Wigner rotations in laser cavities

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The Wigner rotation is important in many branches of physics, chemistry, and engineering sciences. It is a group theoretical effect resulting from two Lorentz boosts. The net effect is one boost followed or preceded by a rotation. While the term ''Wigner rotation'' is derived from Wigner's little group whose transformations leave the four-momentum of a given particle invariant, it is shown that the Wigner rotation is different from the rotations in the little group. This difference is clearly spelled out, and it is shown to be possible to construct the corresponding Wigner rotation from the little-group rotation. It is shown also that the $ABCD$ matrix for light beams in a laser cavity shares the same mathematics as the little-group rotation, from which the Wigner rotation can be constructed.

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I. INTRODUCTION

The term ''Wigner rotation'' is mentioned frequently in many branches of physics. The earliest manifestation of the Wigner rotation is the Thomas precession that we observe in atomic spectra. Thomas formulated this problem 13 years before the appearance of Wigner's 1939 paper [1,2]. The Thomas effect in nuclear spectroscopy is mentioned in Jackson's book on electrodynamics [3]. Recently, as the relativistic effects come to play more prominent roles, the Wigner rotation has become one of the key issues in the field theory of extended objects [4], electron beams [5], relativistic quark models [6,7], nuclear scattering [8], and neutrino physics [9], as well as many other areas of physics, chemistry, and engineering sciences [10].

If we perform two Lorentz boosts in different directions, the result is not a boost but a boost preceded or followed by a rotation. This rotation is commonly known as the Wigner rotation. However, if we trace the origin of this term, Wigner introduced the rotation subgroup of the Lorentz group whose transformations leave the four-momentum of a given particle invariant in its rest frame. The rotation can, however, change the direction of its spin. Indeed, Wigner introduced the concept of a ''little group'' to deal with this type of problem. Wigner's little group is the maximum subgroup of the Lorentz group whose transformations leave the four-momentum of the particle invariant. The particle does not have to be at rest.

The question then is whether the Wigner rotation, as understood in the literature, is the same as the rotation associated with the little group. We address this question and show that there is a nontrivial difference between these two rotations. We show that there is always a Wigner rotation for a given little-group rotation.

Furthermore, in this paper we report that light beams in a laser cavity perform little-group rotations, and thus the corresponding Wigner rotation. It is known that the geometrical optics of laser cavities is a form of lens optics. It is also known that para-axial lens optics can be formulated in terms of the Lorentz group. Thus, we can also formulate the cavity optics in terms of the Lorentz group. We thus expect to find effects of the Lorentz group in cavity optics also, and we report one result in this paper.

As for the mathematical method, the Lorentz group is a sophisticated group based on 4×4 matrices. However, this group shares the same algebraic properties as those of 2×2 unimodular matrices (determinant$=1$) with complex elements or six real parameters. This group is called SL$(2,c)$ and is the underlying language for 2×2 $ABCD$ matrices in optics. If we choose the matrices with real parameters, it forms a subgroup Sp(2) with three independent parameters. This subgroup shares the same algebraic property as the three-dimensional Lorentz group applicable to two spacelike and one timelike coordinates. This group is commonly called O(2,1).

The basic advantage of Sp(2) is its mathematical simplicity, while it is rich in mathematical content. It does not require professional knowledge of group theory to follow the logic based on 2×2 matrices with three independent parameters. This is the reason why it became the standard language in classical and quantum optics. This group is directly applicable to squeezed states of light in the Wigner-function representation of squeezed states [11]. This group has the same algebraic properties as SU(1,1) which is the standard language in the Fock-space representation of squeezed states [11]. Since Sp(2) has a correspondence with O(2,1), the Wigner rotation or the Thomas precession is a meaningful operation in squeezed states of light [12].

Another recent trend is that the Lorentz group is becoming prominent in classical optics, including polarization op-

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tics [13], interferometers [14], and multilayer optics [15,16]. As for lens optics, the formalism starts with 2×2 matrices representing a lens and a translation. Repeated applications of these matrices leads to a 2×2 matrix representing Sp(2). Thus, the fundamental scientific language in lens optics is clearly the group Sp(2) [17,18], which shares the same algebraic properties as those of the Lorentz group O(2,1). We can therefore explain what happens in lens optics in terms of items in special relativity such as Wigner rotation or Iwasawa decomposition. Since laser optics is derivable from lens optics, we can do the same for laser cavities.

