Multivariate Markov processes for stochastic systems with delays: Application to the stochastic Gompertz model with delay

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(Received 11 February 2002; published 26 July 2002)

Using the method of steps, we describe stochastic processes with delays in terms of Markov diffusion processes. Thus, multivariate Langevin equations and Fokker-Planck equations are derived for stochastic delay differential equations. Natural, periodic, and reflective boundary conditions are discussed. Both Ito and Stratonovich calculus are used. In particular, our Fokker-Planck approach recovers the generalized delay Fokker-Planck equation proposed by Guillouzic *et al.* The results obtained are applied to a model for population growth: the Gompertz model with delay and multiplicative white noise.

DOI: 10.1103/PhysRevE.66.011914 PACS number(s): 87.23.Cc, 02.50.Ey, 05.40.Jc

I. INTRODUCTION

The study of open complex systems has become an important aspect of biological physics. In particular, many researches have focused on low-dimensional descriptions of biological systems. These low-dimensional descriptions often involve delays and fluctuation forces. Delays typically arise from the propagation times of forces and the transmission times of information within spatially extended complex systems. Fluctuation forces often account for unspecific fluctuations of structural elements (thermal motion) and fluctuations of task-specific energy and information sources [1–3]. The former kind of fluctuations can be modeled by a heat bath that acts on the system under consideration in terms of additive noise. The latter kind of fluctuations can be conceived as multiplicative noise arising from fluctuating control parameters [4]. The impacts of noise and delays on nonlinear dynamical systems is a classical problem in population dynamics [5–8]. The role of delays in the respiratory system and the visual system has been studied as well [9–11]. There has been a general interest in studying the possible impacts of delays on movement control by artificially introducing delays into visual feedback loops [12–18]. Furthermore, delays of error correction mechanisms have been discussed in the context of polyrhythmic movements [19] and postural sway [20–22] (see, however, Refs. [23,24]). In addition, the relevance of transmission delays for neural networks and ensemble of coupled (neural) oscillators has been investigated in several works [25–33]. For a brief review of biological systems with delays the reader is referred to Ref. [34].

In order to describe complex systems featuring both delays and randomness, we may use generalized master equations [35–37] and stochastic delay differential equations (SDDEs), see, e.g., Refs. [11,15,22,26]. In the linear case, SDDEs can be solved analytically [38–41]. In general, SDDEs recover Langevin equations in the limit of vanishing delays. Recently, Guillouzic *et al.* derived generalized Fokker-Planck equations for SDDEs [40,42]. This correspondence has led to the hope that it might be possible to derive exact solutions for nonlinear stochastic systems with delays by means of Fokker-Planck equations. Since the theory of Fokker-Planck equations for stochastic systems without delays is well-established [43–48], a generalization to systems with delays could provide us with a powerful tool to describe the interplay between randomness and delays in open complex systems such as biological systems. Despite the benefits of the work by Guillouizic *et al.*, their approach is still incomplete. Their generalized Fokker-Planck equation does not provide a closed description of the problem at hand but involves a joint probability density whose evolution equation is not known. Furthermore, the work by Guillouizic *et al.* involves reflective boundary condition. Although most probably their approach can be generalized to other boundary conditions, this has not yet been done.

In the present study a closed description of stochastic processes with delays in terms of multivariate Markov diffusion processes will be derived. These processes will be expressed by means of multivariate Langevin and Fokker-Planck equations (Sec. II A). Boundary conditions will be discussed and the ‘‘delay Fokker-Planck equation’’ proposed by Guillouizic *et al.* will be derived for several boundary conditions using both Ito and Stratonovich calculus (Sec. II B). The power of the Fokker-Planck equation approach by Guillouizic *et al.* will be illustrated by reducing a delay Fokker-Planck equation with state-dependent diffusion term to one with state-independent diffusion term in the context of the Gompertz model for population growth with delay (Sec. II C).

II. MULTIVARIATE MARKOV PROCESSES FOR STOCHASTIC SYSTEMS WITH DELAYS

A. Multivariate Langevin and Fokker-Planck equations

Figure 1 illustrates the basic steps in describing stochastic processes with delays in terms of multivariate Markov processes. Accordingly, the evolution of a stochastic process with delay τ is decomposed into slices or intervals of length τ. The word ‘‘slice’’ reminds us that a process with delay depends on a function ϕ that describes its initial conditions. More precisely, for a complete description of the process we need to attach to each interval of length τ the space of admissible functions ϕ. Therefore, we may speak of a slice rather than an interval. In Fig. 1 the variable $\xi(t)$ and the function $P(x,t)$, respectively, denote the random variable and the probability density of the process of interest at time t. The stochastic evolution equation for ξ is then solved step by step for consecutive slices. This method is called ‘‘method of steps’’ and has been applied, for example, to

