

Theoretical analysis and simulations of the generalized Lotka-Volterra model

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The dynamics of generalized Lotka-Volterra systems is studied by theoretical techniques and computer simulations. These systems describe the time evolution of the wealth distribution of individuals in a society, as well as of the market values of firms in the stock market. The individual wealths or market values are given by a set of time dependent variables w_i , $i=1, \dots, N$. The equations include a stochastic autocatalytic term (representing investments), a drift term (representing social security payments), and a time dependent saturation term (due to the finite size of the economy). The w_i 's turn out to exhibit a power-law distribution of the form $P(w) \sim w^{-1-\alpha}$. It is shown analytically that the exponent α can be expressed as a function of one parameter, which is the ratio between the constant drift component (social security) and the fluctuating component (investments). This result provides a link between the lower and upper cutoffs of this distribution, namely, between the resources available to the poorest and those available to the richest in a given society. The value of α is found to be insensitive to variations in the saturation term, which represent the expansion or contraction of the economy. The results are of much relevance to empirical studies that show that the distribution of the individual wealth in different countries during different periods in the 20th century has followed a power-law distribution with $1 < \alpha < 2$.

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I. INTRODUCTION

In recent years there has been considerable interest in the collection and analysis of large volumes of economic data. Such data includes the distributions of the income and wealth of individuals [1–8], the market values of publicly traded companies as well as their short and long term fluctuations [9–12]. A common observation is that the distributions of economic data exhibit a power-law behavior of the form

$$P(w) \sim w^{-1-\alpha}, \quad (1)$$

where the variable w represents the wealth of an individual or market value of a company and α is the exponent that provides the best fit to the empirical data. Empirical studies show that the distribution of the wealth of individuals in different countries follows the power-law behavior described by Eq. (1), with $1 < \alpha < 2$ [1–5,8]. These results stimulated theoretical studies in an attempt to construct models that reproduce the power-law behavior and predict the value of α [13–19].

In this paper we study a stochastic dynamical model, based on the Lotka-Volterra system that gives rise to the power-law distribution of Eq. (1). The model consists of coupled dynamic equations, which describe the discrete time evolution of the basic system components w_i , $i=1, \dots, N$. The structure of these equations resembles the logistic map and they are coupled through the average value $\bar{w}(t)$. The dynamics includes autocatalysis both at the individual level and at the community level as well as a saturation term. To model the nonstationary conditions we introduce a time dependent parameter into the saturation term in each of these equations. We find that the system components spontaneously evolve into a power-law distribution of the form of Eq. (1), even in the presence of nonstationary external condi-

tions. Furthermore, it is shown analytically that the exponent α depends only on the ratio of the constant drift component (social security) and the fluctuating component (investments). It is found to be insensitive to variations in the saturation term that describes the level of economic activity and varies between periods of prosperity and depression.

The paper is organized as follows. In Sec. II we present the generalized Lotka-Volterra model under nonstationary conditions. Analytical results and predictions are presented in Sec. III and compared with the results of numerical simulations in Sec. IV. A summary is presented in Sec. V.

II. THE MODEL

The generalized Lotka-Volterra system [13–15,18–20] describes the evolution in discrete time of N dynamic variables w_i , $i=1, \dots, N$. In ecological systems, w_i represents the population size of the i th specie, while in economic systems it may represent the wealth of an individual investor or the market value of a publicly traded firm. At each time step t , an integer i is chosen randomly in the range $1 \leq i \leq N$, which is the index of the dynamic variable w_i to be updated at that time step. A random multiplicative factor $\lambda(t)$ is then drawn from a given distribution $\Pi(\lambda)$, which is independent of i and t . It will later be convenient to express this multiplicative factor by

$$\lambda(t) = \langle \lambda \rangle + \eta(t), \quad (2)$$

where $\langle \lambda \rangle$ is the average value of $\Pi(\lambda)$ and

$$D = \langle \lambda^2 \rangle - \langle \lambda \rangle^2 \quad (3)$$

is its standard deviation. The system is then updated according to

$$w_i(t+1) = [1 + \lambda(t)]w_i(t) + \sum_{j=1}^N a_{i,j}w_j(t) - \sum_{j=1}^N a_{j,i}w_i(t) - \sum_{j=1}^N c_{i,j}w_i(t)w_j(t),$$

