Krook collisional models of the kinetic susceptibility of plasmas

M. Opher, G. J. Morales,* and J. N. Leboeuf
Department of Physics and Astronomy, University of California at Los Angeles, Los Angeles, California 90095

(Received 23 January 2002; published 29 July 2002)

An assessment is made of Krook collisional models used to describe the kinetic behavior of collective oscillations, i.e., when Landau damping and collisions must be considered, as is often the case for low-frequency waves. The study focuses on an early energy-conserving model [B. D. Fried, A. N. Kaufman, and D. L. Sachs, Phys. Fluids **9**, 292 (1966)] that is shown to be identical to a more modern version used in drift-wave stability studies [G. Rewoldt, W. M. Tang, and R. J. Hastie, Phys. Fluids **29**, 2893 (1986)]. The inadequacy of the simpler, and often used, nonconserving model is illustrated. Comparisons are established with recent collisional studies of ion acoustic waves [V. Yu. Bychenkov, J. Myatt, W. Rozmus, and V. T. Tikhonchuk, Phys. Plasmas **1**, 2419 (1994)] and electron plasma waves [C. S. Ng, A. Bhattacharjee, and F. Skiff, Phys. Rev. Lett. **83**, 1974 (1999)]. A connection is also established with contemporary studies of condensed matter and quantum liquids [K. Morawetz and U. Fuhrmann, Phys. Rev. E **61**, 2272 (2000); **62**, 4382 (2000)]. A useful empirical fit is found that corrects the Braginskii susceptibility to incorporate the kinetic behavior associated with the Krook kinetic susceptibility.

DOI: 10.1103/PhysRevE.66.016407 PACS number(s): 52.25.−b

I. INTRODUCTION

One of the more difficult and frustrating problems encountered in the theoretical study of plasmas is how to describe simultaneously the intrinsic kinetic behavior of the medium and the effect of collisions. This situation is particularly of interest in the description of waves having relatively low frequencies (e.g., ion acoustic waves and Alfvén waves), and in situations that lead to large collisionality, as may be the case at the edge of magnetically confined plasmas.

The two extreme limits of collisionless and highly collisional plasmas are very well understood. They both admit to compact analytical descriptions that allow the prediction and interpretation of a wide class of phenomena having experimental relevance. For prevailing thermal equilibrium conditions, in the collisionless regime the plasma susceptibility χ_0 is represented by a universal function, namely, the plasma dispersion function $Z(s)$ [1]. The argument of this function is the quantity $s=\omega/(\sqrt{2}ka)$, where ω, k represent the frequency and wave number of the relevant fluctuation, and the quantity a is used in this study (for historical reasons) to represent the appropriate thermal velocity.

In the opposite limit of large collisionality, and for fluctuations that satisfy the condition $s\gg 1$, the transport formalism developed by Braginskii [2] can be linearized to extract a collisional susceptibility, denoted here by χ_B. This quantity depends on s as well as on the ratio of ν_c/ω, where ν_c is the Coulomb collision frequency. While this approach correctly includes the velocity dependence of the Coulomb collisions, it is constrained by the range of phase velocities to which it can be applied.

For situations that require the description of collective oscillations in which the value of ν_c/ω is significant and/or s is not large, the predictions based on Z and χ_B, unfortunately are not reliable. At this stage of development no suitable generalization of Z and/or χ_B has been obtained that continuously bridges the gap between these regimes. An example of a calculation of this type is the pioneering work [3] of Kivelson and Dubois based on a Balescu-type kinetic equation to obtain the susceptibility in the limit of $\nu_c/\omega \ll 1$ for ion acoustic waves.

Another example of a technique based on the kinetic description was introduced [4] by Koch and Horton. This approach retains the electron pitch-angle scattering in its exact Fokker-Planck form. By suitable linearization, the susceptibility can be calculated by a continued fraction method that results in a generalization [5] of the Z function. A technical shortcoming of this approach is that the convergence of the continued-fraction method becomes very slow for small values of ν_c/ω. Another hurdle in its application is that no rigorous proof has been given for the analytic continuation of the equivalent Z function for purely damping problems. However, the method is well suited for the calculation of instabilities [5–8] that do not require this step.

To obtain semiquantitative descriptions of collisional situations that are, at least, correct from the kinetic perspective, an extensive literature has developed in which the Krook [9] collisional model is used. The appeal of this model is its algebraic simplicity, which in its more popular and rudimentary form (nonconserving) amounts to the replacement of $\omega \rightarrow \omega + i\nu$ in the argument of the Z function. Here we do not attempt to single out particular studies, since it would miss the point, but rather just mention that this simple approach is commonly found in numerous and important applications including studies of parametric instabilities, drift-wave instabilities, radio frequency plasma heating, and laser-plasma interactions. The inadequacy of this approach was pointed out in condensed matter studies by Mermin [10] who removed this defect by adding a term that relaxes the density matrix to the local equilibrium distribution. This is equivalent to the number-conservation procedure used by plasma researchers. A quantum mechanical formulation of the improved procedure has been presented by Das [11] to examine the behavior

*Corresponding author. Email address: morales@physics.ucla.edu

of Friedel oscillations in metals.

In a theoretical study [12] by Fried, Kaufman, and Sachs (FKS) in 1966, a derivation was presented of a more advanced, energy-conserving Krook model that yields a reasonably compact expression for the susceptibility in terms of combinations of the plasma dispersion function Z. Since the analytic properties of the Z function are well established, the properties of the kinetic-collisional susceptibility are well grounded and the expression reduces continuously to the collisionless limit. This result does not suffer from the convergence problem associated with the continued-fraction approach of Koch and Horton [4]. However, the FKS calculation, being a Krook model, assumes that the collision frequency is velocity independent, hence it misses subtle features unique to Coulomb collisions. It should also be mentioned that number-conserving (but not energy-conserving) Krook models of collective oscillations are presented in some textbooks [13,14].

In surveying the literature (including textbooks) associated with the use of Krook models to describe wave phenomena, we have been surprised to find that in spite of its early introduction, the FKS expression for χ has not been numerically examined. In fact, we have identified that in 1986, Rewoldt, Tang, and Hastie (RTH) introduced an energy-conserving Krook model to assess the role of collisional models on calculations of drift-wave instabilities related to tokamaks, apparently unaware of the FKS early study.

The present numerical study is motivated by the lack of a definitive analytic solution for the unified description of kinetic phenomena and collisions, together with the common practice to use Krook collisional models to extract guidance in wave-related problems. Specifically, we present here a numerical survey of the susceptibility obtained by FKS for the more prominent collective oscillations encountered in the study of plasmas. In addition, we make comparisons of predictions based on the FKS model with contemporary studies [15,16] of collisional effects on waves.