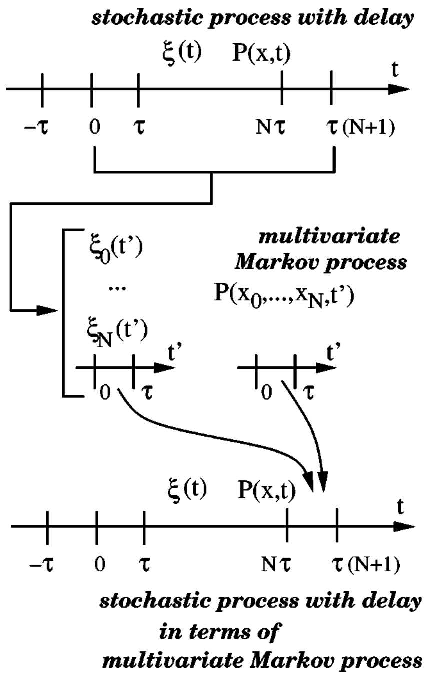

FIG. 1. Description of a stochastic process with delay in terms of a multivariate Markov process.

construct explicit solutions for linear SDDEs [39] and the "light tower model" of interacting neurons [32] and to demonstrate that within each slice a stochastic process with delay can be considered as a Markov (Feller) process [49]. Following these previous studies, we will derive multivariate Markov processes for stochastic systems with delays (cf. Fig. 1). Subsequently, for each slice we will express the stochastic processes with delays in terms of the corresponding multivariate Markov processes. Let us now proceed as outlined.

First, we discuss a stochastic process with delay and additive noise. Let $\xi(t)$ denote a dimensionless random variable which is described by the one-dimensional SDDE

$$\frac{d}{dt}\xi(t)=h(\xi(t),\xi(t-\tau))+\sqrt{Q}\Gamma(t),\quad t\geqslant 0,$$

$$\xi(t)=\phi(t),\quad t\in[-\tau,0], \tag{1}$$

where $h(x,y)$ is a drift function, $\tau>0$ is the delay, $\phi(t)$ describes the initial conditions of $\xi(t)$, and $\sqrt{Q}\Gamma$ represents a fluctuation force composed of the fluctuation strength $Q>0$ and the δ-correlated Langevin force Γ [45] with $\langle\Gamma(t)\Gamma(t')\rangle=\delta(t-t')$. In what follows, $\delta(z)$ denotes the δ function and the brackets $\langle A\rangle$ applied to a random variable A correspond to the ensemble average of A. Introducing for $t\in[-\tau,(N+1)\tau]$ and $n=-1,0,1,\ldots,N$ new random variables ξ_n defined by

$$\xi_n(t'):=\xi(t)\wedge t'=t-n\tau,\quad t\in[n\tau,(n+1)\tau], \tag{2}$$

we obtain $\xi_{-1}(t'):=\phi(t'-\tau)$ and

$$\frac{d}{dt'}\xi_n(t')=h(\xi_n(t'),\xi_{n-1}(t'))+\sqrt{Q}\Gamma(t'+n\tau),$$

$$n=0,\ldots,N \tag{3}$$

for $t'\in[0,\tau]$. For the sake of convenience, we define the Langevin forces Γ_n by $\Gamma_n(t'):=\Gamma(t)$ and rewrite Eq. (3) as

$$\frac{d}{dt'}\xi_n(t')=h(\xi_n(t'),\xi_{n-1}(t'))+\sqrt{Q}\Gamma_n(t'). \tag{4}$$

Note that by this definition the Langevin forces Γ_n satisfy $\langle\Gamma_i(t)\Gamma_k(t')\rangle=\delta_{ik}\delta(t-t')$, where δ_{ik} is the Kronecker symbol. The Langevin equations (4) defined on $t'\in[0,\tau]$ describe a particular kind of multivariate Markov processes, namely, multivariate Markov diffusion processes [4]. Having derived Eq. (4) from Eq. (1), we show now that we can conclude in the opposite way as well. That is, we derive now the stochastic process with delay (1) from the Markov processes (4). To this end, we need to discuss initial conditions and, therefore, define for $t'\in[0,\tau]$ and $n=0,1,\ldots,N$ the joint probability densities $W_n(x,y,t')$ by $W_n(x,y,t'):=\langle\delta(x-\xi_n(t'))\delta(y-\xi_{n-1}(t'))\rangle$. Then, we require that at each computational level n the initial joint probability density $W_n(x,y,0)$ agrees with the final joint probability density $W_{n-1}(x,y,\tau)$ of the previous computational level $n-1$. More precisely, we define

$$W_0(x,y,0):=\delta(x-\phi(0))\delta(y-\phi(-\tau)),$$

$$W_n(x,y,0):=W_{n-1}(x,y,\tau),\quad n=1,\ldots,N. \tag{5}$$

The multivariate Markov process given by Eqs. (4) and (5) is equivalent to the SDDE (1) for $t\in[-\tau,(N+1)\tau]$ which can be illustrated by solving iteratively Eqs. (4) and (5) for each level n upto $n=N$. For example, for $n=0$ from Eqs. (4) and (5) it follows that

$$\frac{d}{dt'}\xi_0(t')=h(\xi_0(t'),\phi(t'-\tau))+\sqrt{Q}\Gamma_0(t'),$$

$$\xi_0(0)=\phi(0), \tag{6}$$

which agrees with Eq. (1) for $t'=t$. Solving the zeroth computational level (6), we obtain $W_0(x,y,t')$ and, in particular, $W_0(x,y,\tau)$. Consequently, for $n=1$ from Eqs. (4) and (5) we obtain

