Microextensive chaos of a spatially extended system

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By analyzing chaotic states of the one-dimensional Kuramoto-Sivashinsky equation for system sizes L in the range $79 \leqslant L \leqslant 93$, we show that the Lyapunov fractal dimension D scales *microextensively*, increasing linearly with L even for increments ΔL that are small compared to the average cell size of 9 and to various correlation lengths. This suggests that a spatially homogeneous chaotic system does not have to increase its size by some characteristic amount to increase its dynamical complexity.

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An important phenomenon associated with sustained nonequilibrium systems is spatiotemporal chaos, a chaotic dynamical state that is spatially disordered [1,2]. An open question is how best to characterize spatiotemporal chaos so that theory can be quantitatively compared with experiment and experiment with simulation. Presently, there is no fundamental theory of nonequilibrium systems to indicate the appropriate quantities to measure and so researchers have borrowed ideas from condensed matter physics, fluid dynamics, nonlinear dynamics, and statistics. Commonly used ways to characterize spatiotemporal chaos include critical exponents [3], the two-point correlation time τ_2 [4], the largest Lyapunov exponent λ_1 [2], the Lyapunov fractal dimension D, the two-point correlation length ξ_2 [5], the dimension length ξ_δ [2,6] and other lengths [2,7,8]. However, calculations have shown that these quantities do not always lead to the same conclusions, e.g., there are systems for which the length ξ_2 diverges while the length ξ_δ remains finite as some parameter is varied [7]. Further research is therefore needed to understand the particular features of spatiotemporal chaos that are measured by any one of these quantities and how these quantities are related to one another.

In the following, we report results that provide an insight about how the dynamical complexity of a nonequilibrium system depends on the volume of the system, and about the interpretation of the dimension length ξ_δ. In 1982, Ruelle conjectured [9], and numerical calculations later confirmed [7,10–12] that the dimension D of a sufficiently large *spatially homogeneous* chaotic system should increase extensively, i.e., linearly with its volume V. Using an argument similar to that used by Landau and Lifshitz to explain the extensivity of additive quantities in thermodynamics [13], this extensivity of D can be understood heuristically as a consequence of spatiotemporal disorder. If two subsystems of a spatiotemporal chaotic system are sufficiently far apart, their coupling is weak because of the disorder and so their dynamics contribute independently and additively to the overall fractal dimension.

This picture of weakly interacting subsystems raises the question of how precisely does the fractal dimension D increase with increasing system volume in the extensive regime. One possibility is that the curve $D(V)$ may be linear only on average and has a staircase-like structure, with the steps corresponding to new degrees of freedom that appear once the system volume has increased sufficiently to include a new subsystem. The widths ΔV of the steps would then define a length scale $(\Delta V)^{1/d}$ (where d is the spatial dimensionality of the system) that would be interesting to compare with the lengths mentioned above (ξ_2, ξ_δ, etc.). Possible steplike features in the $D(V)$ curve might also be associated with the appearance of new linearly unstable modes of some nonlinear state, since the number of such modes can increase linearly on average with increasing volume [14]. Another possibility is that the curve $D(V)$ is extensive only on average but its deviation from linearity is too irregular to characterize by a single length scale. A fourth possibility is that there are no length scales associated with how D increases with V and the curve $D(V)$ is exactly linear for arbitrarily small increases in V, a situation that one could call *microextensive chaos*. In this case, it would be interesting to understand how the geometric structure of the chaotic attractor in phase space changes with V so as to produce such an exact linear behavior.

In this paper, we numerically integrate the one-dimensional Kuramoto-Sivashinsky (KS) equation—a widely studied continuum model of spatiotemporal chaos [2]—to investigate how the Lyapunov fractal dimension $D(L)$ of a homogeneous chaotic system varies with the system size L for increments ΔL that are small compared to the lengths mentioned above (ξ_2, ξ_δ, etc). With one exception [11], all prior numerical studies used increments ΔL that were large compared to these lengths and the detailed form of $D(L)$ was not determined. We show below that, in fact, the Lyapunov fractal dimension D increases linearly with L even for system increments ΔL that are tiny compared to the average cell size and to various correlation lengths. The spatiotemporal dynamics of the one-dimensional KS equation therefore provides an example of microextensive chaos. We conjecture that this will be a general property of chaotic homogeneous nonequilibria media in a sufficiently large domain.