In Sec. II, we show that the Thomas precession is a special case of the Wigner rotation. We also discuss in detail the rotation contained in Wigner's little group for massive particles. It is shown that this little-group rotation is not the Wigner rotation as known in the literature, but these two rotation angles are related. In Sec. III, we use a group theoretical technique to achieve a simplied derivation of the $ABCD$ beam transfer matrix for laser cavities. This matrix takes the same form as that of Wigner's little-group transformation matrix. In Sec. IV, we discuss how we can derive the parameters of the Wigner rotation from the geometry of the laser cavity.

II. WIGNER ROTATIONS AND LITTLE GROUPS

In the literature, the Wigner rotation comes from two successive noncollinear Lorentz boosts. If we boost along the z axis first and then make another boost along a direction that makes an arbitrary angle with the z axis in the zx plane, the result is another Lorentz boost preceded by a rotation. This rotation is known as the Wigner rotation in the literature.

In the metric (t,z,x,y), the rotation matrix which performs a rotation around the y axis by an angle ϕ is

$$R(\phi)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

and its inverse is $R(-\phi)$. The boost matrix along the z direction takes the form

$$B(0,\eta)=\begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

If this boost is made along the ϕ direction, the matrix is

$$B(\phi,\eta)=R(\phi)\ \ B(0,\eta)\ \ R(-\phi), \tag{3}$$

and its inverse is $B(\phi,-\eta)$.

Let us start with a particle at rest with its four-momentum

$$P_a=(m,0,0,0). \tag{4}$$

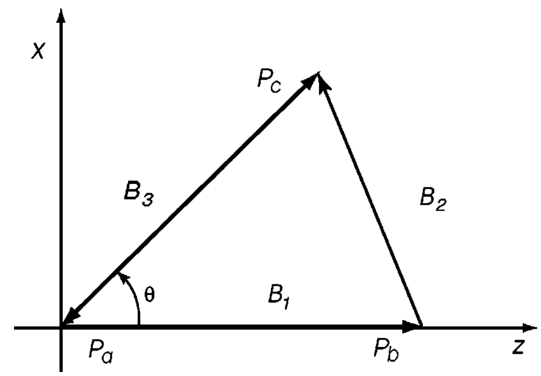

FIG. 1. Closed Lorentz boosts. Initially, a massive particle is at rest with its four-momentum P_a. The first boost B_1 brings P_a to P_b. The second boost B_2 transforms P_b to P_c. The third boost B_3 brings P_c back to P_a. The particle is again at rest. The net effect is a rotation around the axis perpendicular to the plane containing these three transformations. We may assume for convenience that P_b is along the z axis, and P_c in the zx plane. The rotation is then made around the y axis.

Then we boost this four-momentum along the z direction to make

$$P_b=m(\cosh\eta,\sinh\eta,0,0). \tag{5}$$

The corresponding Lorentz-boost matrix of the Sp(2) group is

$$B_1=\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \tag{6}$$

The kinematics is illustrated in Fig. 1.

In deriving the above result, it is sufficient to use 3×3 matrices applicable to the three-dimensional space of (t,z,x). The group of these 3×3 matrices is called O(2,1). If we use Sp(2), the 3×3 matrix algebra of O(2,1) can be reduced to the algebra of 2×2 matrices. This is a significant mathematical simplification. Furthermore, this correspondence allows us to interpret the physics of Lorentz transformations in terms of what we observe in optics laboratories, and vice versa. With this point in mind, we shall exclusively use 2×2 matrices of Sp(2) in the rest of this paper.

If we rotate P_b around the y axis by an angle θ, then the resulting four-momentum is

$$P_c=m\big(\cosh\eta,(\sinh\eta)\cos\theta,(\sinh\eta)\sin\theta,0\big). \tag{7}$$

The rotation matrix that performs this operation is equivalent to

$$R(\theta)=\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{8}$$

Instead of this rotation, we propose to obtain this four-vector by boosting the four-momentum of Eq. (5). It is tedious but straightforward to calculate this boost matrix, and this calculation was carried out by Han *et al.* in 1987 [19]. Let us call this boost matrix B_2. In the 2×2 formalism, B_2 takes the form

$$B_2=\begin{pmatrix} a_- & b \\ b & a_+ \end{pmatrix}, \tag{9}$$

with

$$a_\pm=\cosh(\lambda/2)\pm\sin(\theta/2)\sinh(\lambda/2),$$

$$b=\cos(\theta/2)\sinh(\lambda/2),$$

$$\lambda=2\tanh^{-1}\{(\tanh\eta)\sin(\theta/2)\}. \tag{10}$$

Next, we boost the four-momentum of Eq. (7) to that of Eq. (4). The particle is again at rest. The boost matrix is

$$B_3=R(\theta)B_1^{-1}R(-\theta). \tag{11}$$

It is straightforward to calculate this 2×2 matrix from the boost matrix of Eq. (6) and the rotation matrix of Eq. (8).

