Nonreciprocal microwave band-gap structures

P. A. Belov, S. A. Tretyakov, and A. J. Viitanen
Department of Electrical and Communications Engineering, Helsinki University of Technology, P.O. Box 3000, FIN-02015 HUT, Finland
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An electrically controlled nonreciprocal electromagnetic band-gap material is proposed and studied. The new material is a periodic three-dimensional regular lattice of small magnetized ferrite spheres. In this paper, we consider plane electromagnetic waves in this medium and design an analytical model for the material parameters. An analytical solution for plane-wave reflection from a planar interface is also presented. In the proposed material, a new electrically controlled stop band appears for one of the two circularly polarized eigenwaves in a frequency band around the ferrimagnetic resonance frequency. This frequency can be well below the usual lattice band gap, which allows the realization of rather compact structures. The main properties of the material are outlined.

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I. INTRODUCTION

Microwave and photonic band-gap materials are extensively studied in the literature (see, for a review, [1]). They are periodic structures made of various dielectric or metal inclusions. Naturally, all the electromagnetic phenomena in such materials are reciprocal. In this study, we introduce a nonreciprocal microwave analogy of photonic band-gap materials. Here we will show that a nonreciprocal stop band material can be realized using small magnetized ferrite spheres as periodically positioned inclusions. The new crystal has several important features: (1) the properties (in particular, location of the stop band) are electrically controlled; (2) the stop band occurs at frequencies at which the period is much smaller than the wavelength (a possibility for reducing device size); and (3) nonreciprocal properties can be exploited to design novel devices. More generally speaking, combining the periodicity resonance and the ferrimagnetic resonance gives more flexibility in the design of materials with the desired properties of stop bands.

The additional low-frequency stop band that appears near the resonant frequency of a single spherical inclusion has the same nature as in photonic crystals of metallic (modeled by the free-electron plasma constitutive relations) spheres [2]. In that paper, arrays of small resonant spheres were considered using numerical simulations (as in [3] and [4]) and the Maxwell Garnett mixing rule. The main difference between phenomena in arrays of metal spheres and ferrite spheres is in the nonreciprocal nature of the spheres. This leads to new nonreciprocal properties of the crystal as a whole, seen in the properties of eigenwaves and in reflection from a crystal boundary. From the point of view of applications, the new structure allows electrical control of its properties.

Regarding the modeling approach, the stop band near the resonance of a single inclusion was explained in [2] as an effect of mode hybridization accomplished in the effective-medium treatment. Moreover, it was assumed that the periodic structure is incidental for this phenomenon. Here we make a more detailed analytical investigation of the modal structure near and inside this band gap, comparing the results with the effective-medium (Maxwell Garnett) approach. The results reveal important differences in the results, which show that the periodicity is also quite important near the resonance of a small inclusion.

In this paper, we build an analytical model for plane-wave propagation in nonreciprocal crystals of ferrite spheres, and give numerical examples for dispersion curves and the reflection coefficient from a half space. The present analysis is restricted to the propagation along the geometrical axis of the structure. The bias field that magnetizes the spheres is along the same axis. The size of ferrite spheres is assumed to be small compared to the wavelength and to the lattice period. This restriction allows us to derive the dispersion relation in analytical form, which gives a clear and general physical interpretation of the main properties of the crystal without recourse to numerical techniques like those used in [3] and [4].

The main theoretical problem in studying electromagnetic properties of arrays of small particles is the calculation of the local field exciting the inclusions. For dense arrays (lattice constant is smaller or comparable with the wavelength) the interaction dyadic can be approximately found in analytical form [5,6]. Analytical approximation of the local field makes it possible to develop simple boundary conditions that simulate electromagnetic properties of nonreciprocal dipole arrays. In our studies of planar regular arrays of small ferrite spheres [7], we have found that very broadband nonreciprocal polarization transformation can be realized in reflected or transmitted fields. In this study, we will make use of the analytical model for the local field, together with numerical summation of Floquet modes. One planar array of ferrite spheres can be modeled by a generalized impedance condition [7]. Further, the interaction between planar arrays in a three-dimensional "crystal" can be taken into account by a modification of the boundary condition for every layer, as was done for reciprocal arrays in [8]. The resulting theory gives us a model that can be applied both for dense arrays and for resonant distances between the inclusions.