$$\frac{d}{dt'}\xi_1(t')=h(\xi_1(t'),\xi_0(t'))+\sqrt{Q}\Gamma_1(t') \tag{7}$$

with $\xi_0(t')$ given by Eq. (6) and the initial condition $\langle\delta(x-\xi_1(0))\delta(y-\xi_0(0))\rangle=\langle\delta(x-\xi_0(\tau))\rangle\delta(y-\phi(0))=W_0(x,y,\tau)$ [i.e., $\xi_1(0)$ is distributed like $\xi_0(\tau)$ and $\xi_0(0)$ assumes the fixed value $\phi(0)$]. Under these initial conditions Eq. (7) agrees with the SDDE (1) for $\xi_1(t')=\xi(t),t'=t-\tau,t'\in[0,\tau]$, and $t\in[\tau,2\tau]$.

We recognize that we cannot solve the two-dimensional Markov process described by Eq. (4) for $n=N$ at once. We need to evaluate all preceding levels $n<N$ in order to obtain the initial condition for the level $n=N$ and the evolution of ξ_{N-1}. In other words, by means of a multivariate Markov process we can only solve a SDDE step by step (whence the name "method of steps").

The next objective is to take multiplicative noise into account. To this end, we supplement Eq. (1) with a state-dependent noise term $g(x,y)$ leading to

$$\frac{d}{dt}\xi(t)=h(\xi(t),\xi(t-\tau))+g(\xi(t),\xi(t-\tau))\Gamma(t). \quad (8)$$

Following Mohammed [50], the multiplicative noise term can be interpreted as a stochastic integral (Lebegues-Stieltjes integral) just as in the case of systems without delay [4,45]. The reason for this is that (i) $\xi(t)$ and $\xi(t-\tau)$ can be considered as two random variables $\eta_1(t):=\xi(t)$ and $\eta_2(t)=\xi(t-\tau)$ and (ii) the stochastic integral for $g(\eta_1(t),\eta_2(t))\Gamma(t)$ is defined irrespective of the vanishing or nonvanishing of correlations between η_1 and η_2. Consequently, there is both an Ito and a Stratonovich interpretation of the SDDE (8). Let $w(t)$ denote the Wiener process given by $w(t)=\int^t\Gamma(z)dz$, then Eq. (8) can be regarded as

$$\xi(t+\epsilon)=\xi(t)+\int_t^{t+\epsilon}h(\xi(s),\xi(s-\tau))ds + {}_S^I\!\int_t^{t+\epsilon}g(\xi(s),\xi(s-\tau))dw(s), \quad (9)$$

where ${}_S^I\!\int\cdots$ refers to the Ito and Stratonovich interpretation [4,45]. Just as for SDDEs with additive noise, we can show the equivalence between the SDDE (8) for $t\in[-\tau,(N+1)\tau]$ and the multivariate Markov diffusion processes for $t'\in[0,\tau]$ described by

$$\frac{d}{dt'}\xi_n(t')=h(\xi_n(t'),\xi_{n-1}(t')) + g(\xi_n(t'),\xi_{n-1}(t'))\Gamma_n(t'), \quad n=0,\ldots,N. \quad (10)$$

Again, Eq. (10) is solved iteratively by means of the initial conditions (5).

Introducing the closed and open Θ functions $\Theta_{\rm cl}$ and $\Theta_{\rm op}$ defined by $\Theta_{\rm cl}(z)=1$ for $z\in[0,\infty)$ and zero otherwise and $\Theta_{\rm op}(z)=1$ for $z\in(0,\infty)$ and zero otherwise, we can express $\xi(t)$ for $t\in[-\tau,N\tau]$ in terms of ξ_n by

$$\xi(t)=\sum_{n=-1}^{N}\xi_n(t-n\tau)\Theta_{\rm cl}(t-n\tau)\Theta_{\rm op}(n\tau+\tau-t). \quad (11)$$

Likewise, the probability density $P(x,t)=\langle\delta(x-\xi(t))\rangle$ can be obtained from the joint probability densities W_n according to

$$P(x,t)=\sum_{n=-1}^{N}\Theta_{\rm cl}(t-n\tau)\Theta_{\rm op}((n+1)\tau-t) \times\int W_n(x,y,t-n\tau)dy. \quad (12)$$

