Granular superconductors and a sandpile model with intrinsic spatial randomness

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(Received 16 November 2001; revised manuscript received 15 May 2002; published 28 August 2002)

We present a model for investigation of self-organized criticality applicable to a real physical system: a granular superconductor. The model demonstrates self-organized behavior even in circumstances when other models do not.

DOI: 10.1103/PhysRevE.66.026128 PACS number(s): 05.65.+b, 74.50.+r

I. INTRODUCTION

The concept of self-organized criticality (SOC) proposed in 1987 by Bak, Tang, and Wiesenfeld (BTW) [1] is now extensively used in various fields of science for explanation of behavior of many-body systems. According to this concept, there are a number of giant dissipative dynamical systems that are able to accumulate small external perturbations. Under their action, these systems naturally evolve into a critical state; the latter is a self-reproducing one and can persist without fine tuning of external parameters. The critical state under consideration is an ensemble of metastable states. During the evolution process, the critical system migrates from one metastable state to another by means of so-called "avalanches." Avalanches may be of small or large size, but both of these are initiated by small external local perturbations. Such a critical state is called a self-organized one, and the mathematical criterion of self-organization is the power-law behavior of probability density of avalanche sizes.

Despite the wide range of dynamical systems that behave like self-organized ones, the classical sandpile model and its modification [1–3] still remain the main objects for theoretical investigations of self-organized criticality. The experimental studies of this phenomenon were carried out on a real sandpile only [4]. Therefore, the problem of finding a useful physical system with self-organization, available for experimental investigations of SOC, remains very real. It was shown in Refs. [5–9] that some of the most appropriate objects for this purpose are granular superconductors.

It is known that a granular superconductor is a set of superconducting grains jointed by Josephson junctions. The interest in investigating magnetic properties of such systems increased greatly after the discovery of high-temperature superconductivity (HTSC) because most HTSC materials were initially realized as granular systems. In most of the theoretical papers [10–13], a granular superconductor is considered as an ordered array of Josephson junctions [a multijunction superconducting quantum interference device (SQUID)] that can be described by a system of differential equations for gauge-invariant phase differences.

It was found in Ref. [13] that the granular system, like the II-type hard superconductors, is able to reach the critical state, which is self-reproduced. The properties of an arising critical state depend strongly on the main system parameter $V \sim j_c a^3/\phi_0$ (a is the grain size, j_c is the intergranular critical current density, and ϕ_0 is the magnetic flux quantum). When the parameter V is large, the system has a large number of metastable states. This situation is similar to the one for a self-organized system.

Earlier, in Refs. [5–9], we studied theoretically and by computer simulations the critical state in multijunction SQUIDs. We found that under certain conditions the critical state of the system is self-organized. In the cases considered, the perturbations were generated by current injection or by a varying external magnetic field. The avalanches manifest themselves as voltage pulses and the integrated avalanche voltage plays a role of avalanche size. Therefore, the self-organized criticality can display itself as a power-law behavior of probability density of integrated avalanche voltage.

In addition, it was shown in Refs. [5,6] that for a large value of V, the system of differential equations describing the dynamics of SQUIDs can be substituted by the simplified system of maps (algorithm). In some cases, the obtained systems of maps have analogs among the earlier proposed sandpile models. For example, in the case of a two-dimensional multijunction SQUID with an injection of current in a randomly chosen junction, the system of maps coincides with an algorithm of an Abelian sandpile model [2] with junction currents as "heights." In a one-dimensional situation we obtained an algorithm of a non-Abelian one-dimensional sandpile model [3]. In this case the currents play the role of pile slopes, and the magnetic field magnitudes are the heights of the piles. It was also shown in Refs. [8,9], that the properties of simplified models of SQUIDs are equivalent to those of the original systems that are described by differential equations. However, the physical properties of real granular superconductors lying in the basis of our models provide a number of new interesting features that cannot be observed in classical sandpile models [5–9].

In this paper we present a substantially modified model of a multijunction SQUID—one-dimensional multijunction SQUID with random arrangement of junctions placed in an increasing magnetic field. The principal difference of this system from previously studied models of SQUIDs is the random location of junctions. Such a situation is natural for real SQUIDs because the manufacturing of an ideally ordered Josephson junction array is a very difficult technical problem.