In the low-frequency regime, we can identify two solutions with two plane circularly polarized eigenwaves in the corresponding effective medium. Comparing the eigennumbers with the propagation factors for the axial propagation in ferrite media $q=\omega\sqrt{\epsilon(\mu_s\pm\mu_a)}$, we find the effective diagonal (μ_s) and off-diagonal (μ_a) components of the permeability tensor of the crystal. For larger distances between

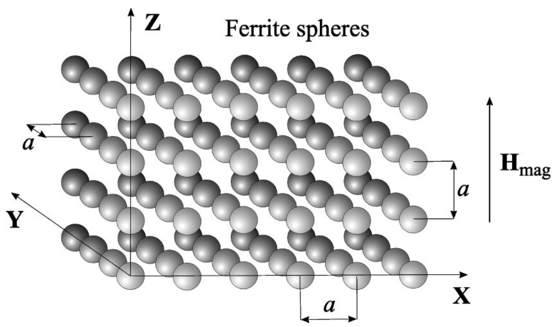

FIG. 1. Inner geometry of the medium.

crystal planes, the effective medium concept loses its sense, and we study the crystal in terms of its dispersion curves and the properties of the eigenwaves.

II. DIPOLE APPROXIMATION

We consider a regular three-dimensional simple cubic lattice with period a of small magnetized ferrite spheres in free space or an isotropic dielectric. The geometry is illustrated in Fig. 1. In this study, we make an assumption that the size of every particle (ferrite sphere) is small compared to the wavelength in the host medium. Due to the small size, electric polarizability of the ferrite spheres can be neglected in the frequency band of interest (around the ferrimagnetic resonance and at low frequencies).

The magnetic dipole moment of each particle is $\mathbf{m}=\bar{\bar{\alpha}}\cdot\mathbf{H}_{\rm loc}$, where $\bar{\bar{\alpha}}$ is the polarizability dyadic of an inclusion and $\mathbf{H}_{\rm loc}$ is the local magnetic field. For nonreciprocal particles, dyadic $\bar{\bar{\alpha}}$ has a nonzero antisymmetric part. If the sphere is magnetized along some axis z, the polarizability dyadic $\bar{\bar{\alpha}}$ is planar in the plane orthogonal to z, and it is given by (e.g., [9])

$$\bar{\bar{\alpha}}=\alpha\bar{\bar{I}}_t+j\alpha_a\bar{\bar{J}}, \tag{1}$$

where the transverse unit dyadic $\bar{\bar{I}}_t=\bar{\bar{I}}-\mathbf{u}_z\mathbf{u}_z$ and the antisymmetric dyadic $\bar{\bar{J}}=\mathbf{u}_z\times\bar{\bar{I}}$. The coefficients read

$$\alpha=\mu_0\frac{\omega_0\omega_M}{\omega_0^2-\omega^2}V, \tag{2}$$

$$\alpha_a=-\mu_0\frac{\omega\omega_M}{\omega_0^2-\omega^2}V. \tag{3}$$

Here, ω_0 is the Larmor precession frequency proportional by constant bias magnetic field $\mathbf{H}_{\rm mag}$, $\omega_M=\gamma M_s$, γ is the gyromagnetic ratio between the magnetic moment and the angular momentum of the electron, and M_s is the saturation magnetization. Furthermore, V is the sphere volume. In the coordinate system corresponding to circularly polarized vectors $\mathbf{e}_\pm=(\mathbf{u}_x\mp j\mathbf{u}_y)/\sqrt{2}$, this polarizability dyadic takes a simple form:

$$\bar{\bar{\alpha}}=(\alpha+\alpha_a)\mathbf{e}_-\mathbf{e}_++(\alpha-\alpha_a)\mathbf{e}_+\mathbf{e}_-. \tag{4}$$

Due to the energy conservation law discussed in detail in [7], the polarizability of lossless inclusions must satisfy the following conditions:

$$\frac{1}{2}\,{\rm Im}\{\bar{\bar{\alpha}}^{-1}+(\bar{\bar{\alpha}}^{-1})^T\}=\frac{k^3}{6\pi\mu_0}\bar{\bar{I}}, \tag{5}$$

$${\rm Re}\{\bar{\bar{\alpha}}^{-1}-(\bar{\bar{\alpha}}^{-1})^T\}=0. \tag{6}$$

Here, $k=\omega\sqrt{\epsilon_0\mu_0}$ is the free-space wave number, and T denotes transposition. In the lossless case, the imaginary part of α and the real part of α_a are responsible for dipole scattering from individual spheres. The above energy conservation requirements are valid in regular crystals, where there is no scattering.

Expression (1) together with (2), (3) is approximate and does not take into account radiation losses, so we should make corrections to these formulas in order to satisfy Eqs. (5) and (6). We can apply these conditions in any coordinate system with the same result. For the system corresponding to circularly polarized basis vectors we find, using Eqs. (4) and (5), the following relation:

$${\rm Im}\left\{\frac{1}{\alpha\pm\alpha_a}\right\}=\frac{k^3}{6\pi\mu_0}. \tag{7}$$

Using this result we can write, instead of formula (1), an expression that correctly takes into account the radiation losses:

$$\bar{\bar{\alpha}}=\left(\frac{\mathbf{e}_+\mathbf{e}_-}{\alpha-\alpha_a}+\frac{\mathbf{e}_-\mathbf{e}_+}{\alpha+\alpha_a}+j\frac{k^3}{6\pi\mu_0}[\mathbf{e}_+\mathbf{e}_-+\mathbf{e}_-\mathbf{e}_+]\right)^{-1}. \tag{8}$$

We will use this formula in our theory in order to satisfy the energy balance exactly. Of course, it is an approximate approach because we use formulas (2), (3), assuming that they give correct values of the real parts of the inverse polarizability.

