Nonlinearity effects in the kicked oscillator

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The quantum kicked oscillator is known to display a remarkable richness of dynamical behavior, from ballistic spreading to dynamical localization. Here we investigate the effects of a Gross-Pitaevskii nonlinearity on quantum motion, and provide evidence that the qualitative features depend strongly on the parameters of the system.

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The dynamical behavior of quantized area-preserving maps has proven to be one of the most relevant fields in the discipline of quantum chaos (see [1]); in particular, the discovery of *quantum dynamical localization* [2], namely, the quantal suppression of classical deterministic diffusion, has provoked a vast amount of theoretical and experimental work. The paradigmatic example in this field is the quantum kicked rotator (see [3]), obtained upon quantization of the classical standard map [4]. From a classical point of view the system is of Kolmogorov-Arnold-Moser (KAM) nature: for small values of the stochasticity parameter global transport is inhibited by invariant curves: once the last invariant torus is destroyed, transport properties abruptly change and, in typical situations of strong chaos, there is a diffusive spreading in momentum [4,5], which characterizes also the quantum motion for times shorter than the break time t_b, where the quantum localization regime sets in and the momentum spreading is suppressed [6]. We remark that this picture is valid for *generic* values of the effective Planck's constant; quantum resonant motion, characterized by ballistic spreading, appears when $\hbar$ assumes rational values [2,8].

Another example of a quantum system originating from a two-dimensional area-preserving map is the kicked harmonic oscillator (see [9–11] and references therein): the classical Hamiltonian is

$$\mathcal{H}(p,x,t)=\frac{1}{2m_0}p^2+\frac{m_0}{2}\omega_0^2x^2+\varepsilon\cos(k_0x)\,\delta_{T_0}(t), \quad (1)$$

where the time dependence is through the periodic δ function

$$\delta_{T_0}(t)=\sum_{m=-\infty}^{\infty}\delta(t-T_0m).$$

By rescaling the variables $\tilde{x}=k_0x$ and $\tilde{t}=\omega_0t$ we realize that the dynamics is dependent only on the parameters

$$\mathcal{K}=\frac{\varepsilon k_0^2}{m_0\omega_0}, \quad T=T_0\omega_0. \quad (2)$$

In particular, the *resonant* case ($T=2\pi p/q$) is characterized by the presence of a stochastic web (for arbitrarily small values of $\mathcal{K}$) supporting unbounded transport [12,13], while in the nonresonant case a threshold $\mathcal{K}(E_0)$ exists below which unbounded transport is not sustained [14].

The kicked harmonic oscillator has been proposed as a model of different physical phenomena from electronic transport in semiconductor superlattices [15] to ion traps [16]. In the latter case the harmonic potential is representative of the ion trap, while the kicking term arises from a time periodic standing wave laser field. Obviously, such examples require a proper quantum mechanical treatment of the Hamiltonian (1), the corresponding Schrödinger equation being (once expressed in the dimensionless variables $\tilde{t}=\omega_0t$ and $\tilde{x}=\sqrt{m_0\omega_0/\hbar x}$)

$$i\frac{\partial}{\partial\tilde{t}}\psi=\left(-\frac{1}{2}\frac{\partial^2}{\partial\tilde{x}^2}+\frac{1}{2}\tilde{x}^2+\sigma\cos(\xi\tilde{x})\,\delta_T(\tilde{t})\right)\psi \quad (3)$$

so that the quantum dynamics depends upon three parameters,

$$\sigma=\frac{\varepsilon}{\hbar}, \quad \xi=k_0\sqrt{\frac{\hbar}{m_0\omega_0}}, \quad T=T_0\omega_0. \quad (4)$$

Once again, the behavior is quite sensitive to number theoretic properties of T: in particular, the *crystal* cases [10] $T=2\pi/q$ with $q\in\{1,2,3,4,6\}$ admit a one-parameter group of commuting generalized translations commuting with the Hamiltonian (exceptional parameter values [10] may also lead to two-parameter groups); the corresponding dynamical behavior is diffusive (or ballistic in the exceptional cases). We remark that the $q=4$ case corresponds to the (symmetric) kicked Harper model [17]. Outside resonant parameter values there are indications of a localization-delocalization transition for resonant noncrystal cases [11], while the simulations reported in [10] for nonresonant cases suggest dynamical localization [18] (however, we observed a delocalization transition in the irrational case, too).

Recently, it has been suggested [19] that the widespread interest and experimental activity in Bose-Einstein condensation [20] (see also recent experiments with condensates in

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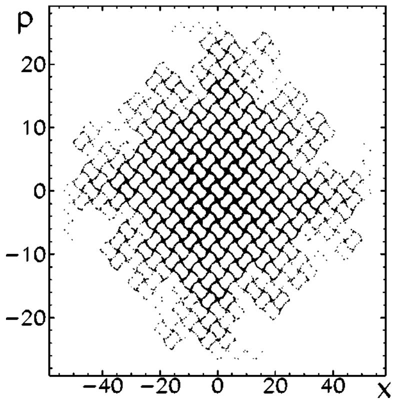

FIG. 1. Classical transport along the stochastic web.

