Two-dimensional Heisenberg model with nonlinear interactions

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We investigate a two-dimensional classical N-vector model with a nonlinear interaction $(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)^p$ in the large-N limit. As observed for $N=3$ by Blöte *et al.* [Phys. Rev. Lett. **88**, 047203 (2002)], we find a first-order transition for $p>p_c$ and no finite-temperature phase transitions for $p<p_c$. For $p>p_c$, both phases have short-range order, the correlation length showing a finite discontinuity at the transition. For $p=p_c$, there is a peculiar transition, where the spin-spin correlation length is finite while the energy-energy correlation length diverges.

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The two-dimensional Heisenberg model has been the object of extensive studies that mainly focused on the $O(N)$-symmetric Hamiltonian

$$H=-N\beta\sum_{\langle ij\rangle}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, \qquad (1)$$

where $\boldsymbol{\sigma}_i$ is an N-dimensional unit spin and the sum is extended over all lattice nearest neighbors. The behavior of this system in two dimensions is well understood. It is disordered for all finite β [1] and it is described for $\beta\to\infty$ by the perturbative renormalization group [2–4]. The square-lattice model has been extensively studied numerically [5–10], checking the perturbative predictions [11–15] and the nonperturbative constants [16–18].

In this paper we study a more general Hamiltonian on the square lattice; more precisely, we consider

$$H=-N\beta\sum_{x\mu}W(1+\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+\mu}), \qquad (2)$$

where $W(x)$ is a generic function such that $W(2)>W(x)$ for all $0\leqslant x<2$, in order to guarantee that the system orders ferromagnetically for $\beta\to\infty$. A particular case of the Hamiltonian (2) has been extensively studied in the years, the case in which $W(x)$ is a second-order polynomial. Such a choice of $W(x)$ gives rise to the so-called mixed $O(N)$-$\mathbb{R}P^{N-1}$ model [19–28], which is relevant for liquid crystals [29–34] and for some orientational transitions [35].

In a recent paper [36], the authors analyzed a model with $W(x)=ax^p+b$ and found an additional first-order transition for large enough p. Here, we will study the same model, finding an analogous result for $p>p_c\approx4.537\,857$ a first-order transition appears, the correlation length—and in general, all thermodynamic quantities—showing a finite discontinuity. Note that the appearance of a first-order transition in nonlinear models is not a new phenomenon. Indeed, for $N=\infty$ it was already shown in Ref. [20] that a first-order transition appears in mixed $O(N)$-$\mathbb{R}P^{N-1}$ models for certain values of the couplings. It is of interest to understand the behavior for $p=p_c$. For such value of p, Ref. [36] found a peculiar phase transition; while the spin-spin correlation length remains finite, the energy-energy correlation length diverges. Here, we will show that the same phenomenon occurs for $N=\infty$. However, at variance with what observed in Ref. [36], the critical theory shows mean-field, not Ising, behavior.

Let us consider the Hamiltonian (2) on a hypercubic d-dimensional lattice. We normalize $W(x)$ by requiring $W'(2)=1$ so that in the spin-wave limit,

$$H=\frac{N\beta}{2}\int dx\,\partial_\mu\boldsymbol{\sigma}\cdot\partial_\mu\boldsymbol{\sigma}. \qquad (3)$$

We also fix $W(1)=0$ so that $H=0$ for a random configuration. Then, we introduce two new fields $\lambda_{x\mu}$ and $\rho_{x\mu}$ in order to linearize the dependence of the Hamiltonian on the spin coupling. We write

$$\exp[N\beta W(1+\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+\mu})] \sim\int d\rho_{x\mu}d\lambda_{x\mu}\exp\left[\frac{N\beta}{2}\lambda_{x\mu}(1+\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+\mu}-\rho_{x\mu}) +N\beta W(\rho_{x\mu})\right]. \qquad (4)$$

As usual in the large-N expansion, we also introduce a field μ_x in order to eliminate the constraint $\boldsymbol{\sigma}_x^2=1$. Thus, we write

$$\delta(\boldsymbol{\sigma}_x^2-1)\sim\int d\mu_x\exp\left[-\frac{N\beta}{2}\mu_x(\boldsymbol{\sigma}_x^2-1)\right]. \qquad (5)$$