The net result of these transformations is $B_3B_2B_1$. This leaves the initial four-momentum of Eq. (4) invariant. Is it going to be an identity matrix? The answer is "No." The result of the matrix multiplications is a rotation matrix of the form given in Eq. (8), but with the rotation angle ω, where

$$\tan\omega=\frac{\sin\theta[\gamma^2\cos^2(\theta/2)+\gamma]}{\cos\theta[\gamma^2\cos^2(\theta/2)+\gamma]+\cosh\eta}, \tag{12}$$

with $\gamma=(\cosh\eta-1)$. This expression can be derived from $\sin(\omega/2)$ given in Ref. [19]. This matrix performs a rotation around the y axis and leaves the four-momentum of Eq. (4) invariant. It can now be written as

$$B_3B_2B_1=R(\omega) \tag{13}$$

or

$$B_2B_1=B_3^{-1}R(\omega). \tag{14}$$

This kinematics is the basis for the Thomas precession [19,20].

Let us examine next why this rotation is called the Wigner rotation. In his 1939 paper [2] on the Lorentz group Wigner did not introduce this rotation. There he introduced the concept of little groups, which are the maximum subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. He observed that the little group for a massive particle is the rotation subgroup of the Lorentz group in the Lorentz frame in which the particle is at rest. This rotation is not the same as the Wigner rotation discussed above.

Wigner's little group is not restricted to particles at rest. Then, is there a little group which leaves the four-momentum P_b of Eq. (5) invariant? In order to answer this question, let us go back to Eq. (13). There we can write B_3 as

$$B(\theta,-\eta)=R(\theta)B(0,-\eta)R(-\theta). \tag{15}$$

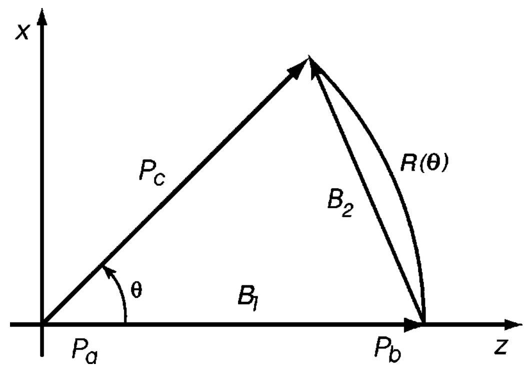

FIG. 2. Little group and Wigner rotation associated with the four-momentum P_b. This momentum can be rotated to P_c by $R(\theta)$. It can then be boosted back to P_b through the inverse of B_2. This operation corresponds to the right-hand side of Eq. (20). The net result is not an identity matrix, but a transformation which leaves the four-momentum P_b invariant. The same effect can be achieved by a Lorentz-boosted rotation matrix that appears in the left-hand side of Eq. (20). The momentum P_b is first boosted to P_a by the inverse of B_1. We can then rotate the system without changing the momentum. This rotation will change the direction of the spin. The particle can then be brought to its initial momentum P_b by the boost matrix B_1. The net result is a transformation that does not change the momentum P_b. The angle α in the transformation of Eq. (20) is precisely the Wigner rotation angle.

Then Eq. (13) becomes

$$R(\theta)B(0,-\eta)R(-\theta)B(\psi,\lambda)B(0,\eta)=R(\omega), \tag{16}$$

which can be written as

$$R(-\theta)B(\psi,\lambda)=B(0,\eta)R(\omega-\theta)B(0,-\eta). \tag{17}$$

The inverse of this expression is

$$B(\psi,-\lambda)R(\theta)=B(0,\eta)R(\alpha)B(0,-\eta) \tag{18}$$

with

$$\alpha=\theta-\omega \quad \text{or} \quad \theta=\alpha+\omega. \tag{19}$$

Here, we have introduced the angle α as a redefinition of the Wigner rotation angle for a given value of θ, but it has its own physical significance: When applied to P_b, both the right-hand side and the left-hand side of Eq. (18) leave P_b invariant. This kinematics is clearly illustrated in Fig. 2.

Then, we can write

$$B_1R(\alpha)B_1^{-1}=B_2^{-1}R(\theta), \tag{20}$$

which enables us to calculate the rotation angle α in terms of η and θ:

$$\tan\alpha=\frac{\sin\theta\cosh\eta}{\cos\theta\cosh^2\eta+(\cosh^2\eta-1)\sin^2(\theta/2)}. \tag{21}$$

This expression can also be derived from the 1986 paper by Han *et al.* where $\cos\alpha$ is given [21].

