Quantifying stock-price response to demand fluctuations

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We empirically address the question of how stock prices respond to changes in demand. We quantify the relations between price change G over a time interval Δt and two different measures of demand fluctuations: (a) Φ, defined as the difference between the number of buyer-initiated and seller-initiated trades, and (b) Ω, defined as the difference in number of shares traded in buyer- and seller-initiated trades. We find that the conditional expectation functions of price change for a given Φ or Ω, $\langle G\rangle_\Phi$ and $\langle G\rangle_\Omega$ (‘‘market impact function’’), display concave functional forms that seem universal for all stocks. For small Ω, we find a power-law behavior $\langle G\rangle_\Omega \sim \Omega^{1/\delta}$ with δ depending on Δt ($\delta \approx 3$ for $\Delta t = 5$ min, $\delta \approx 3/2$ for $\Delta t = 15$ min and $\delta \approx 1$ for large Δt). We find that large price fluctuations occur when demand is very small—a fact that is reminiscent of large fluctuations that occur at critical points in spin systems, where the divergent nature of the response function leads to large fluctuations.

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Stock prices respond to fluctuations in demand, just as the magnetization of an interacting spin system responds to fluctuations in the magnetic field. Periods with a large number of market participants buying the stock imply mainly positive changes in price, analogous to a magnetic field causing spins in a magnet to align. Thus, understanding the dynamics of stock price fluctuations involves quantifying and understanding the relationship between price fluctuations and demand. Here, we quantify how price fluctuations depend on demand [1–3], and find a strikingly nonlinear relationship with a specific functional form that is not altogether unlike the dependence of magnetization on field strength. Our findings for the behavior of this dependence near zero demand are consistent with the intriguing possibility that large price fluctuations and their scale-free behavior arise not merely from external influences, but also from the ‘‘singular’’ response of the cooperative system, just as singularities near critical points of magnets arise from the intrinsic behavior of the system itself.

To quantify fluctuations in demand, we distinguish *buyer-initiated* and *seller-initiated* trades defined by which of the two participants in the trade, the buyer or the seller, is more eager to trade. When such a distinction does not exist, we label the trade as *indeterminate*. We identify buyer- and seller-initiated trades using the bid and ask quotes $S_{\mathrm{B}}(t)$ and $S_{\mathrm{A}}(t)$ at which a market maker is willing to buy or sell, respectively. For records of the bid-ask quotes, prices, and number of shares traded, we analyze the data for the 116 most-frequently traded US stocks from for the 2 yr period 1994–1995 [4]. Using the mid-value $S_{\mathrm{M}}(t) = [S_{\mathrm{A}}(t) + S_{\mathrm{B}}(t)]/2$ of the prevailing quote [5–7], we label a trade buyer initiated if $S(t) > S_{\mathrm{M}}(t)$, and seller initiated if $S(t) < S_{\mathrm{M}}(t)$. For trades occurring exactly at $S_{\mathrm{M}}(t)$, we use the sign of the change in price from the previous trade to determine whether the trade is buyer or seller initiated, while if the previous trade is at the current trade price, the trade is labeled indeterminate [5,8]. Accordingly, for each trade i, we define the variable

$$a_i \equiv \begin{cases} 1 & \text{(buyer initiated)} \\ 0 & \text{(indeterminate)} \\ -1 & \text{(seller initiated)}. \end{cases} \tag{1}$$

We quantify demand fluctuations by analyzing two quantities: (a) the *number imbalance* (difference between the number of buyer-initiated and seller-initiated trades [9,10] in a time interval $[t, t+\Delta t]$),

$$\Phi = \Phi_{\Delta t}(t) \equiv \sum_{i=1}^{N} a_i , \tag{2a}$$

and (b) the *volume imbalance* (difference between the number of shares traded in buyer-initiated and seller-initiated trades in the interval Δt),

$$\Omega = \Omega_{\Delta t}(t) \equiv \sum_{i=1}^{N} q_i a_i , \tag{2b}$$

where q_i is the number of shares traded in trade i, and $N = N_{\Delta t}(t)$ is the number of trades in Δt.