With these transformations we can rewrite the partition function as

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$$Z=\int \prod_{x\mu} [d\rho_{x\mu} d\lambda_{x\mu}]\prod_{x} [d\mu_x d\boldsymbol{\sigma}_x] e^{NA}, \tag{6}$$

where

$$A=\frac{\beta}{2}\sum_{x\mu} [\lambda_{x\mu}+\lambda_{x\mu}\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+\mu}-\lambda_{x\mu}\rho_{x\mu}+2W(\rho_{x\mu})] -\frac{\beta}{2}\sum_{x} (\mu_x\boldsymbol{\sigma}_x^2-\mu_x). \tag{7}$$

We perform a saddle-point integration by writing

$$\lambda_{x\mu}=\alpha+\hat{\lambda}_{x\mu},$$
$$\rho_{x\mu}=\tau+\hat{\rho}_{x\mu},$$
$$\mu_x=\gamma+\hat{\mu}_x. \tag{8}$$

A standard calculation gives the following saddle-point equations [37]:

$$d\beta(1-\tau)+\frac{1}{\alpha}[(2d+m_0^2)I(m_0^2)-1]=0,$$
$$\alpha-2W'(\tau)=0,$$
$$\frac{\beta}{2}-\frac{1}{\alpha}I(m_0^2)=0, \tag{9}$$

where we set $\gamma=\alpha(2d+m_0^2)/2$,

$$I(m_0^2)=\int \frac{d^dp}{(2\pi)^d}\frac{1}{\hat{p}^2+m_0^2}, \tag{10}$$

and $\hat{p}^2=4\Sigma_\mu \sin^2 p_\mu/2$. The variable m_0 has a simple interpretation; it is related to the spin-spin correlation length by $\xi_\sigma=1/m_0$. From Eq. (9) we obtain finally

$$\beta=\frac{I(m_0^2)}{W'(\tau)}, \tag{11}$$

where

$$\tau=\tau(m_0)\equiv 2+\frac{m_0^2}{2d}-\frac{1}{2dI(m_0^2)}. \tag{12}$$

The corresponding free energy can be written as

$$F=-\beta d W(\tau)+\frac{1}{2}\ln I(m_0^2)+\frac{1}{2}L(m_0^2), \tag{13}$$

where

$$L(m_0^2)=\int \frac{d^dp}{(2\pi)^d}\ln(\hat{p}^2+m_0^2). \tag{14}$$

Focusing now on the two-dimensional case, let us show that, for any $W(x)$, the spin-spin correlation length is always finite, i.e., $\xi_\sigma=\infty$, so that $m_0=0$, only for $\beta=\infty$. Note first that $\tau=2$ (respectively $\tau=1$) for $m_0=0$ (respectively $m_0=\infty$) and that $\tau(m_0)$ is a strictly decreasing function of m_0. Thus, $W'(\tau)$ is finite for all m_0. Then, since $I(0)=+\infty$, we find that $\xi_\sigma=\infty$ only if $\beta=\infty$, i.e., ξ_σ is finite for all finite β.

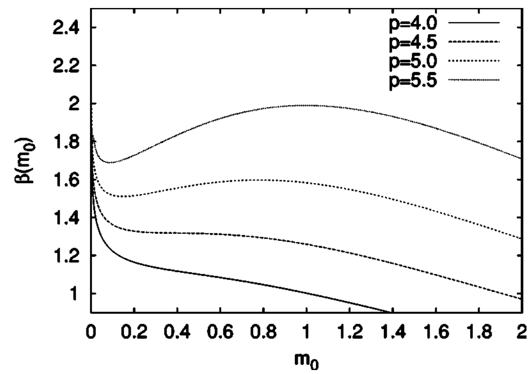

FIG. 1. Function $\beta(m_0)\equiv I(m_0)/W'(\tau)$ vs m_0, for $p=4$, 4.5, 5, and 5.5. For any $p, \beta(m_0)\to\infty$ for $m_0\to 0$.