$$w_j(t+1) = w_j(t), \quad j = 1, \dots, N; \quad j \neq i, \quad (4)$$

where $a_{i,j}$ and $c_{i,j}$ are constants. This is an asynchronous update mechanism. The first term on the right hand side of Eq. (4) describes the effect of stochastic autocatalysis at the individual level. In an ecological system this term represents variations in the population of a given specie, including births and deaths that might be affected by external conditions but are not affected by the interaction with other species. In a stock-market system it represents the increase (or decrease) by a random factor $\lambda(t)$ of the capital of the investor i between time t and $t+1$. The second and third terms in Eq. (4) describe the interaction between different dynamical variables. In an ecological system, the second term represents the dependence of population i on the availability of food, in the form of population j . The third term represents the fact that population i itself may be the food of some other species. In an economic system the second and third terms represent trade between investors or firms i and j , such as buying and selling, respectively. The fourth term in Eq. (4) describes saturation effects due to the competition for limited resources. In an ecological model, this term implies that large populations tend to exhaust the available resources on which they depend. The saturation parameters $c_{i,j}$ are large for populations i and j that consume the same type of food. In an economic system this term has to do with the saturation due to the finite size of the economy.

To simplify the analysis, we will consider in this paper a simple case in which the w_i 's interact in a uniform way with each other. This case is obtained by choosing $a_{i,j} = a/N$ and $c_{i,j} = c/N$. With this choice, Eq. (4) will be reduced to

$$w_i(t+1) = [1 + \lambda(t)]w_i(t) + a\bar{w}(t) - cw_i(t)\bar{w}(t),$$

$$w_j(t+1) = w_j(t), \quad j = 1, \dots, N; \quad j \neq i, \quad (5)$$

where

$$\bar{w}(t) = \frac{1}{N} \sum_{i=1}^N w_i(t) \quad (6)$$

is the average value of the dynamical variables at time t . Here the random term $\lambda(t)$ was shifted to $\lambda(t) - a$, but its distribution around the average (and thus the value of the standard deviation D) remained unchanged. The second term in Eq. (5) may now describe the effect of autocatalysis at the community level. In an economic model, this term can be related to the social security policy or to general publicly funded services that every individual receives. It prevents an individual w_i from falling below a certain fraction of the average \bar{w} . The third term in Eq. (5) describes saturation or the competition for limited resources. It has the effect of

limiting the growth to values sustainable for the current conditions and resources. Within the ecological context, here the interactions between populations are uniform, describing the case in which all of them consume the same type of food. We refer to Eq. (5) as the generalized Lotka-Volterra system because when averaged over i and over $\lambda(t)$, this system tends to approach a Lotka-Volterra-like equation [21,22]

$$w(t+1) = (1 + \langle \lambda \rangle + a)w(t) - cw^2(t), \quad (7)$$

where $w(t) \equiv \bar{w}(t)$. Computer simulations show that after some equilibration time the system described by Eq. (5) approaches steady-state conditions. Even at steady state, \bar{w} exhibits fluctuations. However, its average over long time scales approaches a constant value given by

$$\langle \bar{w} \rangle_t = (\langle \lambda \rangle + a)/c. \quad (8)$$

In previous studies of the system described by Eq. (5) the parameters a and c were considered as constants, corresponding to steady conditions of the market. In fact, the typical dynamics of microscopic market models [8,13,14,23] is generically *not* in a steady state. The effect of varying market conditions can be studied by considering the parameters a and c and the distribution $\Pi(\lambda)$ as slowly varying functions of time. We will show below that the systems described by Eq. (5) lead, under very general conditions, to a power-law distribution of the w_i 's of the form of Eq. (1). Moreover, it will be shown that the exponent α is insensitive to variations in the parameter c , namely it depends only on a and D . In order to examine the effect of variations in the economic conditions we will now introduce an explicit time dependence into the third term, as well as a more general dependence on the w_j 's. The dynamic equation will now take the form

$$w_i(t+1) = [1 + \lambda(t)]w_i(t) + a\bar{w}(t) - C(w_1, \dots, w_n, t)w_i(t),$$

$$w_j(t+1) = w_j(t), \quad j = 1, \dots, N; \quad j \neq i, \quad (9)$$

where $C(w_1, \dots, w_n, t)$ is a general function of the w_j 's that includes an explicit time dependence.