For completeness we mention that it is also possible to construct Krook collisional models that conserve momentum as is sketched in the textbook by Miyamoto [17]. However, when applying the Krook models to describe the damping (or growth) of waves associated with the loss of electron momentum, the standard practice (also followed in Miyamoto's text in describing drift waves) is to ignore this feature. For this reason, the comparative studies presented in this paper do not include this option.

A recent study of the general response function for interacting quantum liquids [18] has considered the limiting case of a nondegenerate plasma to explore the relative importance of the number-, energy-, and momentum-conserving methods on the shape of the plasma resonance. It is found by a numerical study of the type pursued in the present paper that the number-, and energy-conserving models shift the resonance toward smaller frequencies while the incorporation of momentum balance diminishes this effect. Further insight into the role of momentum conservation in one-component systems has been obtained in a study of the response of interacting Fermi gases [19].

The paper is organized as follows. Section II catalogues the analytic expressions for the susceptibility, which follow from the various collisional models to be compared later. Section III demonstrates the inadequacy of the nonconserving model, exhibits the behavior of the energy-conserving model, illustrates the identity between the FKS and RTH results, and introduces an empirical correction that brings the Braginskii result into close agreement with the energy-conserving Krook model. The effects of Krook-type collisions on ion acoustic waves, electron plasma waves, and Alfvén waves are presented in Sec. IV. A comparison is made in Sec. V between the predictions of the energy-conserving model and contemporary studies on ion acoustic waves by Bychenkov, Myatt, Rozmus, and Tikhonchuk [15] (BMRT) and on electron plasma waves by Ng, Bhattacharjee, and Skiff [16] (NBS). Conclusions are given in Sec. VI.

II. SUSCEPTIBILITY MODELS

For reference we reproduce in this section (without derivation) the results previously obtained for the kinetic susceptibility using different models for the Krook collision operator. The interested reader should consult the original reference to obtain the details leading to these expressions.

In the absence of collisions the kinetic susceptibility follows from the general expression given in Landau's landmark paper [20], which when evaluated for a zeroth-order Maxwellian distribution function is completely determined by the plasma dispersion function [1]

$$\chi_0 = \left(\frac{k_D}{k}\right)^2 [1 + sZ(s)], \tag{1}$$

where $k_D \equiv \omega_p/a$, and $s \equiv \omega/(\sqrt{2}ka)$, with, a, the thermal velocity of the species having plasma frequency ω_p. Generalization to multiple species can be achieved by summation over the contribution of the individual species having a form similar to Eq. (1).

For clarity, we emphasize that in the present notation the zeroth-order Maxwellian distribution function has the form

$$f_0 = \frac{n_0}{(2\pi a^2)^{1/2}} \exp\left(-\frac{v^2}{2a^2}\right), \tag{2}$$

with n_0 the density of particles, and the corresponding Debye wave number is $k_D = \omega_p/a$. The connection to the temperature T is through $a = (T/m)^{1/2}$, where m is the mass of the species of interest. Also, the relationship to the dielectric coefficient is $\epsilon = 1 + \chi$.

The popular, nonconserving Krook model alluded to in the Introduction, results in

$$\chi_{\rm NC} = \left(\frac{k_D}{k}\right)^2 [1 + sZ(\xi)], \tag{3}$$

where $\xi \equiv (\omega + i\nu)/(\sqrt{2}ka)$. Here ν is the Krook "collision frequency," which is not a well-defined quantity. However, what is often done, for the sake of making progress in the absence of a general formalism, is to identify ν with the Coulomb collision frequency ν_c.

A significant modification is obtained by imposing the constraint in the Krook collision operator that the number of particles should be rigorously conserved [13,14]. This results in a susceptibility

$$\chi_C = \left(\frac{k_D}{k}\right)^2 \frac{1+\xi Z(\xi)}{1+\gamma Z(\xi)}, \tag{4}$$

where $\gamma = i\nu/(\sqrt{2}ka)$.

By demanding that the Krook collisional model simultaneously conserve the number of particles and the energy, FKS [12] derived the following expression for the susceptibility (which we have independently rederived):

$$\chi_{\mathrm{FKS}} = \left(\frac{k_D}{k}\right)^2 \frac{1+\xi Z_c}{1+\gamma Z_c}, \tag{5}$$

in which the function Z_c now plays a role analogous to the Z function in the more limited, number-conserving model, and where

$$Z_c \equiv \frac{\gamma(Z^2-\xi^2)+[\frac{3}{2}-\gamma\xi(\xi^2-\frac{1}{2})]Z}{[(\xi^4-\xi^2+\frac{5}{4})\gamma Z+\frac{3}{2}+\gamma\xi(\xi^2-\frac{1}{2})]}, \tag{6}$$

in which the argument of all the Z functions is again the quantity ξ.

In a study aimed at providing guidance in choosing model collision operators for toroidal-geometry kinetic calculations, RTH [7] also presented a derivation of a number- and energy-conserving Krook model that results in the following susceptibility:

$$\chi_{\mathrm{RTH}} = \frac{sZ_0[1+i\frac{2}{3}\gamma(\frac{5}{4}Z_0+Z_4-Z_2)]-i\frac{2}{3}\gamma s(Z_2-\frac{1}{2}Z_0)^2}{(1+i\gamma Z_0)[1+i\frac{2}{3}\gamma(Z_4-Z_2+\frac{5}{4}Z_0)]+\frac{2}{3}\gamma^2(Z_2-\frac{1}{2}Z_0)^2}, \tag{7}$$

in which the functions Z_n have ξ as their argument and are defined as

$$Z_n(\xi) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} du\, \frac{u^n \exp(-u^2)}{u-\xi}. \tag{8}$$

It should be mentioned that the general expression originally derived by RTH included the effects of density gradients and electromagnetic effects because they were interested in applications that pertained to gradient-driven electromagnetic instabilities. The susceptibility given by Eq. (7) is extracted from their results by setting the diamagnetic frequency ω_* to zero and dropping the contributions proportional to the magnetic vector potential $A_\parallel$, as is appropriate for the comparisons of interest in the present study.

It should be emphasized that it is not at all evident by comparing Eqs. (5) and (7) that the two independent studies of energy-conserving Krook models yield the same result, thus a comparative numerical study of these two expressions seems warranted. The relevant result is presented in the following section.

To complement the previous kinetic studies that attempt to incorporate collisional effects through Krook models, it is useful to consider the susceptibility that is obtained in the simplest linearization of the transport formalism developed by Braginskii [2] for Coulomb collisions. Neglecting the effect of heat diffusion, yields

$$\chi_B = \left(\frac{k_D}{k}\right)^2 \frac{1}{\left(1-\dfrac{2s^2}{b'}\right)-i\dfrac{2\Gamma_c(0.51)s^2}{b'}}, \tag{9}$$

where $b' = 1+\frac{2}{3}(1.71)^2$ and $\Gamma_c \equiv \nu_c/\omega$, with ν_c the Coulomb collision frequency, which is not an arbitrary quantity, as is the case for the ν entering in Eqs. (3)–(5) and (7).