We want now to discuss the behavior for $\beta\to\infty$. From Eq. (11), we see that $\beta\to\infty$ for $m_0\to 0$ and possibly for $m_0\to\overline{m}_i$, where $W'[\tau(\overline{m}_i)]=0$. If there is more than one solution, the relevant one corresponds to the lowest free energy. Now, for $\beta\to\infty$, we can simply write [38] $F\approx -2\beta W(\tau)$. Since $\tau(0)=2$ and $W(2)>W(\tau)$ for all $0\leq\tau<2$ because of the ferromagnetic condition, the relevant solution is the one with $m_0\to 0$. Then, using

$$I(m_0^2)=-\frac{1}{2\pi}\ln\frac{m_0^2}{32}+O(m_0^2\ln m_0^2) \tag{15}$$

for $m_0\to 0$, we obtain

$$m_0^2=32e^{-2\pi\beta+\pi W''(2)/2}[1+O(\beta^{-1})], \tag{16}$$

in agreement with the standard perturbative renormalization-group predictions [39].

Let us now discuss the possibility of first-order phase transitions, which may arise from the presence of multiple solutions to Eq. (11). As in Ref. [36], we consider

$$W(x)=\frac{2}{p}\left(\frac{x}{2}\right)^p-\frac{2^{1-p}}{p}. \tag{17}$$

In Fig. 1 we plot the function $\beta(m_0)\equiv I(m_0^2)/W'(\tau)$, for $p=4$, 4.5, 5, 5.5. For $p=4$, 4.5, for each β there is a unique solution m_0 and thus there are no phase transitions. On the other hand, for $p=5$, 5.5, there is the possibility of multiple solutions, in which case the most relevant is the one that gives the lowest free energy. For $p=5$, we plot the free energy in Fig. 2. We observe a first-order transition for $\beta\approx 1.543$ with a finite discontinuity of the correlation length, $\Delta\xi_\sigma\approx 16.2$, and of all thermodynamic quantities. A numerical analysis of the gap equation (11) shows that a first-order transition exists for all $p>p_c\approx 4.537\,857$. For $p=p_c$, the thermodynamic functions are nonanalytic for $\beta=\beta_c\approx 1.334\,72$. In this case,

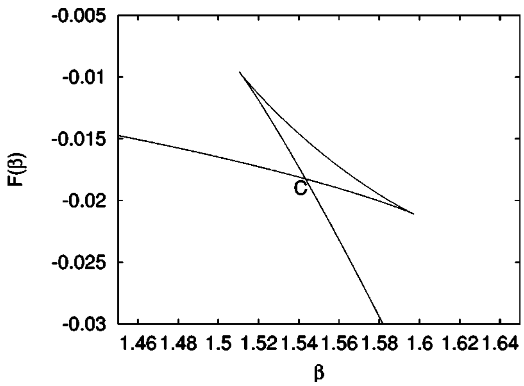

FIG. 2. The free energy $F(\beta)$ for $p=5$. There is a critical point C for $\beta_c \approx 1.543$.

$$\beta-\beta_c \approx -0.035\,726(m_0-m_{0c})^3+O[(m_0-m_{0c})^4], \tag{18}$$

where $m_{0c}\approx 0.387\,537$. Consequently, repeating the discussion of Ref. [20],

$$\xi_\sigma(\beta)\approx 2.5804+7.8682(\beta-\beta_c)^{1/3}+\cdots, \tag{19}$$

$$E(\beta)\approx 0.162\,274+0.314\,385(\beta-\beta_c)^{1/3}+\cdots, \tag{20}$$

$$C(\beta)\approx 0.104\,795(\beta-\beta_c)^{-2/3}+\cdots, \tag{21}$$

where E and C are, respectively, the energy and the specific heat per site. Note that $C(\beta)$ diverges at the critical point, indicating that, although spin-spin correlations are not critical, criticality is observed for energy-energy correlations. Indeed, consider

$$D_Q(k)=\sum_{x\mu\nu} e^{ik\cdot(x-y)}\langle Q(1+\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+\mu});Q(1+\boldsymbol{\sigma}_y\cdot\boldsymbol{\sigma}_{y+\nu})\rangle, \tag{22}$$