III. THEORETICAL ANALYSIS

In order to study the dynamics of the generalized Lotka-Volterra model, it will be convenient to denote the change of w_i in a single time step by $\Delta w_i(t) = w_i(t+1) - w_i(t)$. We introduce a set of normalized variables

$$x_i = \frac{w_i}{w}, \quad i = 1, \dots, N. \quad (10)$$

The change $\Delta x_i(t) = x_i(t+1) - x_i(t)$ in a single time step is given by

$$\Delta x_i \cong \frac{1}{w} \Delta w_i - \frac{w_i}{w^2} \Delta \bar{w} \quad (11)$$

up to first order in powers of the Δw_i 's. Considering the time dependence of the average \bar{w} , one should remember that at any time step t of Eq. (9) only one of the w_i 's is chosen randomly and updated. Moreover, there is no correlation between the chosen w_i and $\lambda(t)$. Thus, the time evolution of \bar{w} should be considered on a longer time scale of the order of N moves. However, for simplicity we evaluate $\Delta \bar{w}$ by averaging Eq. (9) for Δw_i over $i=1, \dots, N$ at a given time t . We make an independent random choice of $\lambda(t)$ for each i , which we denote by $\lambda_i(t) = \langle \lambda \rangle + \eta_i(t)$. The time dependence of \bar{w} is given by

$$\begin{aligned} \Delta \bar{w} = & \frac{1}{N} \sum_{j=1}^N \eta_j(t) w_j(t) + [\langle \lambda \rangle + a] \bar{w}(t) \\ & - C(w_1, \dots, w_n, t) \bar{w}(t). \end{aligned} \quad (12)$$

The dynamics of the x_i 's is thus given by

$$\Delta x_i = x_i(t) \left[\eta(t) - a - \frac{1}{N} \sum_j \eta_j(t) x_j(t) \right] + a. \quad (13)$$

Consider the sum

$$r(N) = \sum_{j=1}^N \eta_j(t) x_j(t). \quad (14)$$

If the x_j 's exhibit a distribution of the form

$$P(x) \sim x^{-1-\alpha}, \quad (15)$$

then the second moment of the distribution of $r(N)$ satisfies [24,25]

$$\langle r^2(N) \rangle^{1/2} = \begin{cases} N^{1/2}, & 2 < \alpha \\ N^{(3-\alpha)/2}, & 1 < \alpha < 2 \\ N, & 0 < \alpha < 1. \end{cases} \quad (16)$$

In the first case, the distribution of the x_j 's exhibits a finite second moment and $r(N)/N \rightarrow 0$ in the limit $N \rightarrow \infty$. In the second case, the second moment of $P(x)$ diverges and $r(N)$ follows a Lévy distribution.

In both cases, namely, for $\alpha > 1$, we obtain that in the (thermodynamic) limit $N \rightarrow \infty$,

$$\frac{1}{N} x_i \sum_j \eta_j(t) x_j(t) \rightarrow 0. \quad (17)$$

Thus, under the assumption that $P(x)$ follows Eq. (15) with $\alpha > 1$, we obtain to a good approximation that for large values of N

$$\Delta x_i = [\eta(t) - a] x_i + a, \quad i = 1, \dots, N. \quad (18)$$

We see that the dynamics of the normalized variable x_i is reduced to a set of identical decoupled linear Langevin equations that do not depend on the function $C(w_1, \dots, w_n, t)$ or on the mean value $\langle \lambda \rangle$ of the multiplicative noise. These equations can be cast into a general framework of multiplicative processes of the form