Again, the expression given by Eq. (9) has a form quite different from that of Eq. (5), thus a numerical comparison is also worthwhile.

III. NUMERICAL COMPARISON OF COLLISIONAL MODELS

We compare first the various predictions for the susceptibility based on different Krook collisional models, i.e., the quantities χ_0, χ_{NC}, χ_O, χ_{FKS}, presented in Sec. II. The comparison is achieved by displaying the dependence of the real and imaginary parts of χ as a function of the scaled phase-velocity parameter $s=\omega/(\sqrt{2}ka)$. Figure 1 displays the behavior obtained for significant collisionality, i.e., $\nu/\omega = 1.0$. Two important features can be immediately extracted from Fig. 1. One is that the predicted behavior for the nonconserving Krook model is topologically different from the underlying collisionless result, both in the real and imaginary parts. In marked contrast, the conserving Krook models are seen to continuously evolve from the ideal collisionless result as the value of the collision frequency is increased. It is evident from this display that in this case the much-desired simplicity achieved by using the nonconserving Krook model leads to fundamentally incorrect behavior over a broad range of fluctuation wave numbers. In fact, Fig. 1 suggests that in arriving at conclusions of experimental significance, it is better to rely on the prediction of the collisionless result rather than of the highly misleading nonconserving Krook model.

The inadequacy of the nonconserving Krook model was also deduced in the study [7] by RTH. They arrived at this conclusion by examining the dependence on collision frequency of the modes driven unstable by density gradients.

The other important feature that is illustrated by Fig. 1 is that the difference between the number-conserving, and the number-and energy-conserving models is relatively small. Again, the stability study by RTH arrived at a similar conclusion about the small improvement obtained by the inclusion of the more advanced energy-conserving method.

It is valuable to point out that the contemporary study by Morawetz and Fuhrmann [18], motivated by interacting quantum liquids, contains two figures that exhibit behavior

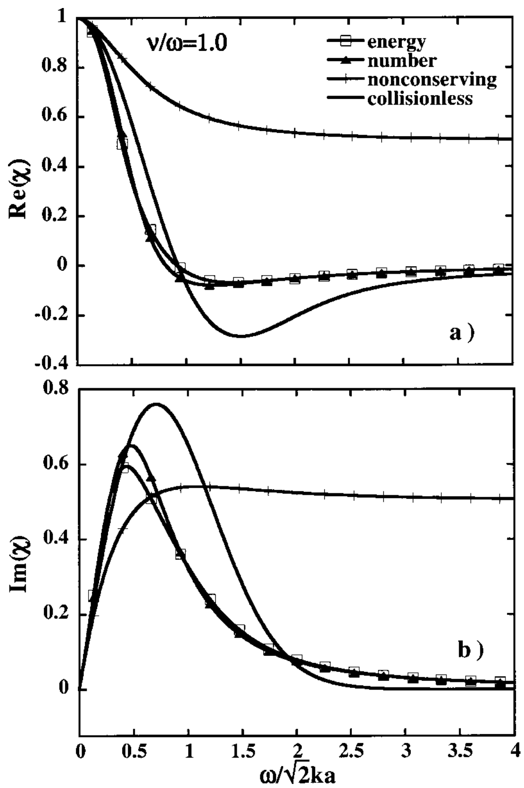

FIG. 1. Dependence on scaled phase velocity of the real (a) and imaginary (b) parts of the kinetic susceptibility predicted for different collisional models, $\nu/\omega=1$. In the abscissa label, the square root only includes the 2 in this and following relevant figures.

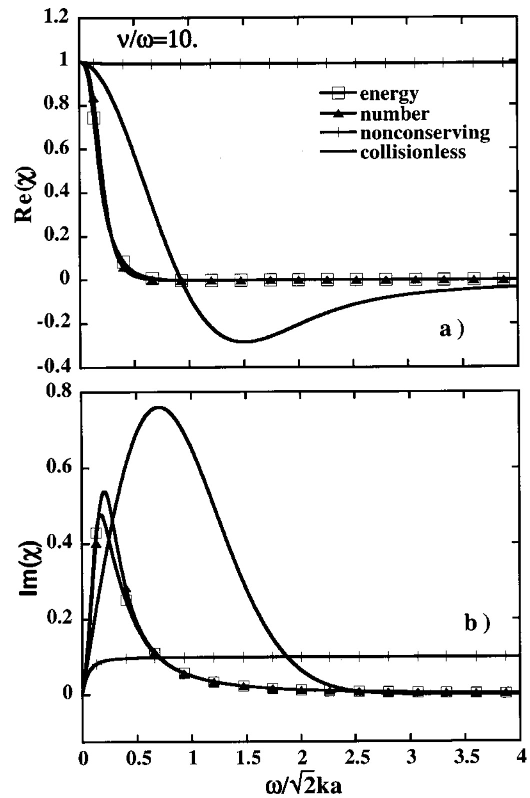

FIG. 2. Dependence on scaled phase velocity of the real (a) and imaginary (b) parts of the kinetic susceptibility predicted for different collisional models, $\nu/\omega=10$.

similar to that displayed in Fig. 1. The approach in that study is based on the Lindhard random-phase approximation and the Skyrme-type functional. Yet when these authors take the limit of a one-component plasma the results are essentially those obtained from the Krook collisional models used in plasma physics.

The important trends identified in Fig. 1 become extremely pronounced in the regime of large collisionality, as is illustrated in Fig. 2 for $\nu/\omega=10$. It is clear from Fig. 2 that when the collisionality is large, the nonconserving model is totally inadequate, and furthermore, there is no practical gain in implementing the energy-conserving constraint; χ_C should be the quantity of choice in practical calculations. In fact, this was the approach followed by Hedrick, Leboeuf, and Spong [8], when performing a survey of the effect of collisional models on the stability of shear Alfvén waves in stellarators. The present study validates that their neglect of the energy-conserving feature in the Krook model is well justified in arriving at their conclusions.

Although we arrive at a perspective for the number-conserving and energy-conserving Krook models, which is in full agreement with the conclusion obtained in the RTH study by completely different methods, the question remains as to what is the relationship between the quantities χ_{FKS} and χ_{RTH} given in Sec. II. To compare these two different-looking results we exhibit the predicted behavior of the real and the imaginary parts of χ as a function of the scaled phase velocity s. The result is shown in Fig. 3 for the choice $\nu/\omega=10$. The solid curve corresponds to χ_{RTH} and the dark triangles to χ_{FKS}. The triangle notation is chosen on purpose to separate the two results. As can be seen, the two quantities are, in fact, identical. This is comforting because they have the same physical origin, although the motivation leading to their study was quite different. For general perspective, we mention that the numerical difference between the two results is within machine precision and that we have verified the result over a broad range of values of ν/ω. The quantities χ_{FKS} and χ_{RTH} are identical.