where $Q(x)$ is an arbitrary regular function. For $N\to\infty$,

$$D_Q(k)=[Q'(\tau)]^2\sum_{\mu\nu}\langle\hat{\rho}_\mu(-k);\hat{\rho}_\nu(k)\rangle, \tag{23}$$

so that

$$ND_Q(0)=\left(\frac{Q'(\tau)}{W'(\tau)}\right)^2 C(\beta). \tag{24}$$

It follows $D_Q(0)\sim(\beta-\beta_c)^{-2/3}$ for any function $Q(x)$. Thus, all correlation functions of the energy show a critical behavior. In order to compute the associated correlation length, we determine $D_Q(k)$ for arbitrary k. We obtain

$$ND_Q(k)=\frac{2[Q'(\tau)]^2[A_2(k)A_0(k)-A_1(k)^2]}{\beta^2[W'(\tau)]^2A_0(k)-\beta W''(\tau)[A_2(k)A_0(k)-A_1(k)^2]}, \tag{25}$$

where

$$A_n(k)=\int\frac{d^2q}{(2\pi)^2}\frac{\left(\sum_\mu \cos q_\mu\right)^n}{[\widehat{(q+k/2)}^2+m_0^2][\widehat{(q-k/2)}^2+m_0^2]}. \tag{26}$$

For $\beta\to\beta_c$ and $k\to 0$, we have

$$D_Q(k)^{-1}=a(\beta-\beta_c)^{2/3}+bk^2+O(k^4), \tag{27}$$

with $a,b\neq 0$. Thus, the energy-energy correlation length $\xi_E(\beta)$ behaves as

$$\xi_E(\beta)\sim(\beta-\beta_c)^{-1/3}, \tag{28}$$

i.e., $\nu_E=1/3$. We thus confirm the results of Ref. [36] on the existence of the critical theory for $p=p_c$, although we disagree on the nature of the critical behavior. Indeed, Ref. [36] suggested $\alpha=1-1/\delta$, with δ assuming the Ising value $\delta=15$. Instead, we find the mean-field value $\delta=3$. It is unclear how to reconcile our large-N result with the argument of Ref. [36]. Indeed, they argue that the transition should be Ising-like because the order parameter is a scalar and confirm numerically this conjecture for $N=3$. Note that the argument applies for all values of N and thus, if the large-N limit is smooth, it would predict Ising behavior even for $N=\infty$. On the other hand, for $N=\infty$ one expects mean-field exponents since fluctuations are neglected. Inclusion of the $1/N$ corrections is expected to change the value of α, but it would make it N *dependent*, and thus definitely not related to the Ising exponents. Therefore, either the limit $N\to\infty$ is singular, or the exponent α for this transition is different from that predicted in Ref. [36]. This issue deserves further investigations.

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[1] N.D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

[2] A.M. Polyakov, Phys. Lett. **59B**, 79 (1975).

[3] E. Brézin and J. Zinn-Justin, Phys. Rev. B **14**, 3110 (1976).

[4] W.A. Bardeen, B.W. Lee, and R.E. Shrock, Phys. Rev. D **14**, 985 (1976).

[5] U. Wolff, Phys. Rev. Lett. **62**, 361 (1989); Nucl. Phys. B **334**, 581 (1990); Phys. Lett. B **248**, 335 (1990).

[6] R.G. Edwards, S.J. Ferreira, J. Goodman, and A.D. Sokal, Nucl. Phys. B **380**, 621 (1992).

[7] J.-K. Kim, Phys. Rev. Lett. **70**, 1735 (1993); Phys. Rev. D **50**, 4663 (1994).

[8] S. Caracciolo, R.G. Edwards, A. Pelissetto, and A.D. Sokal, Phys. Rev. Lett. **75**, 1891 (1995); Nucl. Phys. B (Proc. Suppl.) **42**, 752 (1995).

[9] S. Caracciolo, R.G. Edwards, T. Mendes, A. Pelissetto, and A.D. Sokal, Nucl. Phys. B (Proc. Suppl.) **47**, 763 (1996).