Now that we have given expressions for the angles α and ω, it is worthwhile to check our calculations by carrying out the tangent addition rule:

$$\tan\theta=\tan(\alpha+\omega)=\frac{\tan\alpha+\tan\omega}{1-\tan\alpha\tan\omega}, \tag{22}$$

using the expressions for $\tan\alpha$ and $\tan\omega$ given in Eqs. (12) and (21), respectively. The result is consistent with the addition rule of Eq. (19). Indeed, each little-group rotation is accompanied by a Wigner rotation.

Let us now write the left-hand side of Eq. (20) as

$$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}\begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}. \tag{23}$$

Now, these three matrices can be combined into one matrix:

$$\begin{pmatrix} \cos(\alpha/2) & -e^{\eta}\sin(\alpha/2) \\ e^{-\eta}\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}. \tag{24}$$

This mathematical form is quite common in the literature on lasers [22–24].

Let us go back to Eq. (23). In order to construct the maximum subgroup of the Lorentz group which leaves the four-momentum of the given particle invariant, we bring the particle to its rest frame, and then perform rotations while leaving the four-momentum of the rest particle invariant. We then boost the particle to its original frame. During this process, the four-momentum remains invariant, but its spin orientation will be changed. It is gratifying to note that this is a conjugate transformation from the group theoretical point of view. Indeed, the little group of a massive particle with a nonzero momentum is a conjugate rotation subgroup of the Lorentz group. We shall note in the following sections that the conjugate transformations in the 2×2 matrix representation can play an important role in our understanding of beam transfer matrices.

III. WIGNER'S LITTLE GROUP IN LASER CAVITIES

We are now ready to discuss what is happening in a laser cavity. Let us consider for simplicity a cavity consisting of two identical concave mirrors separated by a distance d. Then the $ABCD$ matrix for a round trip of one beam is

$$\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \tag{25}$$

where R is the radius of the mirror. This form is quite familiar to us from the laser literature [22–24]. However, the crucial question is what happens when this process is repeated many times. This question has also been addressed in the literature. For this purpose, Haus replaces one of the concave mirrors with a flat mirror and repeats the process in order to complete the cycle [23].

In this section, we propose to eliminate the auxiliary flat mirror by using a group theoretical concept, but with a simple matrix algebra. As is illustrated in Fig. 3, the traditional cycle starts from one of the mirrors, but we start here from the midpoint between the mirrors. In order to achieve this mathematically, we write Eq. (25) as

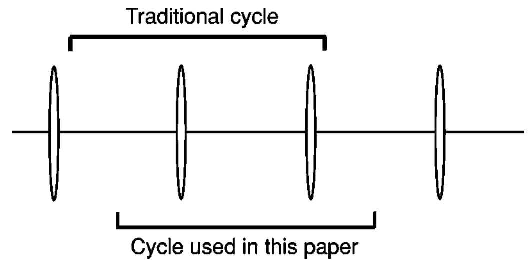

FIG. 3. Cycles in cavity and mirror optics. One complete cycle between the mirrors in a cavity is equivalent to the beam going through two lenses. The issue is where we can start the cycle. It is shown in this paper that we can eliminate the auxiliary mirror needed in the traditional cycle by starting the cycle at the midpoint between the mirrors or the lenses.

$$\begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}\begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix} \times\begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}\begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}. \tag{26}$$

In this way, we translate the system by $d/2$ using a translation matrix, and write the $ABCD$ matrix of Eq. (25) as

$$\begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1-d/R & d-d^2/2R \\ -2/R & 1-d/R \end{pmatrix}\right]^2\begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}. \tag{27}$$

Furthermore, the matrix in the middle can be written as

$$\begin{pmatrix} 1-d/R & d-d^2/2R \\ -2/R & 1-d/R \end{pmatrix} = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix}\begin{pmatrix} 1-d/R & 1-d/2R \\ -2d/R & 1-d/R \end{pmatrix} \times\begin{pmatrix} 1/\sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix}. \tag{28}$$

It is then possible to decompose the $ABCD$ matrix into the "core" matrix C and the "escort" matrix E:

$$EC^2E^{-1}, \tag{29}$$

with

$$C=\begin{pmatrix} 1-d/R & 1-d/2R \\ -2d/R & 1-d/R \end{pmatrix}$$

$$E=\begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix}. \tag{30}$$

If the process is repeated N times, the result is

$$EC^{2N}E^{-1}. \tag{31}$$

With this expression, we can concentrate on the core matrix C, and write this in the form

$$C=\begin{pmatrix} \cos\phi & -e^{\xi}\sin\phi \\ e^{-\xi}\sin\phi & \cos\phi \end{pmatrix} \tag{32}$$

with

$$\cos\phi = 1-\frac{d}{R}, \quad e^{2\xi}=\frac{R}{2d}-\frac{1}{4}. \tag{33}$$

Here both d and R are positive, and the restriction on them is that d be smaller than $2R$. This is the stability condition frequently mentioned in the literature [23,24].