The next comparison pertains to the behavior of the Braginskii susceptibility χ_B given by Eq. (9). Using the same format as in Figs. 1–3, we proceed to display in Fig. 4 the predicted behavior for the real and imaginary parts of χ_B (dashed curve) and χ_{FKS} (continuous curve) for $\nu/\omega=1.0$. It is seen from Fig. 4 that there exist pronounced differences between these expressions in the region where kinetic effects are most important, i.e., near $s\approx 1$. In the spirit in which the Krook model is used, namely, the introduction of a simple collisional model to correct the intrinsic kinetic features, we have been motivated to explore the opposite logic. The question is simply: can the Braginskii χ_B, which treats Coulomb collisions appropriately, be corrected to exhibit the proper kinetic features suggested by the χ_{FKS} in Fig. 4? To explore this possibility we have undertaken a survey of empirical numerical fits whose aim is to bring the Braginskii susceptibility in close agreement with χ_{FKS}, simultaneously for the real and imaginary parts, over a broad range of values of ν/ω.

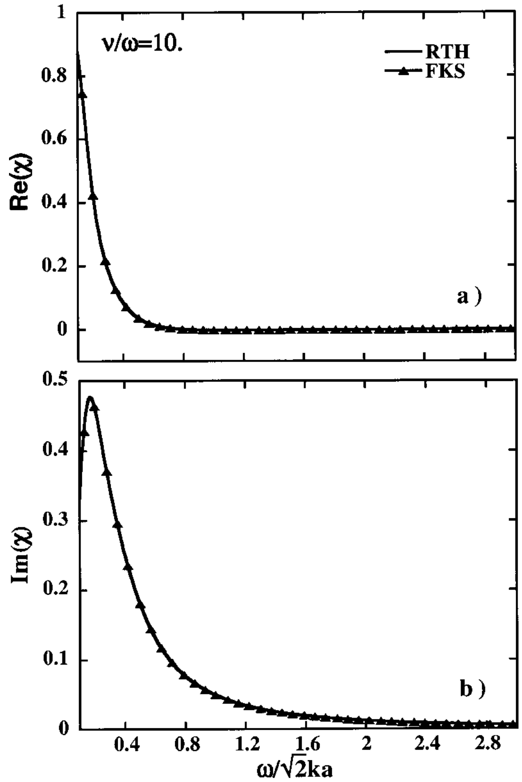

FIG. 3. Demonstration of the identity of the two independently derived results for the susceptibility by FKS [Eqs. (5) and (6)] and by RTH [Eq. (7)] based on the energy-conserving model. (a) Real part and (b) imaginary part; $\nu/\omega = 10$. The solid curve is RTH result and dark triangles are that of FKS.

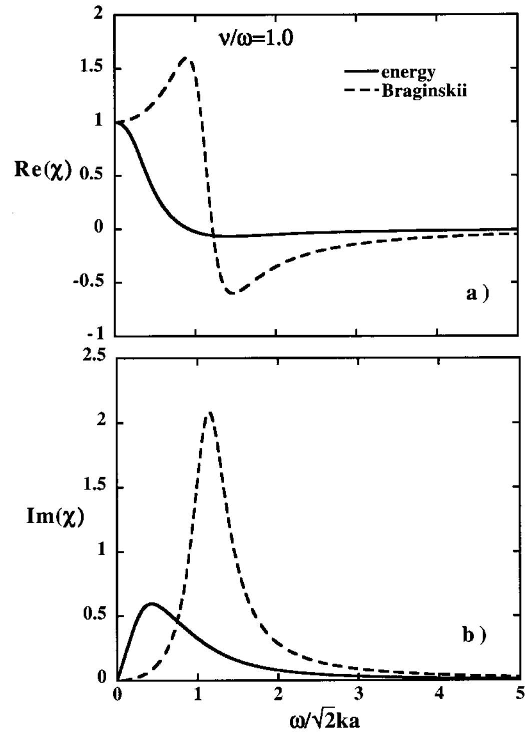

FIG. 4. Comparison of the dependences on scaled phase velocities for the susceptibilities predicted by Braginskii's transport formalism (dashed curve) and the energy-conserving Krook model (solid curve). (a) Real part and (b) imaginary part; $\nu/\omega = 1$.

We have found that the relatively simple replacement in χ_B of ν_c by the expression

$$\nu_c \rightarrow (\nu/0.7s)\exp(3s) \tag{10}$$

brings χ_B in close agreement to χ_{FKS} as is illustrated in Fig. 5. In this figure the dashed curve now corresponds to the "kinetically corrected" Braginskii susceptibility, while the solid curve is the same χ_{FKS} shown in Fig. 4. In our numerical survey we find that as the value of ν/ω is increased the agreement between the two expressions improves significantly.

IV. COLLISIONAL EFFECTS ON PROMINENT MODES

In this section we survey the predictions of the various Krook collisional models for the dispersion relation of prominent collective modes, i.e., ion acoustic waves, electron plasma waves, and shear Alfvén waves. To isolate the collisional effects, we apply the Krook collisional model only to the electron population since they constitute the species that exhibits the most pronounced kinetic effects. Of course, in many plasma situations of great interest the ion contribution can also be overwhelmingly kinetic, but that is not the focus of the present study.

The dispersion relation used to isolate the effect of the Krook collisional models on ion acoustic waves is [21,22]

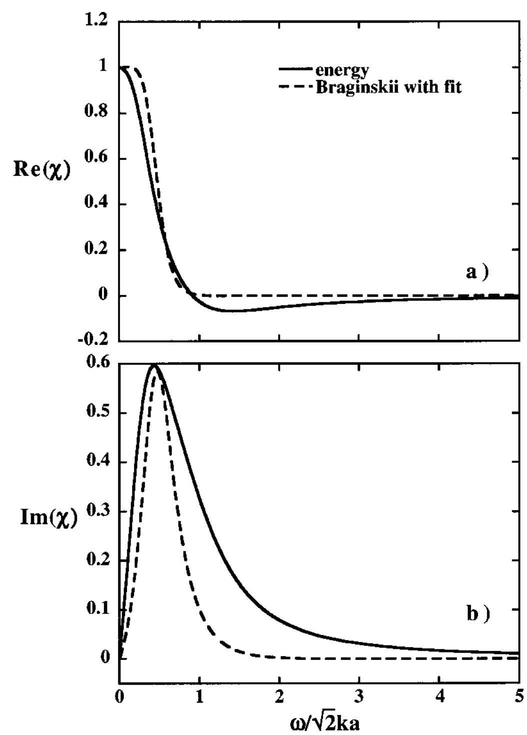

FIG. 5. Correction of Braginskii's fluid susceptibility by a phase-velocity-dependent collision frequency, given by Eq. (10), brings close agreement with the energy-conserving Krook model.