We also introduce the simplified model of the system under consideration, which is analogous to the sandpile model with random toppling rules.

We study an original system described by differential equations and a simplified model by computer simulation,

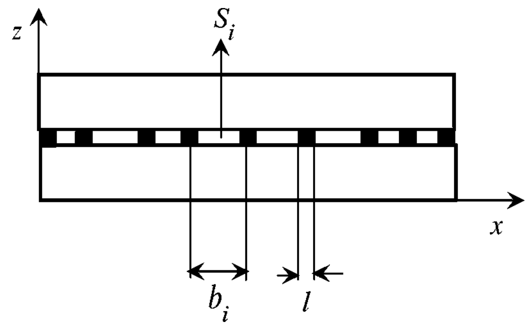

FIG. 1. The (x,z) section of a one-dimensional disordered multijunction SQUID.

and show that the results for both cases coincide. We find that the intrinsic spatial randomness substitutes the external temporal one that is needed for the appearance of self-organized criticality in classical SOC models, and that was introduced by external perturbations that were random in time. As a result, the self-organized critical state arises in a one-dimensional case and for deterministic perturbations. The latter is especially important because in earlier models studied there is no self-organization in a one-dimensional case [1], or a special method of perturbations is required [3] for self-organization to realize.

The paper is organized as follows. In Sec. II, the one-dimensional multijunction SQUID with random location of junctions is considered in detail and the simplified model of our system is constructed. Section III is devoted to the analysis of the algorithm describing the simplified model. In Sec. IV, we present the computer simulations results. In the conclusion, we formulate the main results of the paper.

II. ONE-DIMENSIONAL MULTIJUNCTION SQUID WITH RANDOM LOCATION OF JUNCTIONS

A one-dimensional multijunction SQUID under consideration can be imagined as two superconducting layers that are infinitely long in the y direction and are jointed by the Josephson junctions (Fig. 1). The junctions with size l are placed along the x axis, and the distance between the ith and $(i+1)$th junctions is a random variable b_i. The system is placed into a slowly increasing magnetic field H_{ext} aligned with the y axis. A similar system was considered earlier in Ref. [12].

Following the resistive model of the Josephson junction without thermal fluctuations, the current density j_i is written as

$$j_i = j_c \sin\varphi_i + \frac{\phi_0}{2\pi\rho}\frac{\partial\varphi_i}{\partial t}, \tag{1}$$

where j_c is the critical current density, φ_i is the gauge-invariant phase difference at the ith junction, ρ is the surface resistivity of the junction, and ϕ_0 is the magnetic flux quantum.

The relation between current density for the ith junction and for the magnetic fields in the ith and $(i-1)$th cells can be expressed by Maxwell equation

$$4\pi j_i = (H_i - H_{i-1})\frac{1}{l}, \quad i = 2\ \ldots\ N-1;$$

$$4\pi j_1 = (H_1 - H_{ext})\frac{1}{l};$$

$$4\pi j_N = (H_{ext} - H_{N-1})\frac{1}{l};$$

$$H_i = \Phi_i / S_i; \tag{2}$$

where H_i is the magnetic field in the ith cell (the cells are numbered by nearest left junction), Φ_i is the magnetic flux in the ith cell, $S_i = 2\lambda_L b_i$ is the area of the ith cell, λ_L is the London penetration depth, and l is the junction size. The magnetic flux for the ith cell can be written as

$$\Phi_i = \frac{\phi_0}{2\pi}(\varphi_{i+1} - \varphi_i). \tag{3}$$

From Eqs. (2), (3) we have the following system of equations for φ_i:

$$V\sin\varphi_i + \tau\frac{\partial\varphi_i}{\partial t} = J_i(\varphi_{i+1} - \varphi_i) + J_{i-1}(\varphi_{i-1} - \varphi_i),$$

$$i \neq 1, N;$$

$$V\sin\varphi_1 + \tau\frac{\partial\varphi_1}{\partial t} = J_1(\varphi_2 - \varphi_1) - 2\pi h_{ext};$$

$$V\sin\varphi_N + \tau\frac{\partial\varphi_N}{\partial t} = J_{N-1}(\varphi_{N-1} - \varphi_N) + 2\pi h_{ext};$$

$$V = \frac{16\pi^2 a l \lambda_L j_c}{\phi_0}; \quad \tau = \frac{8\pi a l \lambda_L}{\rho};$$

$$J_i = \frac{a}{b_i}; \quad h_{ext} = \frac{2\lambda_L a}{\phi_0} H_{ext}. \tag{4}$$