Let us next write the core matrix C as

$$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}. \tag{34}$$

Here, a rotation matrix is sandwiched between a squeeze matrix and its inverse. This expression is exactly of the form of Eq. (23) for the little-group rotation.

If the light beam makes one cycle, the effect is C^2, and its expression is

$$C^2=\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}\begin{pmatrix} \cos(2\phi) & -\sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{pmatrix}\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}. \tag{35}$$

Indeed, the beam makes a little-group rotation of 2ϕ when it completes one cycle.

If the light beam makes N round trips, we have to compute C^{2N}, and the result is

$$C^{2N}=\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}\begin{pmatrix} \cos(2N\phi) & -\sin(2N\phi) \\ \sin(2N\phi) & \cos(2N\phi) \end{pmatrix} \times\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} \tag{36}$$

or

$$C^{2N}=\begin{pmatrix} \cos(2N\phi) & -e^{\eta}\sin(2N\phi) \\ e^{-\eta}\sin(2N\phi) & \cos(2N\phi) \end{pmatrix}. \tag{37}$$

In this paper, we noted first that the matrices in lens/mirror optics can be formulated in terms of the three-parameter Sp(2) group. Because of the correspondence between Sp(2) and SO(2,1), we expect Wigner rotations in this branch of physics, and we have shown that light beams perform little-group rotations in the laser cavity.

We considered here only the simplest cavity consisting of two identical mirrors. However, there are other interesting cavities [22] and their combinations. It would be an interesting project to exploit fully the Lorentz-group content of these optical systems.

IV. WIGNER ROTATIONS IN LASER CAVITIES

In Sec. II, we emphasized that the angle α, not ω, is the rotation angle directly associated with Wigner's little group. We could therefore insist that $R(\alpha)$ is the Wigner rotation or the original Wigner rotation, as Han *et al.* did in 1988 [25]. On the other hand, since $R(\omega)$ is widely known as the Wigner rotation in the literature, we choose to call ω the Wigner rotation angle.

Is it possible to construct this angle from one cycle of the beam transfer in a laser cavity? The answer is "Yes." The kinematics of Fig. 2 is essentially the same as that of Fig. 1, as we noted in Sec. II.

The laser cavity gives the two parameters η and α. From them, it is possible to calculate the angle θ. From Eq. (21), the expression for θ becomes

$$\theta = 2\tan^{-1}(\sin(\alpha/2)\sqrt{\cosh\eta}), \tag{38}$$

and, according to Eq. (19), the Wigner rotation angle ω is

$$\omega=\theta-\alpha. \tag{39}$$

Indeed, one Wigner rotation corresponds to the beam going through one cycle in the laser cavity.

V. CONCLUDING REMARKS

The group Sp(2) is an algebra of 2×2 matrices with unit determinant. Its elements are real numbers, and there are three independent parameters. It does not require a group theoretical background to deal with these 2×2 matrices. However, these matrices generate many interesting mathematical results useful in understanding physics.

This group shares the same algebraic property as other groups, such as SU(1,1), which is the basic scientific language of squeezed states of light [11,26]. This Sp(2) group is also the underlying language for classical optics, including multilayer optics and lens optics [16,18]. If expanded to SL(2,c), this group can serve as the basic language for polarization optics and interferometers [13,14]. We have seen in this paper that this group again is the basic language of laser cavities.

In addition, the group Sp(2) shares the same algebraic properties as the group of Lorentz transformations, called O(2,1), applicable to a space consisting of two space dimensions and one time dimension. This allows us to interpret what is happening in optics in terms of the language of special relativity, and vice versa. Indeed, this group is powerful enough to combine relativity and optics into one broad-based scientific discipline.

The term "Wigner rotation" is commonly used in the literature. The reason is that Wigner observed that the little group applicable to a relativistic particle is the three-dimensional rotation group in the Lorentz frame where the particle is at rest. We noted in this paper that there is a difference between this rotation and the Wigner rotation commonly mentioned in the literature. We have clarified this difference in this paper.

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