So far, we have tacitly regarded Eqs. (4) and (8) as two-dimensional Markov processes involving the random variables ξ_n and ξ_{n-1}. Alternatively, we can view the Langevin equations for $n=0,\ldots,N$ as a $(N+1)$-dimensional Markov process. Then, we need to solve this multivariate Markov process for $N=0$ to obtain the initial condition for the Markov process with $N=1$. This Markov process, in turn, gives us the initial condition for the process with $N=2$, and so on. That is, we deal with a hierarchy of Markov processes defined on a phase space that is increased by one dimension with each computational step. On the basis of this interpretation, we can obtain the Fokker-Planck equation that corresponds to the set of Langevin equations (10). For $P^{N+1}(x_0,\ldots,x_N,t')=\langle\delta(x_0-\xi_0(t'))\cdots\delta(x_N-\xi_N(t'))\rangle$ and $t'\in[0,\tau]$ the evolution equation reads [45]

$$\frac{\partial}{\partial t'}P^{N+1}(.,t')=-\sum_{n=0}^{N}\frac{\partial}{\partial x_n}\left\{h(x_n,x_{n-1}) +\frac{\nu}{2}g(x_n,x_{n-1})\frac{\partial g(x_n,x_{n-1})}{\partial x_n}\right\}P^{N+1}(.,t') +\frac{1}{2}\sum_{n=0}^{N}\frac{\partial^2}{\partial x_n^2}[g(x_n,x_{n-1})]^2P^{N+1}(.,t'),$$

$$N=0,1,2,\ldots \quad (13)$$

with $\nu=0$ and $\nu=1$ for Ito and Stratonovich calculus, respectively. Note that we have introduced above the function $x_{-1}(t'):=\xi_{-1}(t')=\phi(t'-\tau)$. Solving iteratively the multivariate Langevin equation (10), we can verify that the initial condition for the $(N+1)$-dimensional Fokker-Planck equation (13) is determined by the final joint distribution of the N-dimensional Fokker-Planck equation for $P^N(x_0,\ldots,x_{N-1},t')$. In detail, we get

$$P^{N+1}(x_0,\ldots,x_N,0)=\delta(x_0-\phi(0))P^N(x_1,\ldots,x_N,\tau) =\delta(x_0-x_{-1}(\tau))P^N(x_1,\ldots,x_N,\tau). \quad (14)$$

Equation (14) describes the connection between two slices (cf. Fig. 1). It tells us how to extend the N-dimensional Markov process described by $P^N(x_0,\ldots,x_{N-1},t')$ to a $(N+1)$-dimensional one given by $P^{N+1}(x_0,\ldots,x_N,t')$ such that the Markov process in the $(N+1)$-dimensional phase space represents a continuation of the Markov process in the N-dimensional phase space. It is clear from Eq. (14) that we need to solve the Fokker-Planck equation (13) iteratively in order to derive the initial condition $P^{N+1}(.,0)$. The continuation condition (14) for the multivariate Fokker-Planck equation (13) can be seen as the counterpart to the continuation condition (5) for the multivariate Langevin equation (10).

Finally, we can read off from Eq. (13) that the Ito and Stratonovich calculus yield the same results for multiplicative noise sources involving exclusively delayed variables, that is, for $g(x,y)=g(y)$. In fact, by computing explicitly the integrals ${}^{I}_{S}\int_t^{t+\epsilon} g(\xi(s-\tau))dw(s)$ for $g(y)=y$, Mohammed showed that in this case the Ito and Stratonovich integrals are equivalent [50].

B. Boundary conditions and generalized delay Fokker-Planck equations

So far, we have avoided a discussion of the set Ω on which the random variable ξ given by Eq. (8) is defined. We consider now three cases.

(i) We assume $\xi\in\Omega=\mathbb{R}$ and require natural boundary conditions, that is, $P(x\to\pm\infty,t)=0$.

(ii) We assume $\Omega=[a,b]$ with $a<b$ and consider periodic boundary conditions, which implies $P(x,t)=P(x+b-a,t)$ as well as $h(x,y)=h(x',y')$ and $h(x,y)=h(x',y')$ with $x'=x+b-a$ and $y'=y+b-a$.

(iii) We put $\Omega=[a,b]$ with $a<b$ again and require reflective boundary conditions, that is, the stochastic process is confined to the interval Ω.

In view of the definition (2), we can conclude that if ξ satisfies a particular boundary condition then the variables ξ_n satisfy that boundary condition as well. Vice versa, if all ξ_n satisfy one of the boundary conditions (i)–(iii) then the multivariate Langevin equation (8) and Eq. (5) describe a stochastic process with delay that satisfies that particular boundary condition.

In order to incorporate boundary conditions into the description of the SDDE (1) via the multivariate Fokker-Planck equation (13) we express the joint probability density $P^{N+1}(x_0,\ldots,x_N,t')$ by virtue of Eq. (2) in terms of the random variable ξ as

$$\begin{aligned}P^{N+1}(x_0,x_1,\ldots,x_N,t')&=\langle\delta(x_0-\xi_0(t'))\delta(x_1-\xi_1(t'))\cdots\delta(x_N-\xi_N(t'))\rangle=\langle\delta(x_0-\xi(t'))\delta(x_1-\xi(t'+\tau))\cdots\delta(x_N\\&\quad-\xi(t'+N\tau))\rangle=\langle\delta(x_0-\xi(t-N\tau))\delta(x_1-\xi(t-(N-1)\tau))\cdots\delta(x_N-\xi(t))\rangle\\&=P^{N+1}(x_0,t';x_1,t'+\tau;\ldots;x_N,t'+N\tau)=P^{N+1}(x_0,t-N\tau;x_1,t-(N-1)\tau;\ldots;x_N,t)\end{aligned}\tag{15}$$