The properties of the system under consideration are strongly dependent on the main parameter V. When $V \gg 1$, each of the elements, and the system as a whole, have a large number of metastable states. It was shown earlier, in Ref. [5], that for $V \gg 1$, phase differences demonstrate very specific behavior. If the junction current density j_i exceeds the critical value j_c, then the phase changes slowly by the value $1/V$ during the long time period T, then, subsequently, quickly "slides" by 2π. Hence, the phase differences can be approximated by the following stepwise function: $\varphi_i \approx 2\pi p_i + (\pi/2)$, where p_i is an integer number. Now we can see from Eq. (3) that the cell magnetic flux can change only by integer number of flux quanta. It was shown in Refs. [5–9] that, in this case, the discrete time $t_k = kT$ can be introduced, where T is the time of slow changing of phases.

Since, during the time period T, phase φ_i can change by 2π if and only if the current density exceeds the critical value, we obtain

$$\varphi_i(k+1)-\varphi_i(k)=2\pi(\theta[z_i-z_c]-\theta[-z_i-z_c]),$$

$$z_i=\frac{8\pi a l\lambda_L}{\phi_0}j_i=z_c\sin\varphi_i+\tau/2\pi\frac{\partial\varphi_i}{\partial t},$$

$$z_c=V/2\pi=\frac{8\pi a l\lambda_L}{\phi_0}j_c, \tag{5}$$

where z_i is the dimensionless junction current density.

As a result, we have, for the dimensionless cell magnetic flux $f_i=\Phi_i/\phi_0$, the following system of maps:

$$f_i(k+1)=f_i(k)+(\theta[z_{i+1}-z_c]-\theta[-z_{i+1}-z_c])$$
$$-(\theta[z_i-z_c]-\theta[-z_i-z_c]),\quad i\neq 0,N;$$
$$f_0(k+1)=f_0(k)+\Delta h_{ext};$$
$$f_N(k+1)=f_N(k)+\Delta h_{ext};$$
$$\Delta h_{ext}=[h_{ext}(k+1)-h_{ext}(k)]. \tag{6}$$

For dimensionless junction current z_i we have the analog of Eq. (2):

$$z_i=h_i-h_{i-1},\quad i\neq 1,N;$$
$$z_1=h_1-h_{ext};$$
$$z_N=h_{ext}-h_{N-1};$$
$$h_i=\frac{2\lambda_L a}{\phi_0}H_i=J_i f_i,\quad i\neq 0,N;$$
$$h_0=f_0;\quad h_N=f_N. \tag{7}$$

The system of maps just obtained in Eqs. (6), (7) serves as ground for a sandpile model with intrinsic spatial randomness.

III. THE SANDPILE MODEL WITH INTRINSIC SPATIAL RANDOMNESS

We can rewrite the system of maps for dimensionless fluxes f_i (6) as an algorithm, usually used to describe a self-organized system model:

perturbation rules: $f_0=f_N=h_{ext}\rightarrow h_{ext}+\Delta h_{ext}$;

toppling rules:

$$\text{if } z_i>z_c,\ \text{then}\quad f_i\rightarrow f_i-1,$$
$$f_{i-1}\rightarrow f_{i-1}+1;$$
$$\text{if } z_i<-z_c,\ \text{then}\quad f_i\rightarrow f_i+1;$$
$$f_{i-1}\rightarrow f_{i-1}-1. \tag{8}$$

This algorithm is an analog of the one for a one-dimensional sandpile model [1] with dimensionless current z_i as a pile slope. However, an analog of the pile height is not f_i but the dimensionless magnetic field h_i (7). Since $h_i=J_if_i$, the algorithm (8) for this variable can be written as

perturbation rules: $h_0=h_N=h_{ext}\rightarrow h_{ext}+\Delta h_{ext}$;

toppling rules:

$$\text{if } z_i>z_c,\ \text{then}\quad h_i\rightarrow h_i-J_i,$$
$$h_{i-1}\rightarrow h_{i-1}+J_{i-1};$$
$$\text{if } z_i<-z_c,\ \text{then}\quad h_i\rightarrow h_i+J_i,$$
$$h_{i-1}\rightarrow h_{i-1}-J_{i-1}. \tag{9}$$

We see that the algorithm (9) is similar to the one for a one-dimensional sandpile [1]. However, since our model takes into account the main peculiarities of behavior of a real physical system, it differs significantly from the model [1].