Our calculations of D versus L yield an additional insight, namely that the onset of extensivity in D is not sharp but occurs only asymptotically with increasing L, after a sequence of alternating windows of stationary, periodic, inter-

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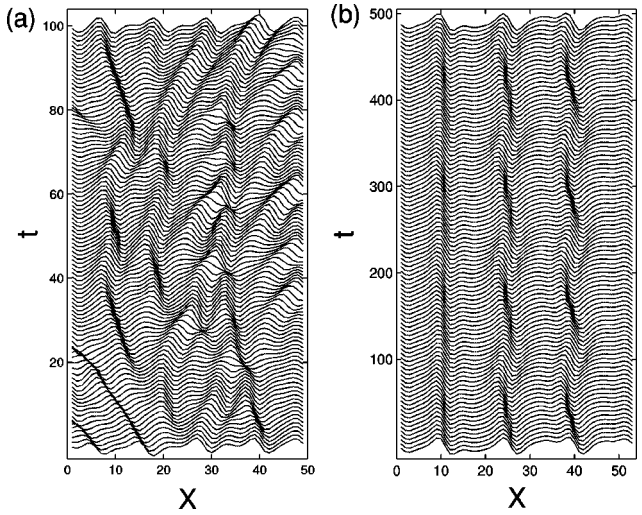

FIG. 1. Space-time evolution of the field $u(t,x)$ for two states of the Kuramoto-Sivashinsky equation (1) with rigid boundary conditions. The space-time resolution was $\Delta t=0.025$ and $\Delta x=0.166$ and the peak-to-peak amplitude is about 4. (a) Chaotic state for $L=50$. Spatial curves are plotted every $\Delta T=1$ time units starting at time $t=50\,000$. (b) A periodic state for $L=54$ with period $\tau=127.6$. Spatial curves are plotted every $\Delta T=5$ time units starting at time $t=80\,000$.

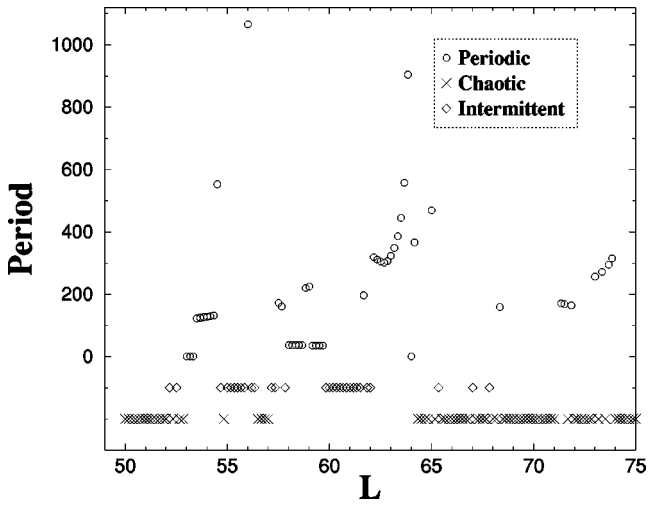

FIG. 2. Periods of numerical solutions to Eq. (1) versus the system length L. Each integration was started from small-amplitude random initial conditions and then integrated 500 000 time units. There is a complex sequence of windows corresponding to chaotic, constant, intermittent, and periodic dynamics. Chaotic and intermittent solutions have been assigned an arbitrary period of -200 and -100, respectively, so that all the data could be compared on one plot.

mittent, and chaotic dynamics. (Such alternating windows have been noted before for the KS equation [15]—and in other systems [16]—but have not been studied with such fine resolution in L as we do here.) These results suggest the possibility that windows of nonchaotic behavior may persist to arbitrarily large values of L but become too narrow to be detected. If true, the dimension $D(L)$ may not be a continuous curve and extensive behavior occurs only between the narrow windows of nonchaotic dynamics.