III. GENERAL EIGENVALUE EQUATION

The dispersion equation for the electromagnetic crystal under consideration can be obtained from the equation for the dipole moment through the local field at some arbitrary chosen (reference) particle. If we assume that an eigenwave traveling along z is described by the wave vector $\mathbf{q}=q\mathbf{u}_z$, the following equation for the dipole moment of the reference particle $\mathbf{m}_0$ holds:

$$[\bar{\bar{\alpha}}^{-1}(\omega)-\bar{\bar{C}}(\omega,q)]\cdot\mathbf{m}_0=0, \tag{9}$$

$$\bar{\bar{C}}(\omega,q)=\sum_{(l,m,n)\neq(0,0,0)}\bar{\bar{G}}(\omega,\mathbf{R}_{l,m,n})e^{-jqan}, \tag{10}$$

where $\bar{\bar{C}}(\omega,q)$ is the dynamic three-dimensional interaction dyadic, $\bar{\bar{G}}(\omega,\mathbf{R})$ is the dyadic Green's function describing the magnetic field produced by a magnetic dipole at the point of space with radius vector $\mathbf{R}$ from the center of the dipole, and $\mathbf{R}_{l,m,n}$ is the radius vector from the center of the reference particle to the particle with indices l, m, n.

The main problem is how to calculate the sums corresponding to the interaction constant. The usual approach for summation of this series is based on summation by layers. In that case we have

$$\bar{\bar{C}}(\omega,q)=\sum_{n=-\infty}^{+\infty}\bar{\bar{\beta}}(n)e^{-jqan}. \tag{11}$$

Here, $\bar{\bar{\beta}}(0)$ is the interaction dyadic corresponding to the zeros layer (the sum of the Green functions from all the particles of layer $n=0$ without the reference particle), and $\bar{\bar{\beta}}(n)$ for $n\neq 0$ are dyadics describing interactions between zero and nth layers. A simple analytical method of calculation of these dyadics for the axial propagation is presented in [5,6,8]. In our planar case, the interaction dyadic is simply $\bar{\bar{\beta}}=\beta\bar{\bar{I}}_t$ with

$$\beta(0)=\frac{\omega}{4\eta a^2}\left(\frac{\cos kR_o}{kR_o}-\sin kR_o\right)+j\left(\frac{k^3}{6\pi\mu_0}-\frac{\omega}{2\eta a^2}\right), \tag{12}$$

where $\eta=\sqrt{\mu_0/\epsilon_0}$ and parameter $R_0=a/1.438$ [5,6], and for $n\neq 0$

$$\beta(n)=\frac{\omega}{4\eta a^2}\Bigg[\frac{R_0'^2}{R_0'^2+(na)^2}\frac{\cos k\sqrt{R_0'^2+(na)^2}}{k\sqrt{R_0'^2+(na)^2}}-\left(1+\frac{(na)^2}{R_0'^2+(na)^2}\right)\sin k\sqrt{R_0'^2+(na)^2}\Bigg]-\frac{1}{4\pi\mu_0}\Bigg[\left(\frac{1}{(|n|a)^3}-\frac{k^2}{|n|a}\right)\cos(kna)+\frac{k}{(na)^2}\sin(kna)\Bigg]-j\frac{\omega}{2\eta a^2}\cos(kna) \tag{13}$$

with $R_0'=a/1.521$ [8].

It is also possible to write an exact expression for $\beta(n)$ $(n\neq 0)$ using the Floquet mode expansion [10]:

$$\beta(n)=-\frac{j}{2\mu_0 a^2}\sum_{m=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\left[k^2-\left(\frac{2\pi m}{a}\right)^2\right]\times\frac{e^{-j\sqrt{k^2-(2\pi m/a)^2-(2\pi l/a)^2}|n|a}}{\sqrt{k^2-\left(\frac{2\pi m}{a}\right)^2-\left(\frac{2\pi l}{a}\right)^2}}, \tag{14}$$

where the square root branch is defined by $\mathrm{Im}\sqrt{2}<0$. For our purposes, this series can be used for practical calculations because of enough quick convergence even for $n=1$. For larger n, only one leading term gives a very accurate result. Expression (13) is reasonable to use in calculations of the layer field at positions near (as compared to the period a) to the array plane, where the Floquet expansion converges very slowly. On the other hand, that closed-form model is not accurate for large distances from the layer plane, which in our case corresponds to $n>1$.