First, we see from Eqs. (4), (6) and from algorithms that the system is perturbed by external magnetic field H_{ext}. This means that perturbations are applied not to a randomly chosen cell [1] but to the boundaries of the system, i.e., it is deterministic.

Next, the increasing magnetic field induces both positive and negative currents in real SQUIDs, and there are two critical current values for real superconductors. As a result, we have the second (negative) critical value for z in Eq. (6). Besides, our system is under the closed boundary conditions. This means that the total system current is conserved. Such a system was considered in detail earlier [8]. It was found that the closed boundary conditions do not prevent the appearance of self-organization because the positive and negative currents can annihilate each other. Thus, in this case the annihilation process effectively replaces the current outflow that takes place in an open system.

Finally, the main difference is that the coefficients J_i are random, whereas in Ref. [1] J_i were equal to unity. The introduction of random J_i leads to nonconservation of a total magnetic field or variable $h=\Sigma_{i=1}^{i=N-1}h_i$. It is seen from Eq. (9) that $h_i+h_{i-1}\rightarrow h_i+h_{i-1}+J_{i-1}-J_i$.

This fact naturally arises from physical principles that require only the conservation of total magnetic flux in the system. This requirement is accomplished. As we see from Eq. (8), only one flux quantum migrates from one cell to another, but due to the differences in cell areas the magnetic field in cells changes by different values. It is also seen that there are integer quantities of flux quanta in each cell. This situation is natural for $V\gg 1$.

Since such a situation is unusual for sandpile models, we consider its interpretation in detail. First, the system under consideration is not an array of sites as in the sandpile model, but a set of cells with different areas.

Second, in terms of sandpile models, in our case the amount of sand that is transferred from one cell to another is not equal to one grain. Such an amount, when toppled from the $(i+1)$th cell to the ith cell, is distributed uniformly at the whole cell area and increases the height h_i by J_i. If the same

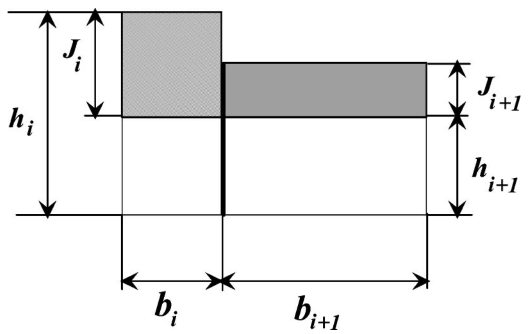

FIG. 2. The sand toppling in the sandpile model with intrinsic spatial randomness. Some amount of sand topples from the ith cell with size b_i to the $(i+1)$th cell with size b_{i+1}. The heights of the cells change by different values because of differences of cell sizes, but the equality $b_iJ_i=b_{i+1}J_{i+1}$ holds.

amount of sand topples to cell $i+1$ with a different area, by the same means, it increases the cell height h_{i+1} by J_{i+1}. It is obvious that $J_{i+1}\neq J_i$, due to the differences in cell squares. This situation is clearly illustrated in Fig. 2.

Thus, in this section we construct the model (9) which we call a *sandpile model with intrinsic spatial randomness*.

In conclusion, we note that the ''intrinsic randomness'' can be introduced for our system by random distribution of current critical values j_c or z_c. However, such randomness does not lead to self-organization.

IV. COMPUTER SIMULATION RESULTS

We studied the original system (4) and obtained a sandpile model (9) with system size $N=129$ by computer simulations. For the system (4) we use the Euler integration scheme with $dt=0.01$, $V=40$, and $\tau=1$. For the algorithm (9) we take $z_c=6.33$. We evaluate our systems in the same way usually used for an SOC system.

(1) Before starting we fix the set of random values J_i, which are unchanged during the simulation process.