$$\Delta x(t) = \eta(t) G(x(t)) + F(x(t)). \quad (19)$$

Equation (18) can then be recovered by taking $F(x_i) = a(1 - x_i)$ and $G(x_i) = x_i$. By using a suitable change of variables to $y = y(x)$ that satisfies

$$\frac{dy}{dx} = \frac{1}{G(x)}, \quad (20)$$

one can reduce Eq. (19) to a Langevin equation in which the term $\eta(t)$ appears as an additive noise, rather than a multiplicative noise such as $\eta(t)G(x(t))$ [26]. The time evolution of $y(t)$ is obtained from Eq. (19) by using the chain differential rule up to second order in Δx (and first order in D),

$$\Delta y \cong \frac{dy}{dx} \Delta x + \frac{1}{2} \frac{d^2 y}{dx^2} \Delta x^2. \quad (21)$$

Inserting $\Delta x(t)$ from Eq. (19) and using the change of variables described in Eq. (20), we obtain

$$\Delta y \cong \frac{1}{G} (F + \eta G) + \frac{1}{2} \frac{d}{dx} \left(\frac{1}{G} \right) (F + \eta G)^2. \quad (22)$$

We now approximate the second order term by averaging over the noise term η that satisfies $\langle \eta \rangle = 0$ and $\langle \eta^2 \rangle = D$. We obtain

$$\Delta y \cong \eta + \frac{F}{G} - \frac{1}{2} \frac{dG}{dx} \left(D + \frac{F^2}{G^2} \right). \quad (23)$$

Assuming that $F^2/G^2 \ll D$, Eq. (23) is reduced to a discrete-time Langevin equation

$$\Delta y \cong \eta + J(y), \quad (24)$$

where the drift force $J(y)$ takes the form

$$J(y) = \frac{F}{G} - \frac{D}{2} \frac{dG}{dx}. \quad (25)$$

The Fokker-Planck equation corresponding to Eq. (24) is [27]

$$\frac{\partial P(y,t)}{\partial t} = - \frac{\partial}{\partial y} [J(y,t) P(y,t)] + \frac{D}{2} \frac{\partial^2 P(y,t)}{\partial y^2}, \quad (26)$$

where $P(y,t)$ is the probability distribution of y at time t . The solution of this equation under the stationary condition $\partial P(y,t)/\partial t = 0$ is

$$P(y) = \exp \left[\frac{2}{D} \int^y J(y') dy' \right]. \quad (27)$$

Thus, the distribution $P(x) = P(y)dy/dx$ of the original variables x_i is

$$P(x) = \frac{1}{G^2(x)} \exp \left[\frac{2}{D} \int^x \frac{F(x')}{G^2(x')} dx' \right]. \quad (28)$$

By taking $F(x) = a(1-x)$ and $G(x) = x$, we obtain a power-law distribution of the form [28]

$$P(x) = x^{-1-\alpha} \exp \left[\frac{1-\alpha}{x} \right], \quad (29)$$

where

$$\alpha = 1 + \frac{2a}{D}. \quad (30)$$

The exponent α thus depends on a single parameter a/D , namely, on the ratio of the global drift coefficient a and the fluctuations measured by D .

Another way to derive Eq. (30) from dynamical models of the form (18) was shown in Refs. [6,16]. It is based on the fact that, under steady-state conditions, linear Langevin equations of the form (18) satisfy [6,16,29]

$$\langle (\eta - a + 1)^\alpha \rangle = 1. \quad (31)$$

Considering $\eta - a$ as a small parameter and expanding Eq. (31) in a power series up to second order, we obtain

$$\alpha = 1 + \frac{2a}{D + a^2}. \quad (32)$$

Assuming that $a^2 \ll D$ we reproduce Eq. (30).