Using the above results, we write the eigenvalue equation (9) for the axial propagation in form

$$[\bar{\bar{\alpha}}^{-1}-C\bar{\bar{I}}_t]\cdot\mathbf{m}_0=0,\quad C(\omega,q)=\sum_{n=-\infty}^{+\infty}\beta(n)e^{-jqan}, \tag{15}$$

which gives nontrivial solutions when the determinant of the dyadic is zero. The problem is easier to analyze in the coordinate system of circularly polarized eigenvectors since this leads to two separate equations for right- and left-hand circularly polarized eigenmodes:

$$\frac{1}{\alpha\mp\alpha_a}=C \tag{16}$$

(the plus sign corresponds to the left-hand circularly polarized solution).

As discussed above, only the real parts of coefficients β are calculated approximately by analytical or numerical means. The imaginary parts are known exactly from the energy conservation law. Actually, it can be easily shown from Eqs. (12), (13), and (15), by evaluating the summation and making use of Eq. (7), that

$$\mathrm{Im}\{\bar{\bar{C}}\}=\mathrm{Im}\{\bar{\bar{\alpha}}^{-1}\}=\frac{k^3}{6\pi\mu_0}\bar{\bar{I}}_t. \tag{17}$$

We should note that in Eq. (16) the imaginary parts cancel out, so this is a purely real equation.

As in [8], we can first consider an approximate solution. From a distance, the separate dipole layers look like "homogeneous infinite plates" with continuous averaged current densities. As a result, the interaction coefficient for $n\neq 0$ is [8]

$$\beta(n)\approx -j\frac{\omega}{2\eta a^2}e^{-jk|n|a}. \tag{18}$$

The first term in the expression for $\beta(0)$ can be combined with the term $\beta(n)$, and the eigenvalue equation is then

$$\mathrm{Re}\left\{\frac{1}{\alpha_s\mp\alpha_a}\right\}=\frac{\omega}{4\eta a^2}\left(\frac{\cos kR_o}{kR_o}-\sin kR_o\right)-\mathrm{Re}\left\{j\frac{\omega}{2\eta a^2}\sum_{n\neq 0}e^{-jk|n|a}e^{-jqna}\right\}. \tag{19}$$

The infinite sum can be written in closed form [8]

$$\sum_{n\neq 0}e^{-jk|n|a}e^{-jqna}=\frac{j\sin ka}{\cos ka-\cos qa}-1, \tag{20}$$

and the dispersion equation becomes

$$\frac{\omega}{2\eta a^2}\frac{\sin ka}{\cos ka-\cos qa}=\mathrm{Re}\left\{\frac{1}{\alpha\mp\alpha_a}\right\}-\frac{\omega}{4\eta a^2}\left(\frac{\cos kR_o}{kR_o}-\sin kR_o\right). \tag{21}$$

This can be rewritten in the classical form known for waveguides with periodic insertions [11]:

$$\cos qa = \cos ka - \frac{2\sin ka}{\mathrm{Re}\left\{\frac{4\eta a^2/\omega}{\alpha \mp \alpha_a}\right\} - \left(\frac{\cos kR_o}{kR_o} - \sin kR_o\right)}. \tag{22}$$

Here, the denominator has the meaning of the normalized equivalent sheet impedance describing every layer of the dipoles [8]. This formula gives the propagation factors in a simple analytical form. We will call it the zero-Floquet mode approach because it does not take into account evanescent Floquet components.

More accurately, using the full expressions (13) or (14) for the interaction coefficient $\beta(n)$ for $n \neq 0$, the dispersion equation reads

$$\frac{\omega}{2\eta a^2}\frac{\sin ka}{\cos ka - \cos qa} = \frac{2\eta a^2}{\omega}\mathrm{Re}\left\{\frac{1}{\alpha \mp \alpha_a}\right\} - \frac{\omega}{4\eta a^2}\left(\frac{\cos kR_o}{kR_o} - \sin kR_o\right) - 2\sum_{n=1}^{\infty}\left(\mathrm{Re}\{\beta(n)\} + \frac{\omega}{2\eta a^2}\sin kna\right)\cos qna. \tag{23}$$

This equation cannot be solved analytically as Eq. (22), and special methods must be applied to solve this nonlinear equation. The solutions give us the propagation constants q of eigenwaves.