(2) Starting from the state in which $\varphi_i=0$ or $z_i=h_i=0$, we perturb the system by increasing the external field h_{ext} by unity, that is, $h_{ext}\rightarrow h_{ext}+1$.

(3) After the perturbation the system is allowed to relax to the next metastable state. We assume that the system reaches the metastable state if $z_i<z_c$ or $(d\varphi_i/dt)<10^{-7}$ for every site. During the relaxation process the value of h_{ext} does not change.

(4) When the dynamics stop and the system reaches the next metastable state, we perturb it again repeating steps 3 and 4.

As in the case of the original system, as in the simplified model case, after the transition process the system reaches the critical state. This state is an ensemble of metastable states in which the variables z_i or $z_i^{st}=z_c\sin\varphi(t_{en})$ (t_{en} is the final moment of the nth avalanche) are positive and closed to a positive current critical value in the right part of the system, and negative and closed to a negative critical value in the left one. One of the large number of metastable states for the sandpile model (9) for scatter of J_i from 1 to 1.5 is shown in Fig. 3. We see from Fig. 3(b) that the distributions of h_i in the positive and negative subsystems are similar to the height distributions in the BTW model. The junction with $i=65$ plays the role of an open boundary for each of the subsystems. An avalanche arising after the perturbation leads the system to the next metastable state. The structure of the state remains the same but the values of h_i and z_i change slightly. In the case of the original system (4) we have the same situation.

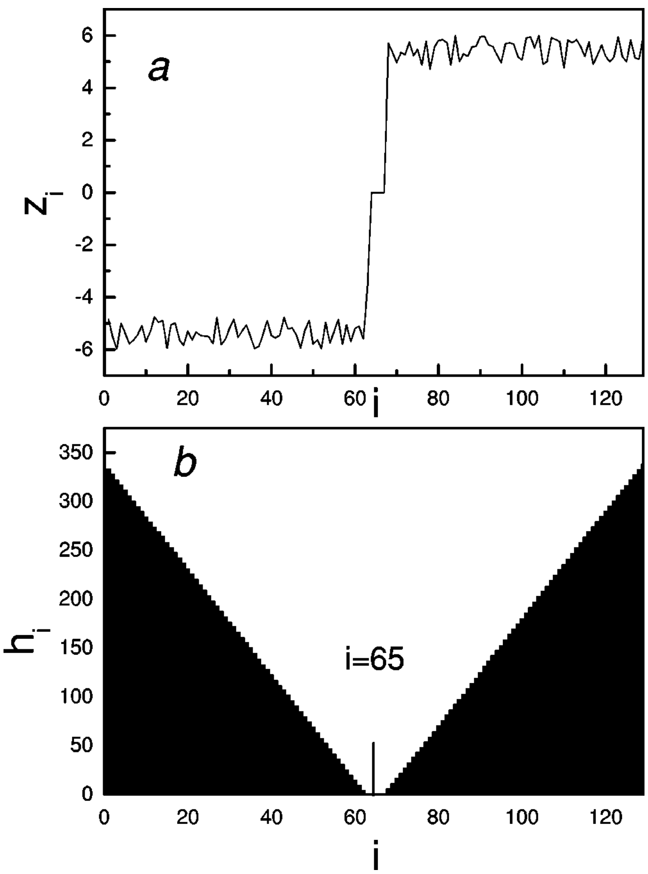

FIG. 3. Distribution of z_i and h_i in the sandpile model with intrinsic spatial randomness with J_i dispersion from 1 to 1.5.

For every avalanche in the critical state for the sandpile model (9), we calculate the quantity that is an analog of the total amount of topplings (an avalanche size in the sandpile model):

$$W_n=\frac{1}{M}\sum_{k=k_{bn}}^{k=k_{en}}\sum_{i=M+2}^{N}\{\theta[z_i(k)-z_c(k)]\}, \qquad (10)$$

where $M=(N-1)/2$, k_{bn} is the initial moment of the nth avalanche, and k_{en} is the final moment of the nth avalanche.

According to Eq. (5), for the system (4) we have an analogous quantity:

$$u_n=\frac{\phi_0}{2\pi M}\sum_{i=M+2}^{N}[\varphi_i(t_{en})-\varphi_i(t_{bn})], \qquad (11)$$

where t_{bn} and t_{en} are beginning and final moments of the nth avalanche.