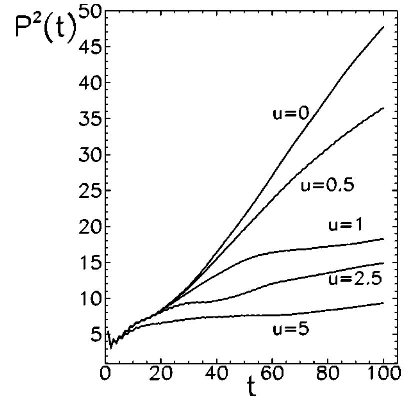

FIG. 2. Quantum transport ($q=4$ one-parameter symmetry group). See the text for details.

weakly chaotic settings [21]) makes it natural to study the effect on Gross-Pitaevskii nonlinearities [22] in the kicked oscillator (thus turning it into a model of a trapped condensate under a laser field in the spirit of [16]). The Gross-Pitaevskii nonlinear correction to the Schrödinger equation is of the form $u|\psi|^2\psi$, where the coefficient u is of the same sign as the scattering length [23] (we will deal mainly with a positive u in what follows). Using the same rescaling in dimensionless variables mentioned in the quantum case, the equation reads

$$i\frac{\partial}{\partial \tilde{t}}\psi=\left(-\frac{1}{2}\frac{\partial}{\partial \tilde{x}^2}+\frac{1}{2}\tilde{x}^2+\sigma\cos(\xi\tilde{x})\,\delta_T(\tilde{t})+v|\psi|^2\right)\psi, \tag{5}$$

where now

$$v=\frac{u}{\hbar}\sqrt{\frac{m_0}{\hbar\omega_0}}. \tag{6}$$

Even if the cubic nonlinearity acts like an effective *repulsive* potential, the main observation in [19], as regards the dynamical effect of the Gross-Pitaevskii (GP) nonlinearity in the crystal $q=6$ case, was its tendency to oppose quantum spreading; it was suggested that this is due to a breakup of quantum symmetries for nonzero nonlinearity. Before presenting the results of our simulations we have to mention that nonlinearity effects were also considered a few years ago for the kicked rotator [24,25] (for a cubic nonlinearity of opposite sign). Here the scenario is quite different. When the nonlinearity is absent the system exhibits quantum dynamical localization: a sufficiently strong nonlinearity may then induce chaotic transitions between localized modes, leading to (subdiffusive) delocalization.

To investigate the effect of the nonlinearity, we studied Eq. (5) in two different regimes: a crystal ($q=4$) case, and an irrational case [$T=\pi/(\sqrt{5}+1)$]. The evolution of the kicked oscillator is conveniently studied by using a discretized propagator [10]

$$\psi(x',t')=\int dx\,\mathcal{G}(x',x;t')\,\psi(x,0) \tag{7}$$

where

$$\mathcal{G}(x',x;t)=C\exp\left\{\frac{im_0\omega_0}{2\hbar\sin(\omega_0 t)}\times[(x^2+x'^2)\cos(\omega_0 t)-2xx']\right\}, \tag{8}$$

and once the discretized positions $x_i=(i-N/2)\Delta_x$ are introduced we have that the propagator $\mathcal{G}$ is unitary if we put

$$\Delta_x=\left(\frac{2\pi\hbar\sin(\omega_0 t)}{m_0\omega_0 N}\right)^{1/2}. \tag{9}$$

In the coordinate representation the action of kicks is multiplicative on the wave function.

Simulation of quantum evolution is considerably more complicated once we introduce a nonzero nonlinearity: to propagate the wave function between kicks we separate the time independent part of the Hamiltonian into the oscillator and the nonlinear part, and use the lowest order split method [26] (this typically requires using ten time steps between consecutive kicks in order to get stable results). As an initial state we consider the ground state of the GP equation (without kicks) shifted into the chaotic region nearest to the origin. The ground states for different values of the nonlinearity parameter are obtained by evolving an eigenstate of the

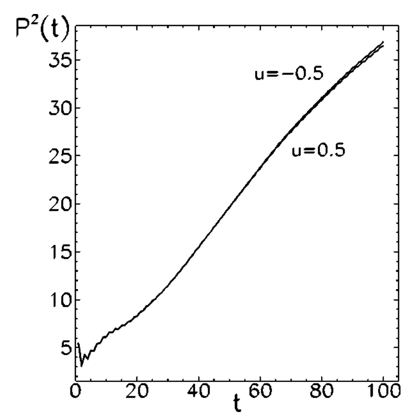

FIG. 3. $P^2(t)$, for $u=+0.5$ and $u=-0.5$, and the same conditions as in Fig. 2.