Our results were obtained by numerical integrations of the one-dimensional Kuramoto-Sivashinsky equation in the form

$$\partial_t u(t,x) = -\partial_x^2 u - \partial_x^4 u - u\,\partial_x u, \quad x \in [0,L], \tag{1}$$

on an interval of length L, with rigid boundary conditions $u=\partial_x u=0$ at $x=0$ and at $x=L$. (Figure 1 shows a chaotic and periodic state for $L=50$ and 54, respectively.) The spatial derivatives were approximated by second-order-accurate finite differences on a uniform spatial mesh, and a standard operator-splitting method was used for the time integration [17]. For given initial conditions and interval length L, we used the Kaplan-Yorke formula [18] to calculate the Lyapunov fractal dimension $D(L)$ in terms of all of the positive and some of the negative Lyapunov exponents λ_i. These exponents were obtained using a standard algorithm that involves integrating many copies of the linearized KS equation along a given orbit of the KS equation [19].

The demonstration of microextensive scaling by the above numerical methods was delicate since the Lyapunov exponents λ_i and so D converge noisily and slowly [20] toward their infinite time limits. As the increment ΔL in system size became smaller, the corresponding increment in dimension ΔD was more difficult to determine since ΔD became comparable to the fluctuations in the dimension curve $D(T)$ as a function of integration time. The exponents λ_i and dimension $D(L)$ were also sensitive to the values of the spatial resolution Δx and temporal resolution Δt, to the renormalization time $T_{\rm norm}$ for the Lyapunov vectors, and to the total integration time. For nearly all runs reported below, we used values of $\Delta x=0.167$, $\Delta t=0.025$, and $T_{\rm norm}=10$ and confirmed the correctness of the corresponding results by comparing the values with spatial and temporal resolutions up to four times larger and for integration times as long as 10^6 time units.

We now turn to our results. Our starting point was the pioneering calculation of Manneville [10], who used numerical integrations of Eq. (1) with rigid boundary conditions to demonstrate for the first time that the fractal dimension D scaled extensively with the system size L. For $L\geqslant 50$, he found that $D=0.230L-2.70$, which implies a dimension length [6] of $\xi_\delta=(dD/dL)^{-1}=1/0.230\approx 4.4$ [22]. This length is somewhat smaller than the average cellular size $\lambda=2\pi/q_{\rm max}=2\sqrt{2}\pi\approx 8.8$ corresponding to the fastest growing linear mode $q_{\rm max}=1/\sqrt{2}$. Based on these results, we chose to calculate the fractal dimension $D(L)$ in constant increments $\Delta L=1.0$ that were much smaller than these lengths. In contrast, the smallest increment used by Manneville was $\Delta L=50$ for which the fractal dimension changes by about 12.

Manneville's linear dependence of D on L suggested that for $L\geqslant 50$, only spatiotemporal chaos exists. In contrast, we find that there is a complicated sequence of different dynamical states over the range $50\leqslant L\leqslant 75$ and then only chaotic states for $75<L<93$ [23]. Figure 2 summarizes our results for the range $50\leqslant L\leqslant 75$ by plotting the period of each state as a function of L. We observe four kinds of states: fixed

points, time-periodic states, chaos, and intermittent states in which one kind of time dependence alternates irregularly with a different kind of time dependence. To combine all the results on a single plot, chaotic states were arbitrarily assigned a period of -200, intermittent states a period of -100, fixed points a period of 0, and periodic states a direct estimate of their period based on repeating features of the time series.