Using the Floquet expansion (14), this equation can be rewritten in closed form:

$$\frac{\omega}{2\eta a^2}\frac{\sin ka}{\cos ka - \cos qa} = \mathrm{Re}\left\{\frac{1}{\alpha \mp \alpha_a}\right\} - \frac{\omega}{4\eta a^2}\left(\frac{\cos kR_o}{kR_o} - \sin kR_o\right) - \frac{1}{2\mu_0 a^2}\sum_{m,l\neq 0}\frac{k^2 - \left(\frac{2\pi m}{a}\right)^2}{\sqrt{\left(\frac{2\pi m}{a}\right)^2 + \left(\frac{2\pi l}{a}\right)^2 - k^2}}$$
$$\times\left[\frac{\sinh\left(a\sqrt{\left(\frac{2\pi m}{a}\right)^2 + \left(\frac{2\pi l}{a}\right)^2 - k^2}\right)}{\cosh\left(a\sqrt{\left(\frac{2\pi m}{a}\right)^2 + \left(\frac{2\pi l}{a}\right)^2 - k^2}\right) - \cos(qa)} - 1\right]. \tag{24}$$

Here, we should stress that this equation has an infinite number of complex roots. Approximately, it has a root near every singularity point $(\mathrm{Im}\{q\} \approx -j2\pi/a\sqrt{m^2+l^2}; m,l = 1,2,...)$. We are interested in the first roots—the real ones (propagating modes) and the complex ones with $\mathrm{Im}\{q\} < 2\pi/a$.

In the properties of the eigenwaves, the nonreciprocal nature of the crystal is seen from the fact that the polarization of eigenwaves traveling along the opposite directions of axis z are different. For waves traveling along the positive z direction, only the right-circular polarized mode has a ferrimagnetic band gap, while for the oppositely bound waves the left-circular polarized dipole moment distribution has the same property. This is because the sense of the dipole moment precession is determined by the direction of the bias field.

IV. REFLECTION COEFFICIENT

If one can neglect higher-order modes generated at an interface between a crystal and free space, the knowledge of the dominant propagation constant of eigenwaves (obtained above) is enough to find the characteristic impedance of the medium and the reflection coefficient from finite-sized samples (for example, from a half space). Let us consider a planar interface between a half space filled with a square cell lattice of ferrite spheres ($z>0$, or index $n \geqslant 0$) and free space. Suppose that a normally incident plane electromagnetic wave $\mathbf{H}e^{-jkz}$ is exciting the medium. The layer-to-layer distribution of the dipole moments in the material is assumed to be a plane wave described by the propagation constant q, which can be found as the dominant solution of dispersion Eq. (23). q is real if there is such a solution of the dispersion equation or it is a complex number with the largest (negative) imaginary part, corresponding to the main evanescent mode with the slowest decay factor. This means that we neglect the transition layer and assume that the first layer of spheres is already excited in the same way as spheres in the bulk. In the theory of dielectrics, it is known that the surface effect correction is actually rather small [12,13].

Let us write the equations for the dipole moment of some Nth layer located deep inside the medium ($N \gg 1$) in terms of the local magnetic field produced by all the spheres and the incident wave:

$$\mathbf{m}_0 e^{-jqaN} = \bar{\bar{\alpha}} \cdot \left(\mathbf{H} e^{-jkaN} + \sum_{n=-N}^{+\infty} \beta(n) e^{-jqa(n+N)} \mathbf{m}_0 \right). \tag{25}$$

Using Eq. (15) we have

$$\sum_{n=-N}^{+\infty} \beta(n) e^{-jqan} \mathbf{m}_0 = \left(\bar{\bar{\alpha}}^{-1} - \sum_{n=-\infty}^{-N-1} \beta(n) e^{-jqan} \right) \cdot \mathbf{m}_0 . \tag{26}$$

For layers with large numbers n in terms $\beta(n)$, the evanescent Floquet modes can be neglected (18), which means that the sum in the right-hand side can be easily calculated:

$$\sum_{n=-\infty}^{-N-1} \beta(n) e^{-jqan} = -j \frac{\omega}{2\eta a^2} \sum_{n=N+1}^{+\infty} e^{-j(k-q)na}$$
$$= -j \frac{\omega}{2\eta a^2} \frac{e^{-j(k-q)a(N+1)}}{1-e^{-j(k-q)a}}. \tag{27}$$

Furthermore, we get

$$\mathbf{m}_0 = j \left[\frac{\omega}{2\eta a^2} \frac{e^{-j(k-q)a}}{1-e^{-j(k-q)a}} \right]^{-1} \mathbf{H}. \tag{28}$$

The reflected magnetic field produced by the excited lattice is

$$\mathbf{H}_R = -\frac{j\omega}{2\eta a^2} \sum_{n=0}^{+\infty} e^{-j(k+q)an} \mathbf{m}_0 = -\frac{j\omega}{2\eta a^2} \frac{\mathbf{m}_0}{1-e^{-j(k+q)a}}. \tag{29}$$

Finally, the reflection coefficient (for magnetic fields) reads

$$R_H = -\frac{1-e^{-j(q-k)a}}{1-e^{-j(q+k)a}} = e^{jka} \frac{\sin[(k-q)a/2]}{\sin[(k+q)a/2]}. \tag{30}$$