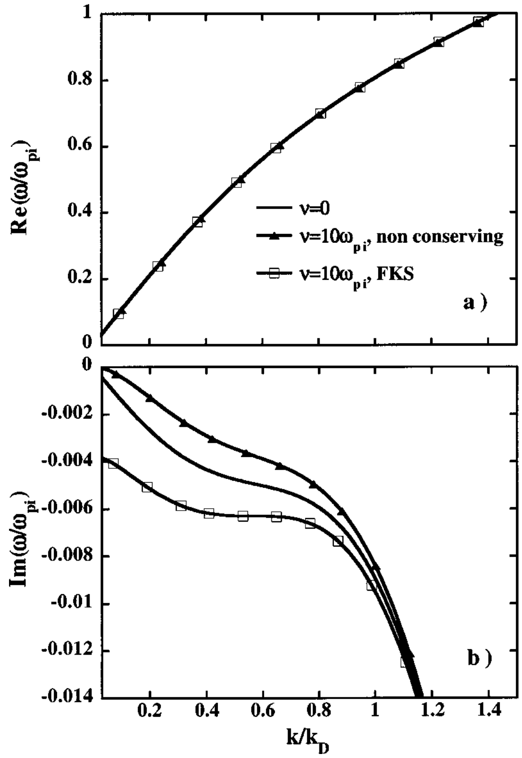

FIG. 6. Dispersion relation of ion acoustic waves predicted by different collisional models. The solid curve is the collisionless case, the curve with dark triangles is the nonconserving model, and the curve with open squares is the FKS prediction. (a) Frequency scaled to ω_{pi} and (b) damping rate scaled to ω_{pi}. The collision frequency is $\nu/\omega_{pi}=10$.

$$1+\chi_i+\chi_j=0, \tag{11}$$

where χ_i is χ_0 of Eq. (1) with ion parameters. In Eq. (11) the χ_j of interest at this stage are χ_0, $\chi_{\rm NC}$, and $\chi_{\rm FKS}$, since χ_C has already been shown to be not very different from $\chi_{\rm FKS}$.

The results obtained from a numerical root-finding study in which k is a real, continuous parameter, and ω is the unknown complex frequency of the ion acoustic mode are shown in Fig. 6. The results correspond to high collisionality, i.e., $\nu/\omega_{pi}=10$ to emphasize the effects. Here ω_{pi} is the ion plasma frequency. In Fig. 6 the continuous curve corresponds to the collisionless result, the curve with the dark triangles is obtained with the nonconserving model, while the curve with open squares follows from the energy-conserving model. Both real and imaginary parts are scaled to ω_{pi} and k to the Debye wave number k_D.

It is seen from Fig. 6(a) that the frequency of ion acoustic waves is independent of electron collisions, hence the collisionless prediction is always excellent. The damping rate, however, is sensitive to electron collisions in the wave number region $k/k_D<0.5$, as is illustrated in Fig. 6(b). As $k/k_D\rightarrow 1$, however, the damping also becomes insensitive to electron collisions. Figure 6(b) corroborates the previous conclusion related to the inadequacy of the nonconserving Krook model. It is seen that for ion acoustic waves this model predicts a lowering of the damping, while the energy-conserving model illustrates that the damping is increased. It should be

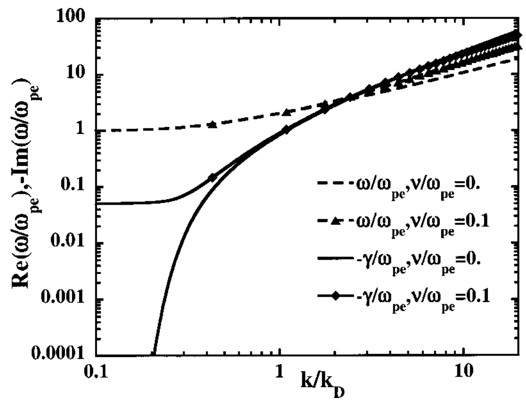

FIG. 7. Dispersion relation of electron plasma waves using the log-log display format introduced by Jackson in which the real and imaginary parts of ω are scaled to ω_{pe} and the wave number k is scaled to the Debye wave number. The collisionless results correspond to the simple dashed and solid curves. The dashed curve with the dark triangles is the real part for the energy-conserving Krook model and the curve with the dark diamonds is the corresponding imaginary part for $\nu/\omega_{pe}=0.1$.

mentioned that Koch and Horton [4] extracted an approximate analytical expression from their electron pitch-angle scattering model that also shows that such collisions enhance the damping of ion acoustic waves.

The next mode examined is the electron plasma oscillations (Langmuir waves). Because there is a long-standing tradition established by Jackson [23] in how to display the frequency and damping rate of electron plasma waves in the same graph using a log-log scale, we proceed here to present the effect of collisions using such a format. The results are shown in Fig. 7, which displays four different curves. Two of them correspond to the collisionless result (the dashed curve for Re ω, the solid curve for Im ω), while the two others are obtained using $\chi_{\rm FKS}$ for a collisionality $\nu/\omega_{pe}=0.1$, where ω_{pe} is the electron plasma frequency. The dashed curve with dark triangles correspond to the Re ω with collisions and the solid curve with dark diamonds to Im ω with collisions. The first impression obtained from Fig. 7 is that the Im ω predicted with the energy-conserving model exhibits a continuous merging with the collisionless damping rate as k increases. This is a highly desirable feature of a model that incorporates collisional and kinetic effects. At this level of collisionality it is seen that for $k/k_D<0.2$ the enhanced collisional damping overwhelms Landau damping, but as $k/k_D\rightarrow 1$ the strong collisionless dissipation (which in this regime is not proportional to $\partial f_0/\partial v$) takes over. In the extremely large wave number limit the energy-conserving collisions cause a slight decrease in the damping and an associated increase in the frequency.

We note that the prediction of the nonconserving model is not included in Fig. 7 because it would be too confusing. We simply state that we have examined its effect, and find that in a more pronounced manner than for the ion acoustic waves, it also leads to unacceptable behavior.