for $t'\in[0,\tau]$ and $t\in[N\tau,(N+1)\tau]$. Accordingly, for $t'\in[0,\tau]$ Eq. (13) reads

$$\begin{aligned}\frac{\partial}{\partial t'}P^{N+1}(x_0,t';\ldots;x_N,t'+N\tau)=\Bigg(&-\sum_{n=0}^{N}\frac{\partial}{\partial x_n}\left\{h(x_n,x_{n-1})+\frac{\nu}{2}g(x_n,x_{n-1})\frac{\partial g(x_n,x_{n-1})}{\partial x_n}\right\}\\&+\frac{1}{2}\sum_{n=0}^{N}\frac{\partial^2}{\partial x_n^2}[g(x_n,x_{n-1})]^2\Bigg)P^{N+1}(x_0,t';\ldots;x_N,t'+N\tau)\end{aligned}\tag{16}$$

and for $t\in[N\tau,(N+1)\tau]$ we obtain

$$\begin{aligned}\frac{\partial}{\partial t}P^{N+1}(x_0,t-N\tau;\ldots;x_N,t)=\Bigg(&-\sum_{n=0}^{N}\frac{\partial}{\partial x_n}\left\{h(x_n,x_{n-1})+\frac{\nu}{2}g(x_n,x_{n-1})\frac{\partial g(x_n,x_{n-1})}{\partial x_n}\right\}\\&+\frac{1}{2}\sum_{n=0}^{N}\frac{\partial^2}{\partial x_n^2}[g(x_n,x_{n-1})]^2\Bigg)P^{N+1}(x_0,t-N\tau;\ldots;x_N,t).\end{aligned}\tag{17}$$

Introducing the probability current

$$S_n[P^{N+1}]:=\left(h(x_n,x_{n-1})+\frac{\nu}{2}g(x_n,x_{n-1})\frac{\partial g(x_n,x_{n-1})}{\partial x_n}-\frac{1}{2}\sum_{n=0}^{N}\frac{\partial}{\partial x_n}[g(x_n,x_{n-1})]^2\right)P^{N+1},\tag{18}$$

Eq. (17) can be written as

$$\frac{\partial}{\partial t}P^{N+1}(x_0,t-N\tau;\ldots;x_N,t)=-\sum_{n=0}^{N}\frac{\partial}{\partial x_n}S_n[P^{N+1}].\tag{19}$$

Natural boundary conditions imply $\forall n:x_n\in\Omega=\mathbb{R},P^{N+1}(x_0,t-N\tau;\ldots;x_n\to\pm\infty,t-(N-n)\tau;\ldots;x_N,t)=0,S_n|_{\partial\Omega}=0$, where $S_n|_{\partial\Omega}$ means that we take the probability current at a boundary value of Ω (here: in the limit of $x_n\to\pm\infty$ for arbitrary n). Periodic boundary conditions lead to $\forall n:x_n\in\Omega=[a,b],P^{N+1}(x_0,t-N\tau;\ldots;x_n,t-(N-n)\tau;\ldots;x_N,t)=P^{N+1}(x_0,t-N\tau;\ldots;x_n+b-a,t-(N-n)\tau;\ldots;x_N,t),S_n|_{\partial\Omega(1)}-S_n|_{\partial\Omega(2)}=0$, that is, the component S_n of the probability current is constant at the boundary of Ω. Likewise, for reflective boundary conditions we obtain $\forall n : x_n \in \Omega = [a,b], S_n|_{\partial\Omega} = 0$ (the probability current S_n must vanish at the boundary of Ω; otherwise, for $t \in [n\tau,(n+1)\tau]$ the random variable ξ could leave the region Ω [51]).

Having defined several boundary conditions we can derive from Eqs. (17)–(19) a generalized Fokker-Planck equation for $P(x,t) = \langle \delta(x-\xi(t)) \rangle$ and $t \in [N\tau,(N+1)\tau]$. To this end, we put $x_N = x$ and integrate Eq. (19) with respect to $x_0, \ldots, x_{N-1}$ which yields for $N \geqslant 1$ the evolution equation

$$\begin{aligned}
\frac{\partial}{\partial t}P(x,t) &= -\frac{\partial}{\partial x}\int_{\Omega^N} S_N[P^{N+1}]\prod_{i=0}^{N-1} \mathrm{d}x_i - \sum_{n=0}^{N-1}\int_{\Omega^N}\frac{\partial}{\partial x_n}S_n[P^{N+1}]\prod_{i=0}^{N-1}\mathrm{d}x_i \\
&= -\frac{\partial}{\partial x}\int_{\Omega}\left\{h(x,y) + \frac{\nu}{2}g(x,y)\frac{\partial g(x,y)}{\partial x}\right\}P_2^{N+1}(y,t-\tau;x,t)\mathrm{d}y \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\int_{\Omega}[g(x,y)]^2 P_2^{N+1}(y,t-\tau;x,t)\mathrm{d}y \\
&\quad - \sum_{n=0}^{N-1}\int_{\Omega^{N-1}}\Big\{\underbrace{S_n|_{\partial\Omega(1)} - S_n|_{\partial\Omega(2)}}_{=0}\Big\}\prod_{i=0;i\neq n}^{N-1}\mathrm{d}x_i
\end{aligned} \tag{20}$$

with $P_2^{N+1}(y,t-\tau;x,t) := \int_{\Omega} P^{N+1}(x_0,t-N\tau; \ldots ;x_{N-2},t-2\tau;y,t-\tau;x,t)dx_0 \cdots dx_{N-2}$. For $N=0$ and $t \in [0,\tau]$ we have