To choose a time scale in which to analyze the dependence of price fluctuations on demand, we first compute the correlation functions (Fig. 1) $\langle \Phi(t) G(t+\tau)\rangle$ and $\langle \Omega(t) G(t+\tau)\rangle$, where $G(t) \equiv G_{\Delta t}(t)$ is the stock price change over

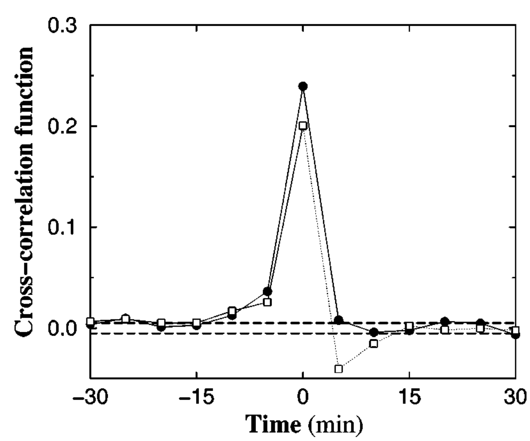

FIG. 1. Cross correlation functions $[\langle \Phi(t) G(t+\tau)\rangle - \langle \Phi(t)\rangle\langle G(t)\rangle]/\sigma_G \sigma_\Phi$ (open circles) and $[\langle \Omega(t) G(t+\tau)\rangle - \langle \Omega(t)\rangle\langle G(t)\rangle]/\sigma_G \sigma_\Omega$ (closed circles) computed using 5 min time series for Φ, Ω, and G. We find short-range time dependence which after ≈ 15 min reaches noise levels (dashed lines).