The shear Alfvén wave of small transverse scale (on the order of the electron skin depth) is a mode of considerable interest [24–26] at the present time. Because of its relatively

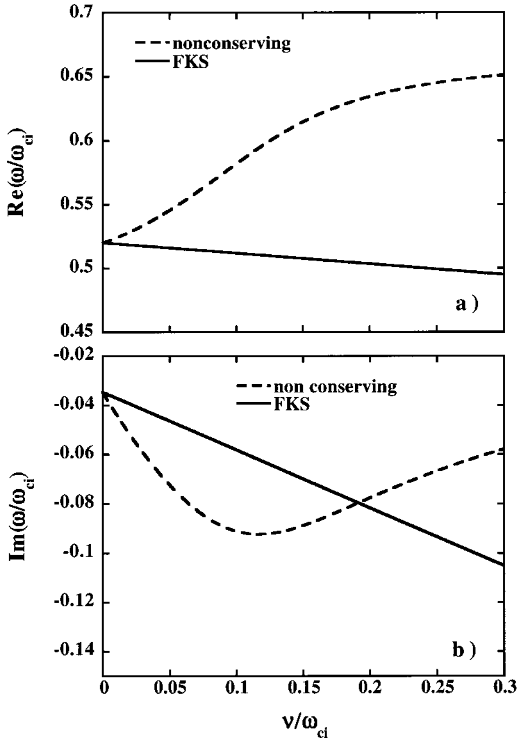

FIG. 8. Dependence on scaled collision frequency (to ω_{ci}) of the dispersion relation of shear Alfvén waves of small transverse scale ($k_\perp c/\omega_{pe}=1$) and for cold ions. The parallel wave number is $k_\| a/\omega_{ci}=0.2$ for a scaled electron plasma beta $\beta=(a/v_A)^2=0.1$ (inertial regime). The real part (a) and imaginary part (b) are scaled to ω_{ci}. The dashed curve is the nonconserving model and the solid curve is the energy-conserving model.

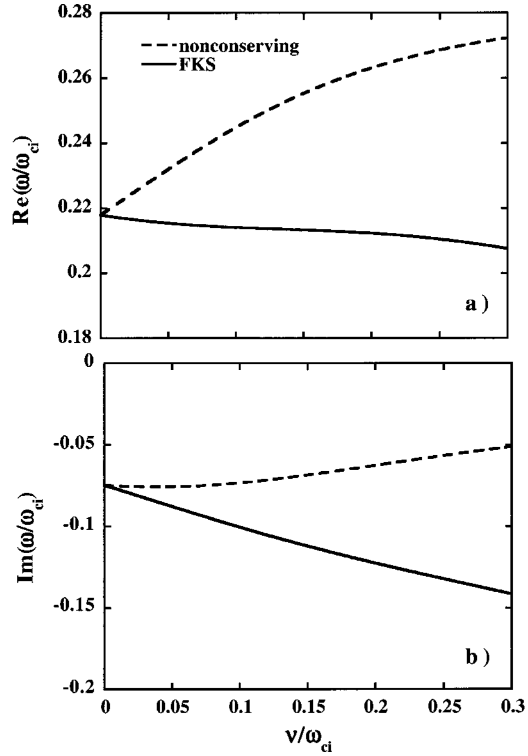

FIG. 9. The same information presented in Fig. 8 but with scaled electron beta $\overline{\beta}=(a/v_A)^2=1.0$, corresponding to a strongly kinetic regime of the shear Alfvén mode.

low frequency this mode is a suitable candidate for a description based on the Krook collisional model. In the limit of cold ions the relevant dispersion relation takes the form

$$k_\|^2=\left(\frac{\omega}{v_A}\right)^2\left[1-\left(\frac{k_\perp c}{\omega}\right)^2\frac{1}{\chi_j}\right], \tag{12}$$

where v_A is the Alfvén speed, c is the speed of light, and χ_j is the electron susceptibility given by one of the expressions presented in Sec. II. In this notation $k_\|$ represents the wave number along the confining magnetic field, and $k_\perp$ the component across the field. Because of the unique topology of this mode, it is more informative to examine the dependence of the frequency and damping coefficient on the strength of the collisionality for fixed values of $k_\perp$ and $k_\|$.

Figure 8 exhibits the dependence of Re ω and Im ω obtained numerically as the value of the collision frequency is increased in $\chi_{\rm NC}$ and $\chi_{\rm FKS}$. All the quantities in Fig. 8 are scaled to the ion cyclotron frequency ω_{ci}. For these results the parameters are $k_\perp c/\omega_{pe}=1$ (yielding strong parallel electric fields), $k_\| a/\omega_{ci}=0.2$ (resulting in frequencies below ω_{ci}), and relatively cold electrons (the inertial regime) in which the parameter $\beta\equiv(a/v_A)^2=0.1$. In Fig. 8 the solid curve corresponds to the energy-conserving model and the dashed curve to the nonconserving model. It is quite evident in this case that $\chi_{\rm NC}$ gives rise to rather bizarre behavior as ν increases. The appropriate trend displayed by $\chi_{\rm FKS}$ is that the frequency exhibits a relatively small change, while the damping coefficient experiences a linear increase leading to an enhancement by a factor of two over the collisionless damping in the range of ν examined.

The complementary behavior obtained as the scaled electron β is increased to $\overline{\beta}=1.0$, is displayed in Fig. 9. In this case, the collisionless result corresponds to a lower frequency and stronger damping, but the same trends obtained for the $\overline{\beta}=1.0$ case are seen for the two collisional models. Again, the energy-conserving model predicts a monotonic increase in the damping coefficient that roughly doubles its value at a collisionality level of $\nu/\omega_{ci}=0.3$.

V. COMPARISON TO CONTEMPORARY STUDIES

In this section we compare the predictions of the energy-conserving Krook model to two contemporary studies in which the issue of modeling collisional effects on wave properties is emphasized. One pertains to ion acoustic waves and the other to electron plasma waves.

In 1994, Bychenkov, Myatt, Rozmus, and Tikhonchuck (BMRT) [15] investigated a quasihydrodynamic description of ion acoustic waves in a collisional plasma. In their approach the ion response is essentially hydrodynamic and a formulation based on a generalized 21-moment closure results in a suitable ion-fluid susceptibility χ_i, which is used to calculate the ion acoustic dispersion relation. The description used for the electron response, however, is not so straightfor-