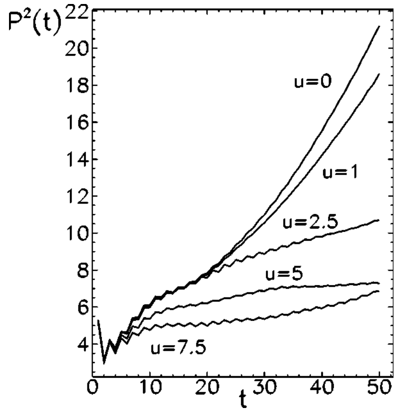

FIG. 4. Quantum transport ($q=4$ two-parameter symmetry group), $\epsilon=0.7$. The configuration space has been discretized with 2^{12} points.

quantum harmonic oscillator under the imaginary time Gross-Pitaevskii equation [27].

The first case we take into account is a *crystal* $q=4$ example. In Fig. 1 we show the classical phase picture, exhibiting unbounded transport along the stochastic web. We put $T=\pi/2$ (and ω_0 fixed in such a way that generalized translations form a one-parameter group of symmetries) and $\epsilon=0.7$ (all other linear quantum parameters are fixed by taking $\hbar=1$ and $\xi=\sqrt{2}$; we will adopt this choice for other examples, too).

This case is characterized by a one-parameter group of symmetries (generalized phase space translations) and thus the quantum case is expected [10] to exhibit a diffusive momentum spreading (the evolution corresponds to the upper line in Fig. 2). The effect of nonlinearity is considered by taking $u=0.5$, 1, 2.5, and 5; the corresponding curves are shown in Fig. 2. Such simulations are performed by using an $N=2^{14}$ discretization of the position variable (which has been checked to provide a reliable choice up to the considered evolution time, by comparing the results with a simulation with twice the number of points). To smooth out oscillations in the evolving patterns we plot the integrated second moment

$$P^2(t)=\frac{1}{t}\sum_{k=0}^{t-1}\langle(p_k-p_0)^2\rangle. \tag{10}$$

The qualitative features confirm the observation in [19], namely, that the most striking effect of the nonlinearity is to oppose quantum delocalization. This has been claimed to be due to symmetry breaking effects of the nonlinearity, inhibiting transport along delocalized Floquet states. We remark that at a classical level also related features have been observed; if noise is added to the kicked Harper map, transport along the stochastic web is slowed down [28].

In principle, a positive nonlinearity acts like a repulsive potential, but here the symmetry breaking effect is not related to the sign of the effective potential, as we see from Fig. 3, where it is shown that the sign of the nonlinearity has a tiny effect on momentum spreading.

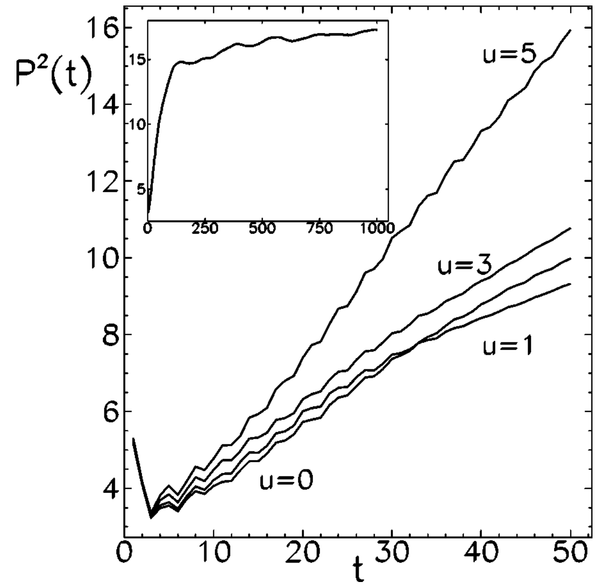

FIG. 5. Quantum transport irrational case. Discretization is over 2^{14} points. The inset shows the kicked oscillator case on a longer time scale.

If this view is correct, the effect must equally appear in the resonant crystal case, where a group of two-parameter symmetries leads to ballistic transport [10]. Such an effect is indeed evident even for short times (see Fig. 4); after a characteristic time scale, which shrinks as the nonlinearity increases, the deviation from the kicked oscillator case is more and more marked as u increases.

A priori, the situation is different when we consider the oscillator outside the *crystal* regime; to this end we analyzed the case in which $\epsilon=1.4$ and $T=\pi/(\sqrt{5}+1)$. In this case the kicked oscillator displays dynamical localization: the striking observation is that here the nonlinearity acts in an opposite way, enhancing the quantum delocalization (see Fig. 5). So, when symmetries are not present in the quantum case, nonlinearity seems to play a completely different role. This is at least in qualitative agreement with what happens to the kicked rotator evolving under a nonlinear Schrödinger equation, or even when noise is superimposed on the quantum evolution [29]. We have checked that the same happens even for higher values of ϵ, when the oscillator undergoes a delocalization transition.

In conclusion, we have analyzed how nonlinearity influences a complex and physically relevant quantum system, the kicked harmonic oscillator. We provide evidence that, at least at moderate times, it opposes quantum diffusion when transport is linked to symmetry properties of the linear Hamiltonian, while it may lead to diffusion enhancement when no symmetry breaking occurs.

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