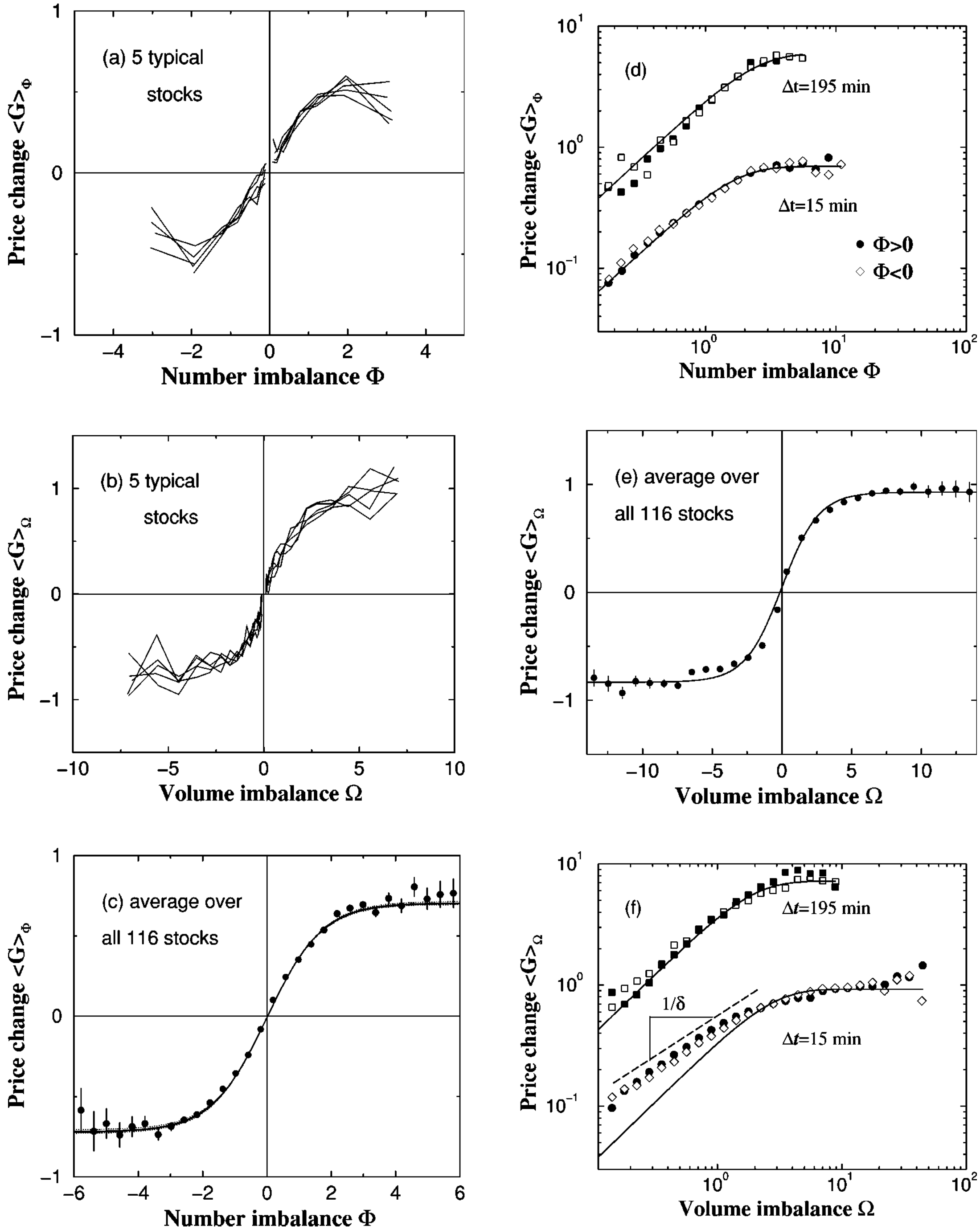

FIG. 2. (a) Conditional expectation $\langle G\rangle_\Phi$ of the price change for a given value of Φ for five typical stocks over a time interval $\Delta t = 15$ min. Both G and Φ are normalized to have zero mean and unit variance. (b) Conditional expectation $\langle G\rangle_\Omega$ for the same five stocks as in part (a). We normalize G to have zero mean and unit variance. Since Ω has a tail exponent $\zeta=3/2$ which implies divergent variance, we normalize Ω by the first moment $\langle|\Omega-\langle\Omega\rangle|\rangle$. (c) $\langle G\rangle_\Phi$ averaged over all 116 stocks studied. The solid curve shows a fit to the function $A_0\tanh(A_1\Phi)$, with $A_0=0.71\pm0.01$ and $A_1=0.58\pm0.01$, where the fit is performed with tolerance $=0.01$ [22]. The dotted lines (nearly indistinguishable from the solid curve) show $A_0\tanh(A_1\Phi)$ for the bounding values of A_0. (d) Same as (c), on a log-log plot for $\Phi>0$ (filled symbols) and $\Phi<0$ (empty symbols) for $\Delta t=15$ min and 195 min (shifted vertically for clarity). The solid curves show fits to $A_0\tanh(A_1\Phi)$, which agree well with the data. (e) Conditional expectation $\langle G\rangle_\Omega$ averaged over all 116 stocks. We calculate G and Ω for $\Delta t=15$ min. The solid line shows a fit to the function $B_0\tanh(B_1\Omega)$. (f) $\langle G\rangle_\Omega$ on a log-log plot for different Δt. For small Ω, $\langle G\rangle_\Omega \simeq \Omega^{1/\delta}$. For $\Delta t=15$ min find a mean value $1/\delta=0.66\pm0.02$ by fitting $\langle G\rangle_\Omega$ for all 116 stocks individually. The same procedure yields $1/\delta=0.34\pm0.03$ at $\Delta t=5$ min (interestingly close to the value of the analogous critical exponent in mean field theory). The solid curve shows a fit to the function $B_0\tanh(B_1\Omega)$. For small Ω, $B_0\tanh(B_1\Omega)\sim\Omega$, and therefore disagrees with $\langle G\rangle_\Omega$, whereas for large Ω the fit shows good agreement. For $\Delta t=195$ min ($\frac{1}{2}$ day) (squares), the hyperbolic tangent function shows good agreement.