IV. NUMERICAL SIMULATIONS AND RESULTS

To examine the theoretical predictions presented in Sec. III, we have performed computer simulations of the generalized Lotka-Volterra system described by Eq. (9) with different choices of $C(w_1, \dots, w_n, t)$. It was found that after some equilibration time the distribution $P(x)$ reaches a steady state, and exhibits a power-law behavior. For $C(w_1, \dots, w_n, t) = c\bar{w}$ [as in Eq. (5)], $\bar{w}(t)$ fluctuates around some average value, given by Eq. (8). For a general function $C(w_1, \dots, w_n, t)$, which exhibits an explicit time dependence, $\bar{w}(t)$ continues to vary according to this function and its temporal average does not reach a steady state.

To examine how robust the power-law distribution is under varying conditions, we have simulated Eq. (9) with

$$C(w_1, \dots, w_n, t) = c_0 \left(1 + \sin \frac{2\pi t}{T} \right) \sum_{j=1}^N w_j^2 \quad (33)$$

(for $c_0 = 0.001$, $T = 2 \times 10^5$ and $\langle \lambda \rangle = 0.002$) and compared the results with the case $C(w_1, \dots, w_n, t) = c\bar{w}$ (for $c = 1$ and $\langle \lambda \rangle = 0.01$). The time dependence of \bar{w} in both cases is shown in Fig. 1(a). The distributions $P(x)$ obtained from the simulations in these two cases are shown in Fig. 1(b). These

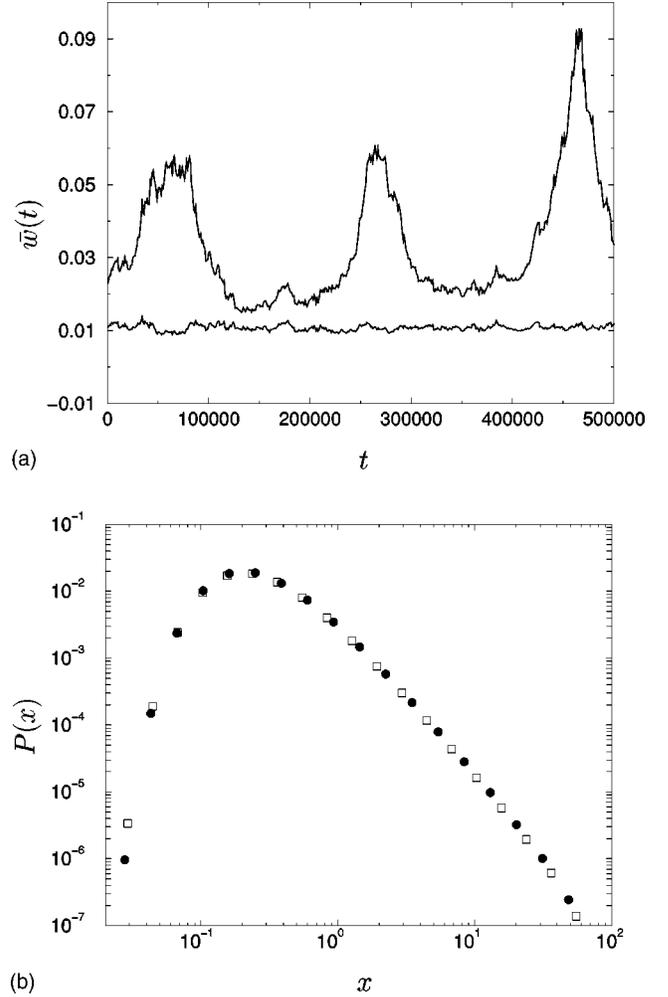


FIG. 1. (a) The time dependence of the average wealth \bar{w} for the model of Eq. (9) with $C(w_1, \dots, w_n, t)$ given by Eq. (33) where $c_0 = 0.001$ and $T = 2 \times 10^5$ (upper curve), and with $C(w_1, \dots, w_n, t) = c\bar{w}$ where $c = 1$ (lower curve). In the first case \bar{w} oscillates, following the time dependence of $C(w_1, \dots, w_n, t)$, while in the second case it only exhibits small fluctuations around a constant value. In both cases $N = 100$, $a = 0.00083$, $D = 0.0033$. (b) The distributions of the variables $x_i = w_i/\bar{w}$ for the simulations shown in the lower curve (squares) and the upper curve (\bullet) in (a). The two distributions are found to be nearly identical, showing an approximate power-law behavior. We thus observe that the exponent α is robust and insensitive to variations in $C(w_1, \dots, w_n, t)$.