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left\{h(x,\phi(t-\tau)) + \frac{\nu}{2}g(x,\phi(t-\tau))\frac{\partial g(x,\phi(t-\tau))}{\partial x}\right\}P(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}[g(x,\phi(t-\tau))]^2 P(x,t). \tag{21}$$

On account of the boundary conditions discussed earlier the surface terms in Eq. (20) vanish—as indicated. The two-point probability densities $P_2^{N+1}(y,t-\tau;x,t)$ represent the projections of the general two-point probability density $P(y,t-\tau;x,t) = \langle \delta(x-\xi(t))\delta(y-\xi(t-\tau)) \rangle$ onto the slices $t \in [N\tau,(N+1)\tau]$. Furthermore, the identity $W_N(x,y,t-N\tau) = P_2^{N+1}(y,t-\tau;x,t)$ holds. From Eq. (5) it then follows that $P_2^{N+1}(y,t-\tau;x,t)|_{t=N\tau} = W_N(x,y,0) = W_{N-1}(x,y,\tau) = P_2^N(y,t-\tau;x,t)|_{t=N\tau}$. Put differently, at the interface $t=N\tau$ between two slices we find $P_2^{N+1}(y,t-\tau;x,t) = P_2^N(y,t-\tau;x,t) = P(y,t-\tau;x,t)$. Consequently, we can replace P_2^{N+1} by $P(y,t-\tau;x,t)$. Then, for $t \geqslant \tau$ Eq. (20) becomes

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\int_{\Omega}\left\{h(x,y) + \frac{\nu}{2}g(x,y)\frac{\partial g(x,y)}{\partial x}\right\}P(y,t-\tau;x,t)dy + \frac{1}{2}\frac{\partial^2}{\partial x^2}\int_{\Omega}[g(x,y)]^2 P(y,t-\tau;x,t)dy. \tag{22}$$

The generalized delay Fokker-Planck equation (22) was previously derived by Guillouzic *et al.* for $g(x,y) = g(x)$ and reflective boundary conditions [40]. Our analysis reveals that Eq. (22) can be viewed as a projection of the closed hierarchy of Fokker-Planck equations (17) with $N = 0,1,2,\ldots$ onto the one-variable probability density $P(x,t)$. As a result of this projection, we deal with an evolution equation for $P(x,t)$ that is not closed because it involves another member of the hierarchy: $P(y,t-\tau;x,t)$.

C. Gompertz model with multiplicative noise and delay

As stated in the Introduction, the impacts of delay and noise on dynamical systems have frequently been studied in the context of population dynamics. Delays are typically related to the maturation (or generation) times of species [7,8]. Noise is a phenomenon that inevitably occurs in population dynamics because populations evolve in close contact with their environments and environmental parameters fluctuate. It is, in particular, multiplicative noise that can affect population growth. Multiplicative noise may reflect fluctuations of growth rate parameters [52] or evolutionary disasters proportional to population sizes [53]. The Gompertz model for population growth [6,7,53] is one of the few models that can be treated analytically when taking impacts of delays and multiplicative noise into account [6,41]. Therefore, it may serve as a benchmark model. The objective now is twofold: to elucidate the nature of the multiplicative noise (Ito vs Stratonovich) and to discuss the Gompertz model within the framework of the delay Fokker-Planck equation (22).

Let $N(t) \geqslant 0$ denote the population size at time t. The deterministic Gompertz model reads $dN/dt = kNG(N)$ with $G(N) = -\ln(N/c)$ and $c > 0$. The factor k corresponds to the growth rate in the linear case $G=1$, whereas G denotes a saturation function leading to a stable fixed point at $N=c$. The saturation function describes the decrease of the effective growth rate due to finite resources and the increase of the population size [i.e., $dN/dt = k_{\mathrm{eff}}N$ and $k_{\mathrm{eff}} = kG(N)$].

However, in general, the saturation function G depends on the history of the population [6,7,54], which can be modeled by introducing a delay $\tau \geqslant 0$ like $G(t)=G(N(t-\tau))$. As a result, the Gompertz model with delay and size-dependent (multiplicative) white noise reads [41]

$$\frac{d}{dt}N(t)=-kN(t)\ln\left(\frac{N(t-\tau)}{c}\right)+\sqrt{Q}N(t)\Gamma(t), \quad t\geqslant 0. \tag{23}$$

For the sake of convenience, for $t\in[-\tau,0]$ we may choose $N(t)=d>0$. Now, the question arises how to interpret the multiplicative noise term $N\Gamma$.