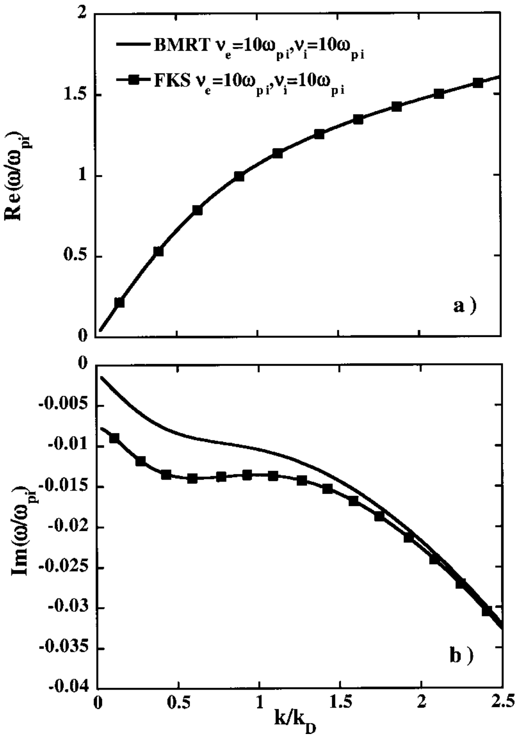

FIG. 10. Comparison of the prediction of the energy-conserving model (curve with dark squares) and results of the BMRT study [13] (solid curve) for the dispersion relation of ion acoustic waves. Real part (a) and imaginary part (b) are scaled to ω_{pi}. Collisionality is $\nu=\nu_i=10\omega_{pi}$ corresponding to the $T_e/T_i>10$ regime.

ward because for this mode, the electrons behave kinetically. To handle this issue BMRT constructed a model that mixes an electron collisional heat conductivity with a heuristic form of Landau damping.

Because in the BMRT study the electron population is treated in a special manner to merge kinetic and collisional effects, it is suggestive that it would be useful to compare their results to the energy-conserving Krook model in which the kinetic behavior is automatically included. To implement a suitable comparison we replace the χ_e used in the BMRT study by χ_{FKS} but retain the 21-moment χ_i that they derived. From these two susceptibilities we then solve numerically for the complex frequency ω as a function of the real wave number k.

Figure 10 displays the results obtained for the choice of ion collision frequency $\nu_i=\nu=10\,\omega_{pi}$. For singly ionized ions, this choice corresponds to an electron to ion temperature ratio $T_i/T_e=(M/m)^{1/3}$, where M and m refer to the ion and electron masses, respectively. This in turn implies a regime of the ion acoustic waves in which the ions behave hydrodynamically and is thus consistent with the BMRT description. The choice of collisionality also permits a cross comparison to the results of Fig. 6 in which the ion behavior is collisionless. In Fig. 10 the continuous curve corresponds to the BMRT prediction while the curve with solid squares is obtained using χ_{FKS}. It is seen from Fig. 10(a) that the frequency of the mode is identical for both methods. This implies that the fluid-electron model introduced by BMRT does an excellent job in describing the effective "electron pressure" over a wide range of wave numbers. From Fig. 10(b) it is found that the energy-conserving Krook model gives a larger (less than a factor of 2) damping for $k/k_D<1.0$, but the overall functional dependence predicted by the two models is quite similar. In particular, for $k/k_D>1.5$ the two models merge into each other, again confirming that the method used by BMRT to incorporate electron Landau damping through a fluid response approximates the actual kinetic response quite well.

From the cross comparison between Figs. 10(a) and 6(a) it can be deduced that the frequency of ion acoustic waves is very robust to the inclusion of collisions of any form. This quantity is intrinsically determined by the collisionless kinetic response of the electrons. From the behavior seen in Figs. 10(a) and 6(b) one deduces that for the level of collisionality examined (i.e., $\nu_i/\omega_{pi}=10$) the dissipation due to ion collisions plays a significant role and leads to an enhancement over the collisionless prediction in the wave number range of $k/k_D<1$ by as large as a factor of 3.

In 1999, Ng, Bhattacharjee, and Skiff (NBS) [16] published a study in which they revisited the damping of electron plasma oscillations in a weakly collisional plasma using the model collision operator introduced by Lenard and Bernstein [27]. The study uses a procedure based on a complete set of Hermite polynomials to derive an infinite determinant from which a recurrence relation is obtained that yields a discrete spectrum of values for the frequency and damping of the electron plasma oscillations. The study reports on the results in the form of a table for the real and imaginary parts of the scaled eigenvalues for different values of the scaled collision frequency covering four orders of magnitude.

Since the NBS study utilizes a model for the merging of kinetic and collisional effects that is considerably different from the approach based on the energy-conserving Krook model, it is of interest to explore how the discrete modes reported by NBS compare to the roots obtained for electron plasma waves based on χ_{FKS}.

To make the comparison meaningful we utilize the same notation and scaled variables introduced by NBS. They define the scaled (complex) frequency as $\Omega\equiv\omega/(\sqrt{2}ka)$ and the scaled collision frequency as $\mu\equiv\nu/(\sqrt{2}ka)$. In addition, they introduce the inverse, scaled wave number as the parameter $\alpha\equiv(k_D/k)^2$. The goal is to identify how the value of Ω varies as ν is changed. In Table II of Ref. [16] NBS present results for $\alpha=9.0$.

With the previous translations in notation, we proceed to solve for the roots predicted using χ_{FKS} and varying ν in very small steps over a very wide dynamic range that requires a semilog display. The results of this numerical survey are shown in Fig. 11, in which, again we duplicate the NBS notation of $\Omega_T\equiv\mathrm{Re}\,\omega$ and $\Omega_i\equiv\mathrm{Im}\,\omega$. The continuous solid curve corresponds to the roots of the energy-conserving Krook model while the dark triangles are the discrete modes reported by NBS. It is seen that the discrete modes found by the NBS formalism closely overlap the continuous roots predicted by the energy-conserving Krook model.

At this stage we refrain from speculation about the meaning of the agreement displayed in Fig. 11, but from the perspective of the present study we conclude that the energy-

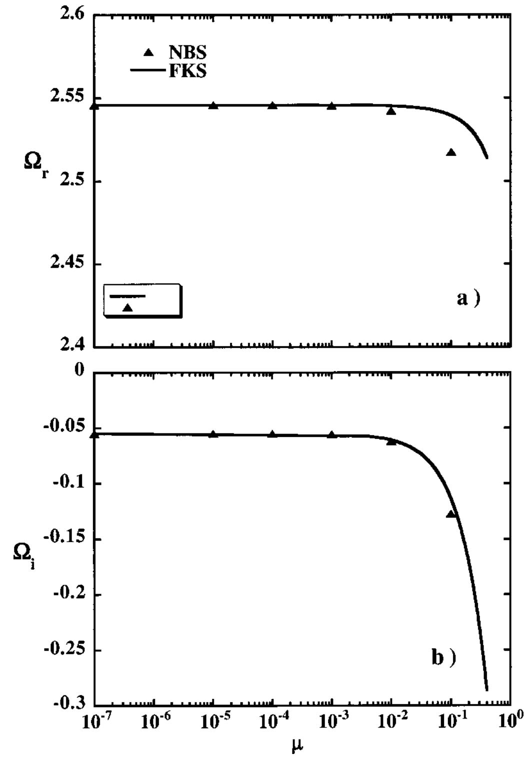

FIG. 11. Comparison of the prediction of the energy-conserving Krook model (continuous solid curve) to prediction of discrete electron plasma modes (dark triangles) according to the NBS study [14]. The notation used is that introduced in Ref. [14], $\Omega \equiv \omega/(\sqrt{2}ka)$, $\mu \equiv \nu/(\sqrt{2}ka)$. Note that the scaled collision frequency is varied logarithmically over a large dynamic range.

conserving Krook model is capable of reproducing a variety of predictions based on seemingly different collisional methods that incorporate the intrinsic kinetic behavior of the plasma.