[10] T. Mendes, A. Pelissetto, and A.D. Sokal, Nucl. Phys. B **477**, 203 (1996).

[11] M. Falcioni and A. Treves, Nucl. Phys. B **265**, 671 (1986).
[12] S. Caracciolo and A. Pelissetto, Nucl. Phys. B **420**, 141 (1994).
[13] S. Caracciolo and A. Pelissetto, Nucl. Phys. B **455**, 619 (1995). The results presented here contained a small numerical error. The correct results are given in Ref. [14].
[14] B. Allés, S. Caracciolo, A. Pelissetto, and M. Pepe, Nucl. Phys. B **562**, 581 (1999).
[15] B. Allés, A. Buonanno, and G. Cella, Nucl. Phys. B **500**, 513 (1997); Nucl. Phys. B (Proc. Suppl.) **53**, 677 (1997).
[16] P. Hasenfratz, M. Maggiore, and F. Niedermayer, Phys. Lett. B **245**, 522 (1990).
[17] P. Hasenfratz and F. Niedermayer, Phys. Lett. B **245**, 529 (1990).
[18] M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Lett. B **402**, 141 (1997).
[19] S. Hikami and T. Maskawa, Prog. Theor. Phys. **67**, 1038 (1982).
[20] N. Magnoli and F. Ravanini, Z. Phys. C **34**, 43 (1987).
[21] K. Ohno, H.-O. Carmesin, H. Kawamura, and Y. Okabe, Phys. Rev. B **42**, 10360 (1990).
[22] H. Kunz and G. Zumbach, J. Phys. A **22**, L1043 (1989); Phys. Lett. B **257**, 299 (1991); Phys. Rev. B **46**, 662 (1992); J. Phys. A **25**, 6155 (1992).
[23] P. Butera and M. Comi, Phys. Rev. B **46**, 11141 (1992).
[24] S. Caracciolo, R.G. Edwards, A. Pelissetto, and A.D. Sokal, Nucl. Phys. B (Proc. Suppl.) **26**, 595 (1992); **30**, 815 (1993); Phys. Rev. Lett. **71**, 3906 (1993).
[25] M. Hasenbusch, Phys. Rev. D **53**, 3445 (1996).
[26] F. Niedermayer, P. Weisz, and D.-S. Shin, Phys. Rev. D **53**, 5918 (1996).
[27] S.M. Catterall, M. Hasenbusch, R.R. Horgan, and R. Renken, Phys. Rev. D **58**, 074510 (1998).
[28] A.D. Sokal and A.O. Starinets, Nucl. Phys. B **601**, 425 (2001).
[29] G. Lasher, Phys. Rev. A **5**, 1350 (1972).
[30] P.A. Lebwohl and G. Lasher, Phys. Rev. A **6**, 426 (1972).
[31] G. Kohring and R.E. Shrock, Nucl. Phys. B **285**, 504 (1987).
[32] Lin Lei, Mol. Cryst. Liq. Cryst. **146**, 41 (1987).
[33] K.M. Leung and Lin Lei, Mol. Cryst. Liq. Cryst. **146**, 71 (1987).
[34] F. Biscarini, C. Zannoni, C. Chiccoli, and P. Pasini, Mol. Phys. **73**, 439 (1991).
[35] T.J. Krieger and H.M. James, J. Chem. Phys. **22**, 796 (1954).
[36] H.W.J. Blöte, W. Guo, and H.J. Hilhorst, Phys. Rev. Lett. **88**, 047203 (2002).
[37] If $W'(1)=0$, there is an additional solution with $\tau=1,\alpha=0,\gamma=\beta^{-1}$.
[38] This is obvious for $m_0\to\overline{m}_i$, since $I(\overline{m}_i)$ and $L(\overline{m}_i)$ are finite. For $m_0\to 0$, it is enough to observe that $L(0)$ is finite, while $\ln I(m_0^2)\approx \ln\beta$ because of the gap equations.
[39] The large-β behavior of ξ_σ for any $N\geqslant 3$ and any potential $W(x)$ is given in Ref. [12].