First, we interpret $N\Gamma$ according to the Stratonovich calculus. In this case, we can perform the variable transformation $\xi:=\ln(N/c)$. The reason for this is the equivalence between the SDDE (23) and the multivariate Langevin equation (23) and the fact that for multivariate Stratonovich-Langvin equations variable transformations of that kind can indeed be performed. Strictly speaking, we transform the Stratonivich-SDDE (23) into a multivariate Stratonovich-Langevin equation of the form (23), carry out the corresponding variable transformations $\xi_n=\ln(N_n/c)$, and transform the multivariate Langevin equation thus obtained into a SDDE again. Thus, we obtain

$$\frac{d}{dt}\xi(t)=-k\xi(t-\tau)+\sqrt{Q}\Gamma(t), \quad t\geqslant 0 \tag{24}$$

with $\xi\in\Omega=\mathbb{R}$ and $\xi(t)=\ln(d/c)$ for $t\in[-\tau,0]$. The stationary solution $P_{\rm st}(x)$ of Eq. (24) can be derived [39,40]. Then, using the inverse transformation $N=c\exp\xi$, the stationary probability density $P_{\rm st}(N)$ of the Gompertz model (23) can be computed [41]. Let us discuss the Gompertz model (23) within the framework of the delay Fokker-Planck equation (22). Then, we have $h(x,y)=-kx\ln(y/c)$, and $g(x,y)=\sqrt{Q}x$ leading to [55]

$$\frac{\partial}{\partial t}P(x,t)=\frac{\partial}{\partial x}\left[kx\int_0^\infty\left\{\ln\left(\frac{y}{c}\right)\right\}P(y,t-\tau;x,t)dy-\frac{Q}{2}xP(x,t)\right]+\frac{Q}{2}\frac{\partial^2}{\partial x^2}x^2P(x,t)=k\frac{\partial}{\partial x}x\int_0^\infty\left\{\ln\left(\frac{y}{c}\right)\right\}P(y,t-\tau;x,t)dy+\frac{Q}{2}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}xP(x,t). \tag{25}$$

By means of the transformations $x'=\ln(x/c),y'=\ln(y/c),R(x',t)dx'=P(x,t)dx$, and $R(y',t-\tau;x',t)dy'dx'=P(y,t-\tau;x,t)dydx$ the relations $xP(x,t)=R(x',t)$ and $xyP(y,t-\tau;x,t)=R(y',t-\tau;x',t)$ can be found. Furthermore, the operator relation $x\partial/\partial x=\partial/\partial x'$ holds. Substituting these results into Eq. (25) gives us

$$\frac{\partial}{\partial t}R(x',t)=k\frac{\partial}{\partial x'}\int_0^\infty y'R(y',t-\tau;x',t)dy'+\frac{Q}{2}\frac{\partial^2}{\partial x'^2}R(x',t). \tag{26}$$

Equations (24) and (26) illustrate that the variable transformation based on the Stratonovich calculus can be performed both for the SDDE and the delay Fokker-Planck equations and, indeed, leads to consistent results [because Eq. (26) is the delay Fokker-Planck equation of Eq. (24)].

Second, we interpret Eq. (23) as Ito-SDDE. The corresponding delay Fokker-Planck equation reads

$$\frac{\partial}{\partial t}P(x,t)=k\frac{\partial}{\partial x}x\int_0^\infty\left\{\ln\left(\frac{y}{c}\right)\right\}P(y,t-\tau;x,t)dy+\frac{Q}{2}\frac{\partial^2}{\partial x^2}x^2P(x,t) \tag{27}$$

[use $\nu=0$ and cf. Eqs. (22) and (25a)]. By means of c^* defined by

$$k\ln\frac{1}{c}=k\ln\frac{1}{c^*}-\frac{Q}{2}, \tag{28}$$

Eq. (27) can be expressed as

$$\frac{\partial}{\partial t}P(x,t)=\frac{\partial}{\partial x}\left[kx\int_0^\infty\left\{\ln\left(\frac{y}{c^*}\right)\right\}P(y,t-\tau;x,t)dy-\frac{Q}{2}xP(x,t)\right]+\frac{Q}{2}\frac{\partial^2}{\partial x^2}x^2P(x,t)=k\frac{\partial}{\partial x}x\int_0^\infty\left\{\ln\left(\frac{y}{c^*}\right)\right\}P(y,t-\tau;x,t)dy+\frac{Q}{2}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}xP(x,t). \tag{29}$$

By means of the transformations $x'=\ln(x/c^*)$ and $y'=\ln(y/c^*)$ the delay Fokker-Planck equation (29) can be transformed into Eq. (26) again. Substituting Eq. (28) into the Gompertz model (23) yields

$$\frac{d}{dt}N(t)=-kN(t)\ln\left(\frac{N(t-\tau)}{c^*}\right)+\frac{Q}{2}N(t)+\sqrt{Q}N(t)\Gamma(t). \tag{30}$$