There are two interesting features of the dynamical states of Fig. 2 in addition to the occurrence of many windows of alternating dynamics for this range of L. First, we found that for a given system size L, numerical integrations using up to seven different random initial conditions (each consisting of uniformly distributed numbers in the interval $[-0.1,0.1]$) led to only one state. Thus empirically there seems to be only one basin of attraction for each system size and we do not expect hysteresis in the range $50<L<75$. Second, we found rather remarkably that the fractal dimension D of each chaotic state in Fig. 2 lay on Manneville's extensive curve $D(L)=0.230L-2.70$ with $D \geqslant D(50)=8.8$ (a least-squares fit of our chaotic states gave the almost identical curve $0.227L-2.85$). Thus the states jump abruptly from low-dimensional $D=1$ periodic states to high-dimensional chaotic states that are scaling extensively with the system size. We did not try to characterize the intermittent states, e.g., by their fractal dimension or by the scaling properties of the fractional duration of a particular phase [21].

Over the range $78<L<93$, only chaotic states were observed. Fig. 3 shows that the corresponding values of the Lyapunov fractal dimension D lie on a straight line that has the same slope (to two digits) and intercept as that found by Manneville over the much larger range $50<L<400$. *Thus the fractal dimension shows microextensive scaling: a linear dependence on L for system increments* ΔL *that are much smaller than any characteristic length scale such as the average cell size or various correlation lengths.* Given the similar result obtained by Xi *et al.* [11] for a different mathematical model of spatiotemporal chaos, we conjecture that microextensive scaling will be a general feature of spatiotemporal chaos in sufficiently large, approximately homogeneous nonequilibrium systems.

In conclusion, we have demonstrated the occurrence of microextensive scaling of the Lyapunov fractal dimension with system size for the one-dimensional Kuramoto-Sivashinsky equation with rigid boundary conditions. This suggests that a spatially extended nonequilibrium dynamical system does not have to increase its volume by some minimal amount for the fractal dimension to increase. Correspondingly, the dimension length ξ_δ does not have some direct physical meaning as the characteristic size of a dynamical subsystem, it is simply determined by the linear growth of D with L. Our calculations suggest two questions for further exploration. One is whether there is a cutoff system size L_c above which only chaotic solutions are found for the KS equation. Second is to understand mathematically how the geometry in phase space of a strange attractor changes with system size L such that the fractal dimension D varies exactly linearly with L.

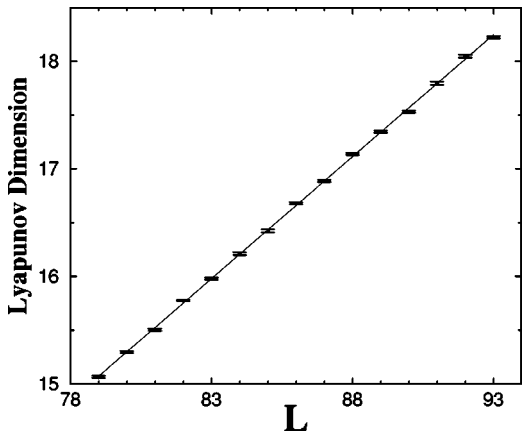

FIG. 3. The Lyapunov fractal dimension D of chaotic solutions to Eq. (1) versus system size L for $79 \leqslant L \leqslant 93$.The dimension values accurately fall on a straight line, demonstrating the occurrence of microextensive scaling.The straight line $D=0.227L-2.85$ was obtained by a least-squares fit to the points and agrees well with Manneville's result [10] of $D=0.230L-2.70$ over the much larger range $50<L<400$. The error bar for each point corresponds to a relative error of at most 0.05% in D. The error bar was determined by the peak-to-peak fluctuations of D versus integration time T.