To prevent possible confusion, we rewrite the reflection coefficient in terms of the electric field (the two values simply differ by sign). For electric fields

$$R = -e^{jka} \frac{\sin[(k-q)a/2]}{\sin[(k+q)a/2]}. \tag{31}$$

Nonreciprocal properties of the crystal are clearly seen from this formula. Because the propagation factor q is different for the two eigenmodes (right- and left-circularly polarized dipole moments of the spheres), the reflection phenomenon is nonreciprocal. Right-hand circularly polarized incident waves effectively excite the spheres, and the reflected plane wave is left-hand circularly polarized. In the opposite case, if the incident plane wave is left-hand circularly polarized, the propagation constant q is very close to k, and the reflection coefficient (31) is very small.

V. CHARACTERISTIC IMPEDANCE AND EFFECTIVE PARAMETERS

Since the reflection coefficient is known, we can introduce normalized wave impedance Z of the medium *defining* it through

$$R = \frac{Z-1}{Z+1}. \tag{32}$$

In doing so, we get

$$Z = 2\,\mathrm{cotan}(ka) \sin(qa/2) e^{jka/2}. \tag{33}$$

The result depends on the reference plane. It appears advantageous to rewrite the above equations for the plane at a half period distance from the first layer grid of the medium:

$$R = -\frac{\sin[(k-q)a/2]}{\sin[(k+q)a/2]}, \tag{34}$$

then

$$Z = \frac{\tan(qa/2)}{\tan(ka/2)}. \tag{35}$$

In this definition, it is obvious that if q is purely imaginary, then

$$Z = j \frac{\tanh(|q|a/2)}{\tan(ka/2)}, \tag{36}$$

and we have full power reflection as from a magnetic wall: $|R| = 1$.

Furthermore, we can *define* effective permittivity and permeability through the wave impedance and the propagation constant:

$$\epsilon_{\mathrm{eff}} = \frac{q}{kZ}, \quad \mu_{\mathrm{eff}} = \frac{qZ}{k}. \tag{37}$$

In this definition, the use of the effective permittivity and permeability together with the standard Maxwell boundary conditions will give the correct reflection coefficient. Spatial dispersion of the medium is seen from dependence of the effective parameters on the propagation constant.

VI. QUASISTATIC LIMIT

In the low-frequency limit, when it can be assumed that $ka \ll 1$ and $qa \ll 1$, the eigenvalue equation can be simplified by approximating the sine and cosine functions for small argument values. The eigenvalue equation reduces to the form

$$\frac{(qa)^2}{2} = \frac{(ka)^2}{2} + \frac{(ka)^2}{2\mu_0 a^3 \left(\mathrm{Re}\left\{ \frac{1}{\alpha \mp \alpha_a} \right\} - C \right)}, \tag{38}$$

$$C=\frac{1}{2\mu_0 a^2}\left[\frac{1}{2R_0}+\sum_{n=1}^{\infty}\left(\frac{R_0'^2}{[R_0'^2+(na)^2]^{3/2}}-\frac{1}{\pi a n^3}\right)\right]. \tag{39}$$

The solution of this equation and the effect of the infinite sum is considered in detail in [8]. As a result, one arrives at the static limit, which coincides with the Lorenz-Lorentz formula $[C=1/(3\mu_0 a^3)]$. Using Eqs. (37) and (35) for small ka and qa, we obtain the relative effective permittivity $\varepsilon_{\rm eff}=1$ and the relative effective permeability $\mu_{\rm eff}=q^2/k^2$. Thus, we obtain a formula that has the same meaning as the Maxwell Garnett mixture rule:

$$\mu_{\rm eff}=1+\frac{1}{\mu_0 a^3\,\mathrm{Re}\left\{\frac{1}{\alpha\mp\alpha_a}\right\}-\frac{1}{3}}. \tag{40}$$

This can be written, using the expression for the polarizability factors, as

$$\mu_{\rm eff}=1+\frac{1}{\frac{a^3}{V}\left(\frac{\omega_0\mp\omega}{\omega_M}\right)-\frac{1}{3}}=\mu_s\pm\mu_a. \tag{41}$$

Here, it is easy to identify the effective diagonal and off-diagonal components μ_s and μ_a of the permeability, respectively (as shown in the above formula), and write them in the classical form [9]:

$$\mu_s=1+\frac{\tilde\omega_0\tilde\omega_M}{\tilde\omega_0^2-\omega^2},\quad \mu_a=\frac{\omega\tilde\omega_M}{\tilde\omega_0^2-\omega^2}. \tag{42}$$

In this formula, $\tilde\omega_0=\omega_0-f\omega_M/3$ is the shifted resonance frequency, $\tilde\omega_M=f\omega_M$, and $f=V/a^3$ is the ferrite volume fraction. This simple and expected result has a clear meaning. Parameter ω_M is proportional to the magnetization density. In the composite, this parameter is reduced by the volume fraction ratio f. The resonance frequency ω_0 is proportional to the external bias field, which is reduced by the Lorentz factor $f/3$.