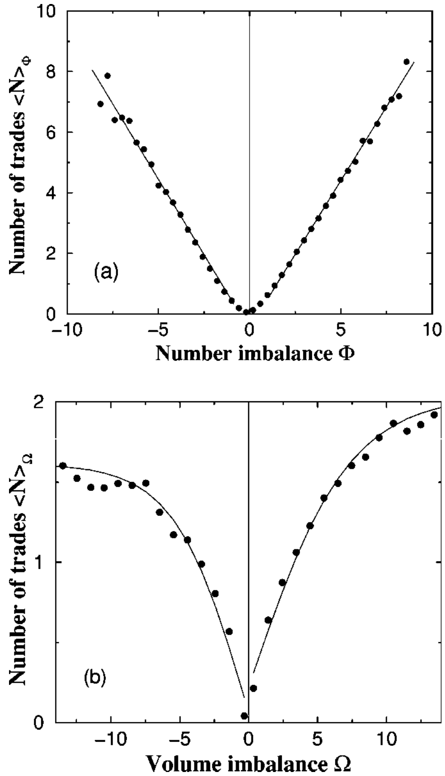

FIG. 3. (a) Conditional expectation $\langle N\rangle_\Phi$ of the number of trades for a given Φ averaged over all 116 stocks, shows approximately linear behavior with increasing Φ. (b) $\langle N\rangle_\Omega$ averaged over all 116 stocks shows strikingly nonlinear behavior. The solid line shows a fit to the function $C_0 - C_1\exp(-C_2\Omega)$ (which has the same large Ω behavior as a hyperbolic tangent). For both parts (a) and (b), we calculate G, Φ and Ω over $\Delta t = 15$ min. Both Φ and G are transformed to have zero mean and unit variance, whereas Ω is normalized by its first moment.

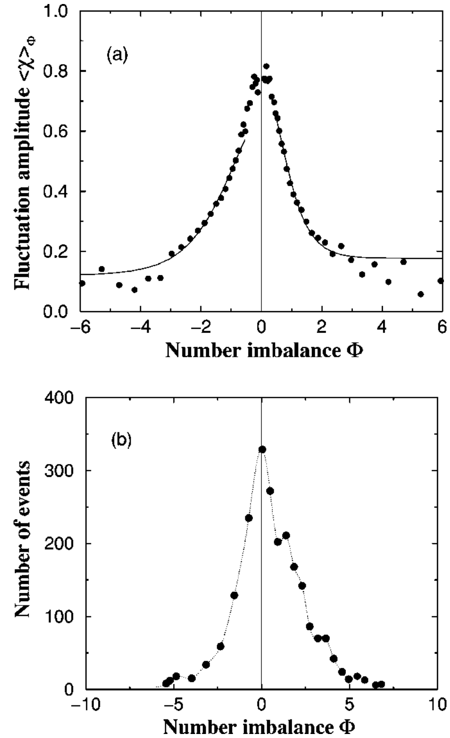

FIG. 4. (a) Conditional expectation $\langle\chi\rangle_\Phi$, where χ is calculated using Eq. (5), shows large values near $\Phi=0$ and decay for increasing Φ. The solid lines show a fit to the function $D_0\,\mathrm{sech}^2(D_1\Phi)$. (b) Number of events with $|G|>5$ standard deviations for a given Φ shows large values at $\Phi=0$.

the interval Δt. We find significant dependence at $\tau=0$, while for $|\tau|>0$, both correlation functions decay rapidly and cease to be statistically significant beyond $\tau \approx 15$ min—thereby setting a short time scale for the response of price changes to fluctuations in demand.

Next, we shall examine the relationships

$$\langle G\rangle_\Phi \equiv \mathbf{E}(G|\Phi), \tag{3a}$$

$$\langle G\rangle_\Omega \equiv \mathbf{E}(G|\Omega), \tag{3b}$$

which give the equal-time expectation values of $G(t)$ for a given $\Phi(t)$ or $\Omega(t)$. Figures 2(a) and 2(b) show $\langle G\rangle_\Phi$ and $\langle G\rangle_\Omega$ for five typical stocks for $\Delta t=15$ min. We find that both $\langle G\rangle_\Phi$ and $\langle G\rangle_\Omega$ are nonlinear, displaying concave curvature with increasing Φ and Ω [11–14], and "flattening" at large values [15].

Figure 2(c) shows the average behavior of $\langle G\rangle_\Phi$ for all stocks. The error bars correspond to one standard deviation for each Φ bin. We find that $\langle G\rangle_\Phi$ is consistent with the functional form

$$\langle G\rangle_\Phi = A_0 \tanh(A_1\Phi), \tag{4}$$

where A_0 is a constant that denotes the level of "saturation," and A_1 determines the average price change for unit change in Φ. In the case of a spin system, the saturation at large values for the analogous curve—magnetization vs field—is due to the fixed number of spins. The apparent saturation of $\langle G\rangle_\Phi$ is surprising in the present context, since there is no clear upper limit either on the price change, or on the number of trades. We find that $\langle G\rangle_\Phi$ for a range of Δt, also displays good agreement with Eq. (4) [Fig. 2(d)].