distributions are found to be nearly identical and exhibit a power-law behavior characterized by the same exponent α . The exponent α is also found to be independent of $\langle \lambda \rangle$. Note that the power-law behavior is maintained even for $C(w_1, \dots, w_n, t) \equiv 0$, where $\bar{w}(t)$ does not reach a steady state and diverges to infinity (for $\langle \lambda \rangle > 0$) or collapses to 0 (for $\langle \lambda \rangle < 0$) [15]. This can lead to changes by orders of magnitude in the total wealth or the population size without affecting the exponent α .

To examine the theoretical prediction for the distribution, given by Eq. (29), and the exponent α , given by Eq. (30), we have compared these predictions to the results of numerical

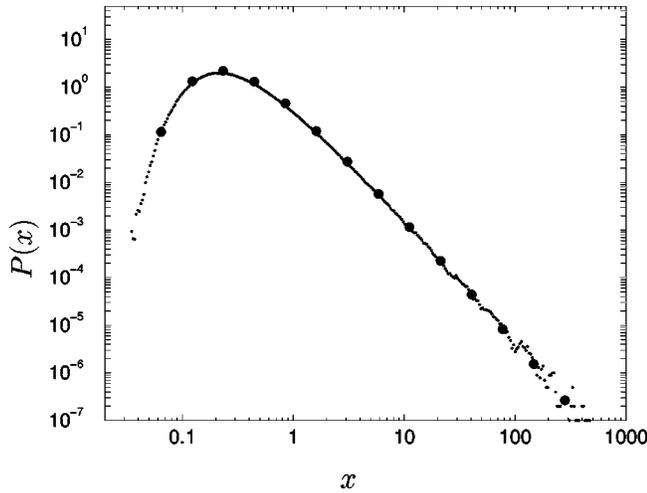


FIG. 2. Results of computer simulations (dots) and theoretical analysis based on Eq. (29) (●) for the distribution of the variables $x_i = w_i/\bar{w}$. The parameters are $N=1000$, $a=0.000\,23$, $D=0.000\,83$, and $C(w_1, \dots, w_n, t) = 0.01\bar{w}$. In both cases a power-law distribution is obtained with an excellent agreement between the theoretical predictions and the simulation results.

simulations. This comparison for the distribution $P(x)$ is shown in Fig. 2. The simulations were done for $N=1000$, $a=0.000\,23$, $C(w_1, \dots, w_n, t) = 0.01\bar{w}$, and $\lambda(t)$ uniformly distributed in the range $0.0 \leq \lambda(t) \leq 0.1$ (namely, $D=0.000\,83$). We found that the simulation results are in very good agreement with the theoretical predictions. A power-law distribution is found for a range of almost three orders of magnitude, with the exponent $\alpha=1.52$. This is close to the theoretical prediction of $\alpha=1+2a/D=1.55$. The distribution has a peak at $x_0 = (\alpha-1)/(\alpha+1)$, which using Eq. (30) can be expressed by $x_0 = a/(a+D)$. Above x_0 the distribution $P(x)$ behaves like a power law, while below it $P(x)$ decays exponentially. This provides an effective lower cutoff for the range of x in which a power-law behavior is observed. This result can be compared to a somewhat simpler model studied earlier, in which the value of the lower cutoff x_{min} is imposed as a constraint [15]. In this model, using the sum rules for the probability and the total wealth, it was found that $x_{min} = 1 - 1/\alpha$. Using Eq. (30) it can be expressed as $x_{min} = 2a/(2a+D)$. These predictions for the lower bounds in the two models satisfy $x_0 < x_{min} < 2x_0$, namely, they are in good agreement in light of the broad distribution of x .