VII. NUMERICAL EXAMPLE

As an illustrative example, we consider a microwave crystal formed by a simple cubic lattice of small monocrystal spheres of yttrium iron garnett ferrite ($Y_3Fe_5O_{12}$). The saturation magnetization of $Y_3Fe_5O_{12}$ corresponds to $\omega_M/(2\pi)=4.9$ GHz, and we choose the bias field so that the resonance frequency $\omega_0/(2\pi)=10$ GHz. The sphere radius is $r=1$ mm and the lattice period is $a=3$ mm. For further comparisons with the composite properties, the resonant curve of an individual ferrite sphere is shown in Fig. 2. The absolute value of the polarizability for right-hand circularly polarized field $\alpha+\alpha_a$ is plotted as a function of the frequency. In the model of lossless scatterers, the quality factor is determined by scattering losses given by formula (8). Note that the resonant curve is rather narrow.

We have made calculations of the band-gap structure of the nonreciprocal crystal under consideration using approaches (22) and (24). In the first case, only the fundamental Floquet mode in the periodical medium was taken into account. The second approach accurately includes all the Floquet modes in the model. A new ferrimagnetic band gap with 0.76 GHz width (9.76–10.52 GHz) was found for the right-circular polarization (see Fig. 3) and, naturally, no band gap was found for the left-circular polarization. All the results below correspond to the right-hand circularly polarized eigenmodes. For the orthogonal polarization, the medium is nearly transparent in the frequency band of analysis. Let us note that the usual lattice band gaps appear at much higher frequencies.

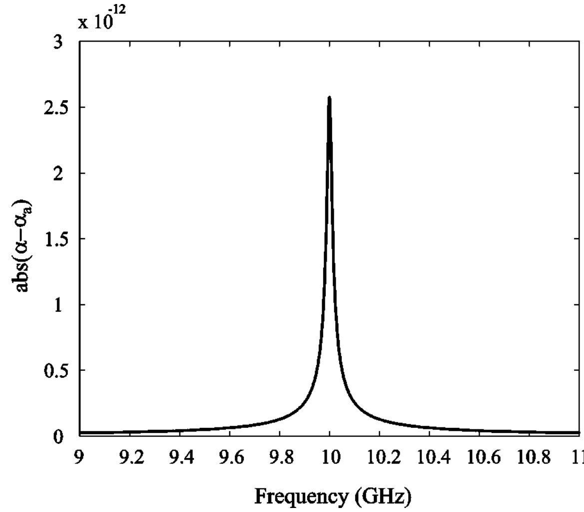

FIG. 2. Resonant curve for a single sphere in free space. Right-circularly polarized incident field.

At frequencies within the band gap, all solutions for the propagation constant are purely imaginary. The imaginary part of the propagation constant (decay constant) is plotted in Fig. 4 for the main evanescent modes with the slowest decay.

Comparing the results that follow from the exact and approximate formulas, we have found that the difference between approximate (22) and exact (23) solutions is no more than 0.1% for the propagating modes (at frequencies outside the band gap). Calculation and comparison of decay constants inside the band gap by formula (22) (zero-Floquet mode approach), (23) (exact approach), and (40) (quasistatic

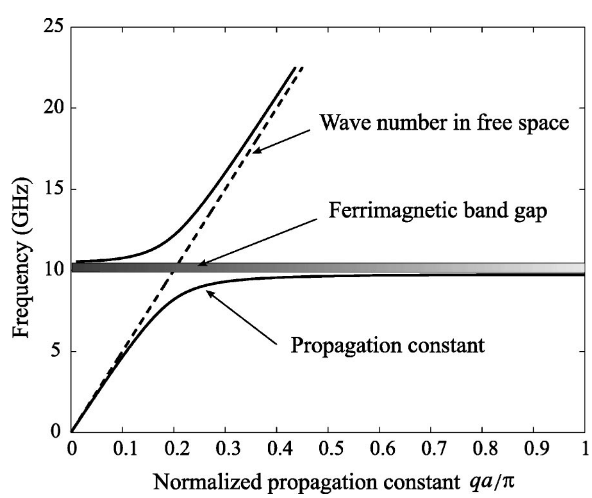

FIG. 3. Band-gap structure (right circular polarization).