From Eqs. (29) and (30) it is clear that the Ito-SDDE (30) yields the same delay Fokker-Planck equation as the Stratonovich-SDDE

$$\frac{d}{dt}N(t)=-kN(t)\ln\left(\frac{N(t-\tau)}{c^*}\right)+\sqrt{Q}N(t)\Gamma(t). \tag{31}$$

Moreover, by means of Eq. (10), we can assign to Eqs. (30) and (31) the multivariate Ito- and Stratonovich-Langevin equations

$$\frac{d}{dt'}N_n(t')=-kN_n(t')\ln\left(\frac{N_{n-1}(t')}{c^*}\right)+\frac{Q}{2}N_n(t)+\sqrt{Q}N_n(t')\Gamma_n(t') \tag{32}$$

and

$$\frac{d}{dt'}N_n(t')=-kN_n(t')\ln\left(\frac{N_{n-1}(t')}{c^*}\right)+\sqrt{Q}N_n(t')\Gamma_n(t'), \tag{33}$$

respectively. Furthermore, we can verify that these two Langevin equations correspond to the same multivariate Fokker-Planck equation (13). Consequently, not only are the two delay Fokker-Planck equations of the Ito- and Stratonovich-Langevin equations (30) and (31) identical but they describe the same stochastic process. Therefore, the stationary solution of the Ito interpretation of the Gompertz model (23) with the fixed point parameter c can be obtained from the solution of the Stratonovich interpretation of the Gompertz model (33) with parameter c^* and vice versa. In addition, comparing Eqs. (30) and (31), we obtain the equivalence

$$\frac{1}{2}QN(t)+\underbrace{\sqrt{Q}N(t)\Gamma(t)}_{\text{Ito}}=\underbrace{\sqrt{Q}N(t)\Gamma(t)}_{\text{Stratonovich}}. \tag{34}$$

On account of this equivalence, we may say that the observed shift of the fixed point is caused by the emergence of the so-called spurious drift [45] given above by $QN/2$.

III. CONCLUSIONS

Using the method of steps, multivariate Langevin and Fokker-Planck equations have been derived for stochastic processes with delay. In doing so, non-Markovian stochastic processes have been assigned to multivariate Markov diffusion processes. This procedure is reminiscent of the treatment of one-dimensional non-Markovian stochastic equations with colored noise that can be mapped onto two-dimensional Markov processes with white noise [45].

The multivariate Markov processes that have been discussed in the present article provide a closed description for stochastic processes with delays and can be solved iteratively under several boundary conditions. The delay Fokker-Planck equation proposed by Guillouzic *et al.* has been identified as the projection of such a multivariate Markov process onto a one-variable stochastic process.

Furthermore, the applicability of the Ito and Stratonovich calculus for stochastic processes with delays has been demonstrated. As a key result, we have found that for delayed random variables the Ito and Stratonovich calculus yield the same results. Nondelayed variables can be treated just as the random variables of ordinary Langevin equations.

For a model describing population growth, namely, the Gompertz model with delay and multiplicative white noise, we demonstrated how variable transformations can be carried out in stochastic delay differential equations and their corresponding delay Fokker-Planck equations. In this context, it has been shown that a multiplicative noise term interpreted according to the Ito calculus can be expressed by a multiplicative noise term interpreted according to the Stratonovich calculus. This relation [cf. Eq. (34)] can easily be generalized to arbitrary multiplicative noise terms $g(x,y)$ because of the equivalence of stochastic delay differential equations and multivariate Markov diffusion processes. Then, we obtain

$$\frac{1}{2}g(x,y)\frac{\partial g(x,y)}{\partial x}\Big|_{x=\xi(t),y=\xi(t-\tau)}+\underbrace{g(\xi(t),\xi(t-\tau))\Gamma(t)}_{\text{Ito}} =\underbrace{g(\xi(t),\xi(t-\tau)\Gamma(t)}_{\text{Stratonovich}}, \tag{35}$$

which recovers for multiplicative noise terms without delay [i.e., for $g(x,y)=g(x)$] the well-known relation (see, e.g., [4] [Sec. 5.4.2] and [40] [Appendix])

$$\frac{1}{2}g(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\Big|_{x=\xi}+\underbrace{g(\xi)\Gamma}_{\text{Ito}}=\underbrace{g(\xi)\Gamma}_{\text{Stratonovich}}. \tag{36}$$

We showed that in order to evaluate a single time-interval of a stochastic process with delay in terms of a multivariate Markov process we need to increase the dimension of that Markov process by one. Consequently, the stationary solution of a stochastic process with delay corresponds to a N-dimensional Markov process where N tends to infinity. Future studies may exploit this observation in order to construct stationary solutions of nonlinear stochastic processes with delay on the basis of solutions of N-dimensional Fokker-Planck equations in the limit $N\rightarrow\infty$.

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[55] Here, $N \in \Omega = [0,\infty)$ with a reflective boundary at $N=0$ and a natural boundary at $N \rightarrow \infty$. For these mixed boundary conditions the currents S_n vanish at the boundaries again [hint: use the arguments for boundary conditions (i) and (iii) in combination]. Consequently, the results obtained in Sec. II B apply in this case, too.