To examine the prediction given by Eq. (30) for the exponent α , we present in Fig. 3 a comparison between this prediction and the numerical results for α as a function of a/D . The numerical results are presented for $N=1000$ (●). The prediction of Eq. (30) (solid line), shows a good agreement with the numerical results for $a/D > 0.2$. The numerical results for the range of small a/D converge to the theoretical prediction as the value of N is increased. As shown in Ref. [15] the infinite system limit, $N \rightarrow \infty$, and the vanishing coupling limit, $a/D \rightarrow 0$, do not commute. On the one hand, for any finite N and $a/D \rightarrow 0$ the exponent $\alpha \rightarrow 0$. On the other hand, for any fixed positive value of a/D (no matter how small) and $N \rightarrow \infty$ the exponent $\alpha \geq 1$. The majority of em-

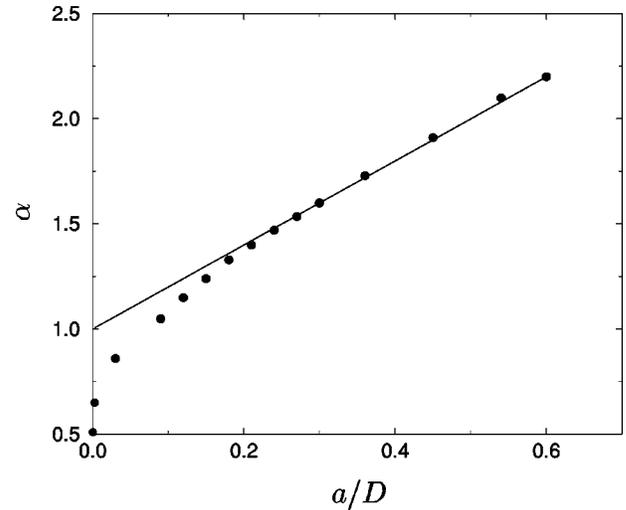


FIG. 3. Simulation results for the exponent α of the power-law distribution of the variables $x_i = w_i/\bar{w}$, $i=1, \dots, N$ as a function of the parameter a/D for $N=1000$ and $C(w_1, \dots, w_n, t) = 0.000\,01\bar{w}$ (●). The theoretical prediction of Eq. (30) (line) is found to be in agreement with the numerical values for $a/D > 0.2$. The agreement for small values of a/D tends to improve as N increases.

pirical results in which $1 < \alpha < 2$, indicate that the second case is highly relevant and that the theoretical predictions of Eqs. (29) and (30) broadly apply. Thus, for $N \rightarrow \infty$, both the exponent α of the power-law decay and the lower bound x_0 depend only on a single parameter a/D . In the economic context, this parameter represents the ratio between the fixed income of minimal wage jobs or social security payments and the level of fluctuations of the speculative income/loss.

V. SUMMARY

We have studied the dynamics of stochastic Lotka-Volterra systems under nonstationary conditions using both analytical and numerical techniques. For this class of models, we found that in order to obtain a power-law distribution, it is sufficient that relative returns of the agents are stochastically equivalent. The assumption that the distribution $\Pi(\lambda)$ of the multiplicative noise is independent of i means that there are no investors or strategies that can obtain “abnormal” returns. This can be related to the “efficient market hypothesis,” which assumes that the market pricing mechanism is so efficient that it reaches the “right price” before any of the agents can take systematic advantage. Therefore, the presence of a power-law distribution may be a sign of “market efficiency,” by analogy with Boltzmann distributions in statistical mechanics systems, which characterize thermal equilibrium. Here we have shown that the power-law distribution is stable even under nonstationary economic conditions, which are represented by the time dependence of the saturation term $C(w_1, \dots, w_n, t)$. We found that even under such conditions the distribution of the (normalized) dynamical variables x_i follows a power-law distribution with an exponent α . An expression for α in terms of the ratio of the parameters a and D was obtained [Eq. (30)]. In the eco-

conomic context, the parameter a represents the minimal wage or social security payments, while D represents the level of fluctuations in speculative income/loss. These results provide the distribution of wealth in a society in terms of the social

security policy and the volatility of the stock market. They also provide a connection between the incomes/wealths of the poorest and the richest sectors of the society as a function of a single parameter.

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