We next focus on $\langle G\rangle_\Omega$ [Fig. 2(e)]. We find that the function $\langle G\rangle_\Omega$, like $\langle G\rangle_\Phi$, is consistent with Eq. (4) [1,16,17]. However, near $\Omega=0$, $\langle G\rangle_\Omega$ shows not a strict linear behavior for small Ω as we expect for $\tanh\Omega$, but rather a power-law $\langle G\rangle_\Omega \sim \Omega^{1/\delta}$ [Fig. 2(f)]. We find that $1/\delta$ depends on Δt [Fig. 2(f)]: $\delta\approx 3$ for $\Delta t=5$ min and $\delta\approx 3/2$ for 15 min, and $\delta\to 1$ for larger Δt (agreeing well with $\tanh\Omega$) [16]. On a trade-by-trade basis, we find values of $1/\delta$ ranging from 0.2 up to 0.6 for different stocks.

Next, we analyze the dependence of the number of trades N on demand fluctuations to quantify how large volume imbalances generate trades. Figure 3(a) shows that the equal-time expectation value $\langle N\rangle_\Phi$ shows a linear increase with Φ. The dependence of N on volume imbalance Ω is nonlinear; $\langle N\rangle_\Omega$ displays a "cusp" at $\Omega=0$ followed by a sharp increase and saturation at large values [Fig. 3(b)]. We further analyze the small-Ω behavior of $\langle N\rangle_\Omega$ and find the relationship $N\sim\Omega^\gamma$ for each stock. We obtain a mean value of $\gamma = 0.17\pm 0.02$ for all stocks analyzed.

In spin systems, the amplitude of spin fluctuations is related to the susceptibility, which quantifies the response of the system to fluctuations in the magnetic field. In our problem, a certain change $\Delta\Phi$ in demand Φ (analog of the field) causes a response $\delta\langle G\rangle_\Phi/\delta\Phi\Delta\Phi$, which we find to be largest at $\Phi=0$ (Fig. 2), suggesting that the nonlinear shape of $\langle G\rangle_\Phi$ can give rise to large fluctuations (large "volatility" [18]) when Φ is small. The average amplitude of fluctuations in $G\equiv\Sigma_{i=1}^{N}\delta p_i$ is given by the variance

$$\chi^2\equiv\langle\delta p_i^2\rangle-\langle\delta p_i\rangle^2, \tag{5}$$

where δp_i is the price change due to trade i and $\langle\cdots\rangle$ denotes the average computed over the interval Δt. Figure 4(a) shows that $\langle\chi\rangle_\Phi$ displays large values near $\Phi=0$ and a rapid decay for increasing Φ. Figure 4(b) shows the dependence on Φ of the number of events with price change $|G|>5$ standard deviations. Interestingly, we find that a majority of the large events occur at $\Phi=0$, consistent with previous empirical results [19] which show that the power-law distribution of price changes [20] mainly arises from χ. Our findings are reminiscent of phase transitions in spin systems, where the divergent behavior of the response function at the critical point (zero magnetic field) leads to large fluctuations [21].

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[16] Regardless of the fitting function, the "flat" asymptotic behavior of $\langle G\rangle_\Omega$ is consistent with what we expect from the distributions of G and Ω. Since q_i has a distribution $P_q(q>x)\sim x^{-\zeta}$ with exponent $\zeta\approx3/2$ within the Lévy stable domain [10], the distribution $P_\Omega\{\Omega>x\}$ also has the same tail exponent, since the variables q_i and a_i in Eq. (2b) have only short-ranged time dependence. Price fluctuations have distribution $P_G(G>x)\sim x^{-\alpha}$ with $\alpha\approx3$ [20]. Therefore, the asymptotic behavior of the function $\langle G\rangle_\Omega$ must be bounded from above by $\Omega^{1/2}$, since for $\langle G\rangle_\Omega\sim\Omega^\delta$ with $\delta>1/2$, the exponent α of $P_G(G>x)$ must be smaller than the empirical value of $\alpha\approx3$. Thus, for the consistency of tail exponents of G and Ω, we must have $\delta\leqslant1/2$ for large Ω, which is consistent with the flat behavior of $\langle G\rangle_\Omega$ for large Ω that we find. Zhang [1] presents an argument for $\langle G\rangle_\Omega\sim\Omega^{1/2}$.
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