **267**, 453 (1999).
[23] M.J. de Oliveira (unpublished).