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[1] M.C. Cross and P.C. Hohenberg, Science **263**, 1569 (1994).
[2] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[3] Philippe Marcq, Hugues Chaté, and Paul Manneville, Phys. Rev. E **55**, 2606 (1997).
[4] D.A. Egolf and H.S. Greenside, Phys. Lett. A **185**, 395 (1994).
[5] B.J. Gluckman, C.B. Arnold, and J.P. Gollub, Phys. Rev. E **51**, 1128 (1995).
[6] Henry S. Greenside, in *Semi-Analytic Methods for the Navier Stokes Equations*, edited by K. Coughlin, CRM Proceedings and Lecture Notes, Vol. 20. (American Mathematical Society, Providence, RI, 1999), URL: http://arXiv.org/abs/chao-dyn/9612004.
[7] Corey O'Hern, David Egolf, and Henry Greenside, Phys. Rev. E **53**, 3374 (1996).
[8] T. Bohr *et al.*, Nature (London) **372**, 48 (1994); Scott M. Zoldi *et al.*, Phys. Rev. E **58**, R6903 (1998); Ralf W. Wittenberg and Philip Holmes, Chaos **9**, 452 (1999).
[9] David Ruelle, Commun. Math. Phys. **87**, 287 (1982).
[10] Paul Manneville, *Macroscopic Modeling of Turbulent Flows*, edited by O. Pironneau, Lecture Notes in Physics, Vol. 230 (Springer, New York, 1985), pp. 319–326.

[11] Haowen Xi, Raúl Toral, J.D. Gunton, and Michael I. Tribelsky, Phys. Rev. E **62**, R17 (2000).
[12] J.-P. Eckmann and C.E. Wayne, J. Stat. Phys. **50**, 853 (1988); K. Kaneko, Prog. Theor. Phys. Suppl. **99**, 263 (1989); P. Grassberger, Phys. Scr. **40**, 346 (1989); L. Keefe, Phys. Lett. A **140**, 317 (1989); M.C. Strain and H.S. Greenside, Phys. Rev. Lett. **80**, 2306 (1998); D.A. Egolf *et al.*, Nature (London) **404**, 733 (2000).
[13] L. Landau and E. Lifshitz. *Statistical Physics*, 3rd ed. (Pergamon, New York, 1980), Vol. 5, Pt. 1.
[14] Y. Pomeau, A. Pumir, and P. Pelce, J. Stat. Phys. **37**, 39 (1984).
[15] P. Manneville, in *Propagation in Systems Far from Equilibrium*, edited by J.E. Wesfried Springer Series in Synergetics, Vol. 41 (Springer-Verlag, New York, 1988), pp. 265–280.
[16] P. Grassberger, Phys. Scr. **40**, 346 (1989); A. Torcini *et al.*, Physica D **53**, 85 (1991); I. Aranson and L. Kramer, Rev. Mod. Phys. **74**, 99 (2002).
[17] Paul Manneville, *Dissipative Structures and Weak Turbulence*. Perspectives In Physics (Academic Press, New York, 1990).
[18] Edward Ott, *Chaos in Dynamical Systems* (Cambridge University Press, New York, 1993).
[19] T.S. Parker and L.O. Chua, *Practical Numerical Algorithms for Chaotic Systems* (Springer, New York, 1989); see, also, the earlier references, I. Shimada and T. Nagashima, Prog. Theor. Phys. **61**, 1605 (1979); G. Benettin *et al.*, Meccanica **9**, 21 (1980).
[20] Isaac Goldhirsch, Pierre-Louis Sulem, and Steven A. Orszag, Physica D **27**, 311 (1994).
[21] Pierre Bergé, Yves Pomeau, and Christian Vidal, *Order Within Chaos* (Wiley, New York, 1984).
[22] It is interesting to note that the number of positive Lyapunov exponents increases more slowly than the dimension D as $N_+ = 0.14L - 1.5$, while the number of unstable modes increases more rapidly as $N_u \approx 0.32L$ [14].
[23] As summarized in Manneville's review article [15], others have studied states and their bifurcations for the 1d KS equation with *periodic* boundary conditions over about the same range in length. A different sequence of different dynamical behaviors is observed compared to what we find in Fig. 2.