VI. CONCLUSIONS

This numerical study has aimed to elucidate some features related to the usage of Krook collisional models in the description of kinetic wave phenomena, which have not been widely discussed. The study focused primarily on an early formulation of an energy-conserving model by Fried, Kaufman, and Sachs [12] (FKS), which has not received much attention. In fact, twenty years after the original FKS study, Rewoldt, Tang, and Hastie [7] (RTH) were motivated, as part of a larger investigation, to examine the role of this type of collision model and obtained an expression that algebraically differs significantly from the FKS form. In this work we have illustrated that these two developments are identical.

A strong conclusion that follows from the present survey is that the nonconserving Krook model is entirely inadequate for the prediction of collisional effects on kinetic wave properties. Another valuable conclusion is that the particle-conserving and the simultaneously particle- and energy-conserving models do not differ significantly in their predictions. Thus, in practical situations in which a simpler and more compact formulation is desirable, the number-conserving approach can be used with confidence. However, it should not be extrapolated that the number-conserving version is suitable for all applications. If heat transport is important, then energy-, momentum-, and number-conserving are required.

The robustness of the frequency of ion acoustic waves to various collision models has been documented. This quantity is essentially determined by the collisionless kinetic response of the electrons.

An empirical fit has been found in which the simple replacement of the Coulomb collision frequency by a kinetic factor brings the Braginskii susceptibility into close agreement with that obtained in the energy-conserving Krook model.

Based on the classic plot format introduced by Jackson [23] for electron plasma waves, a pedagogically valuable presentation that illustrates the continuous transition from collisional to collisionless behavior has been obtained using the energy-conserving model.

A comparison of the predictions of the energy-conserving model to the study by BMRT [15] of ion acoustic waves in a collisional plasma yielded useful insight. One perspective is that the heuristic model for the electron response introduced by BMRT to incorporate Landau damping shows good agreement with the energy-conserving Krook model. The complementary perspective is that in attacking problems of this nature, the use of the heuristic model can be replaced by the suitable Krook model.

A surprising result has been obtained for electron plasma waves in comparing the predictions of the energy-conserving model to a recent study by NBS [16], in which a set of discrete eigenmodes was identified. In spite of the significantly different collisional models used, it is found that the eigenvalues predicted by NBS lie on the continuous curve predicted by the Krook model over a very large range in variation of the collision frequency. This again illustrates the usefulness of the energy-conserving model in describing subtle features that mix the kinetic response and collisions.

It is noteworthy that contemporary studies of condensed matter and quantum fluids [18,19] are addressing analogous issues to those considered in this plasma-oriented study and that similar results are being obtained. Hopefully, the connections mentioned in this study will bring awareness of the commonality of approaches developed independently to describe the different states of matter.

In summary, this study provides through specific numerical comparisons a useful perspective on the limitations and capabilities of the various Krook collisional models. This information should be useful to future studies that try to find a compromise in handling the yet unresolved problem of how to unify the kinetic response and collisions in wave phenomena.

ACKNOWLEDGMENTS

This work was sponsored by The Office of Naval Research and The U.S. Department of Energy (USDOE), Grant No. DE-FG03-01ER54617 at UCLA. The authors thank Dr. Frank Tsung for helpful discussions and suggestions.

[1] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic, New York, 1961).
[2] S. I. Braginskii, in *Review of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[3] M. G. Kivelson and D. F. Dubois, Phys. Fluids **7**, 1578 (1964).
[4] R. A. Koch and W. Horton, Jr., Phys. Fluids **18**, 861 (1975).
[5] J. R. Peñano, G. J. Morales, and J. E. Maggs, Phys. Plasmas **4**, 555 (1997).
[6] J. R. Peñano, G. J. Morales, and J. E. Maggs, Phys. Plasmas **7**, 544 (2000).
[7] G. Rewoldt, W. M. Tang, and R. J. Hastie, Phys. Fluids **29**, 2893 (1986).
[8] C. L. Hedrick, J. N. Leboeuf, and D. A. Spong, Phys. Plasmas **2**, 2033 (1995).
[9] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[10] N. D. Mermin, Phys. Rev. B **1**, 2362 (1970).
[11] A. K. Das, J. Phys. F: Met. Phys. **5**, 2035 (1975).
[12] B. D. Fried, A. N. Kaufman, and D. L. Sachs, Phys. Fluids **9**, 292 (1966).
[13] A. F. Alexandrov, L. S. Bogdankevich, and A. A. Rukhadze, *Principles of Plasma Electrodynamics* (Springer-Verlag, New York, 1984), pp. 90–92.
[14] P. C. Clemmow and J. P. Dougherty, *Electrodynamics of Particles and Plasmas* (Addison-Wesley, Redwood City, CA, 1990), pp. 311–315.
[15] V. Yu. Bychenkov, J. Myatt, W. Rozmus, and V. T. Tikhonchuk, Phys. Plasmas **1**, 2419 (1994).
[16] C. S. Ng, A. Bhattacharjee, and F. Skiff, Phys. Rev. Lett. **83**, 1974 (1999).
[17] K. Miyamoto, *Plasma Physics for Nuclear Fusion* (MIT, Cambridge, MA, 1989), pp. 383–386.
[18] K. Morawetz and U. Fuhrmann, Phys. Rev. E **61**, 2272 (2000).
[19] K. Morawetz and U. Fuhrmann, Phys. Rev. E **62**, 4382 (2000).
[20] L. Landau, J. Phys. (Moscow) **X**, 25 (1946).
[21] B. D. Fried and R. W. Gould, Phys. Fluids **4**, 139 (1961).
[22] L. M. Kovrizhnykh, Zh. Eksp. Teor. Fiz. **37**, 16902 (1959) [Sov. Phys. JETP **37**, 1198 (1960)].
[23] J. D. Jackson, J. Nucl. Energy, Part C **1**, 171 (1960).
[24] A. V. Strelsov and W. Lotko, J. Geophys. Res. **104**, 4411 (1999).
[25] R. L. Lysak, J. Geophys. Res. **104**, 10017 (1999).
[26] D. Leneman, W. Gekelman, and J. Maggs, Phys. Rev. Lett. **82**, 2673 (1999).
[27] A. Lenard and I. B. Bernstein, Phys. Rev. **112**, 1456 (1958).