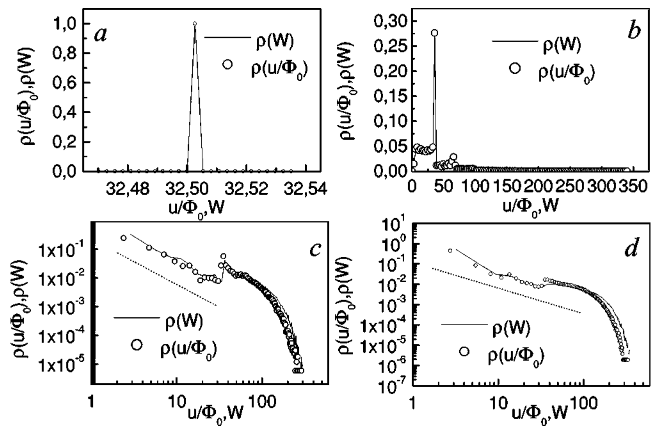

FIG. 4. Probability densities $\rho(u/\phi_0)$ and $\rho(W)$ for different dispersions of interjunction distances. (a) All J_i are the same and equal to 1. (b) J_i are scattered from 1 to 1.01. (c) J_i are scattered from 1 to 1.2, straight line has a slope $\alpha=-1.62$. (d) J_i are scattered from 1 to 1.4, straight line has slope $\alpha=-1.2$.

Note that these quantities have a clear physical meaning. It is an integral voltage of the positive part of the system during the avalanche time. This fact was discussed in detail in [5–9].

We consider our systems for some sets of J_i with different scatter. For every J_i set we calculate the probability densities $\rho(W)$ and $\rho(u/\phi_0)$. The resulting dependencies are shown in Fig. 4. From this figure we see that the results for the sandpile model with intrinsic randomness (9) and for the original system (4) coincide.

Figure 4(a) shows the probability densities for the case where all J_i are equal to unity, i.e., the situation is analogous to that considered in [1]. In this case, no self-organization is observed in either system and only a single metastable state occurs, to which the system returns after every perturbation. All avalanches have the same size $W_0\approx 32.5$, and the probability density has the form of a δ function. Figure 4(b) illustrates the case where the J_i values are randomly chosen in the interval from 1 to 1.01. One can see from Fig. 4(b) that the avalanches in the systems have different sizes, but they are appreciably fewer in number than the number of avalanches of size W_0. As the dispersion of J_i increases, the probability density becomes more and more different from the case of identical J_i, although the peak at u_0/Φ_0 is still seen [Fig. 4(c)]. This peak disappears for the dispersion in the range of 1–1.4, and the probability density becomes a power function with an exponent close to unity [Fig. 4(d)]. Therefore, the self-organization arises in the systems without dispersion of initial conditions and under fully deterministic perturbation.

V. CONCLUSIONS

The main results of the paper can be formulated as follows.

We present our model of a self-organized system, using as a ground a real physical system: one-dimensional multijunction SQUID with random location of junctions placed in increasing magnetic field. This model demonstrates the self-organized behavior in cases where there is no self-organization in earlier proposed models. An intrinsic spatial randomness introduced into the model allows us to obtain self-organization (1) in a one-dimensional case, and (2) under fully deterministic perturbation.

These results are important for experimental investigation of SOC because granular superconductors are convenient objects for experiments. The external conditions considered in our paper are the simplest ones for experimental realization.

Therefore, we can conclude that the self-organized criticality can be experimentally observed in granular superconductors as a power-law behavior of probability density of voltage.

ACKNOWLEDGMENTS

We are grateful to M. A. Pustovoit for valuable remarks. This work was supported by the Russian Foundation for Basic Research (Project Nos. 02-02-16979 and 02-02-06687), the Scientific Council of the ''Superconductivity'' direction of the program ''Topical Directions in Physics of Condensed Media'' (Project No. 96021 ''Profile''), and the state programs ''Investigations of Collective and Quantum Effects in Condensed Matter'' and ''Quantum Macrophysics.'' N.S. would like to thank the Science Support Foundation (Grant for Talented Young Researchers).

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