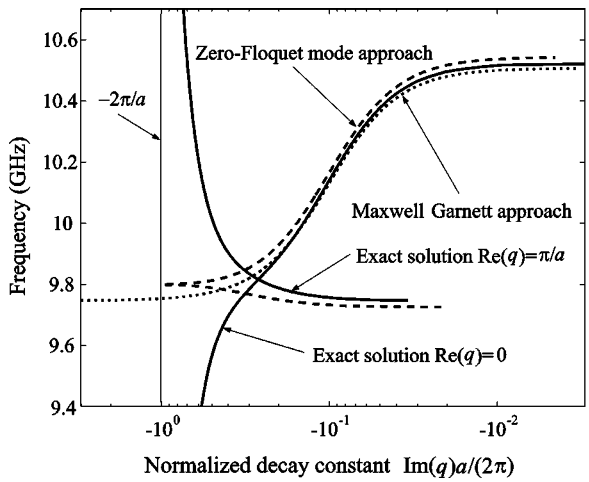

FIG. 4. Structure of the ferrimagnetic band gap: two branches of the exact solution with Re(q)=0 and Re(q)=π/a, respectively (thick lines), together with $q=-2\pi j/a$ asymptote (thin line), compared with the Maxwell Garnett approach (dotted line) and zero-Floquet mode approach (22) (dashed line).

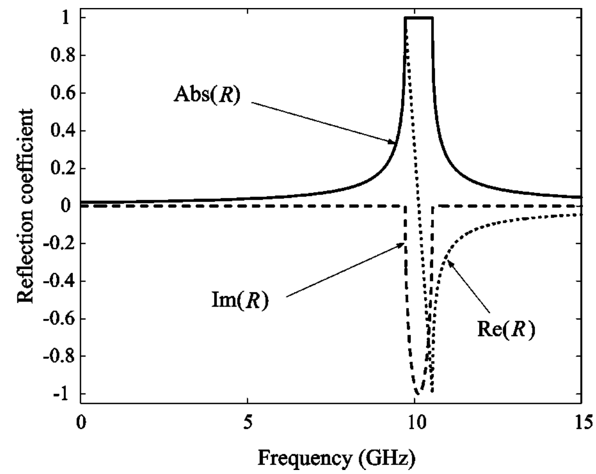

FIG. 5. Reflection coefficient.

approach) lead to the following observations.

(1) All these formulas give us correct information about the frequency position and the width of the ferrimagnetic band gap.

(2) The zero-Floquet mode solution contains two types of evanescent modes: one has a purely imaginary propagation constant Re(q)=0 ($f>9.8$ GHz) and the other has Re(q) $=\pi/a$ ($f<9.8$ GHz).

(3) The exact approach also gives two branches of solutions: one with Re(q)=0 and the other with Re(q)=π/a. These branches continue into the propagation bands ($f<9.76$ GHz and $f>10.52$ GHz).

(4) The quasistatic approach (Maxwell Garnett formalism) gives totally wrong results in a small area near the bottom of the band gap [in particular, we get Re(q)$>\pi/a$ at these frequencies].

Calculations of the reflection coefficient from a half space filled by the nonreciprocal crystal under consideration have been made using formula (34), and the result is plotted in Fig. 5. The reflection coefficient grows with the frequency until the bottom of the band gap is reached, where it becomes equal to $+1$. In the stop band, it becomes complex, and the phase changes inside the band gap so that the reflection coefficient becomes equal to -1 at the top of the band gap. In the upper propagation band, its absolute value decays again as the frequency becomes much higher than the resonant frequency. This expected resonatorlike behavior supports validity of the analysis.

Note that, due to inclusion interactions in a regular lattice, the width of the band gap is dramatically wider than the resonant curve of an isolated sphere, as is seen from comparison with Fig. 2.

VIII. CONCLUSIONS

Nonreciprocal electromagnetic stop band structures have been proposed. A simple analytical theory of dispersion for cubic lattices of small ferrimagnetic spheres has been presented for the axial propagation along the direction of a magnetization field. The dispersion equation has been solved both analytically, using a kind of averaging, and numerically in the exact formulation. The approximate solution leads to a very simple analytical formula for the propagation constant. Numerical calculations show that this approach has a very small mismatch with the exact one for the propagating modes. For the analysis inside the band gap, the exact solution is required, in particular near the lower boundary of the gap. The dispersion curves for the nonreciprocal crystal have been plotted and it has been shown that this crystal has a very interesting band-gap structure. The properties of the crystal depend very heavily on polarization. For the right-circular polarization, an additional band gap corresponding to the ferrimagnetic resonance is found that differs from the classical lattice band gaps. For the left-circular polarization, the ferrimagnetic band gap has not been found, as was expected. The ferrimagnetic band gap is rather wide and its central frequency is easily tunable by the magnetization field. At frequencies inside the gap, the lattice period can still be very small compared to the wavelengths, which is an important feature allowing the design of compact structures. The new band gap is not complete: it exists for only one of the two eigenpolarizations. Although the other propagation directions have not been considered, there is no reason to expect that the gap can exist for all propagation directions in this anisotropic medium.

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