Dynamic stability conditions for Lotka-Volterra recurrent neural networks with delays

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The Lotka-Volterra model of neural networks, derived from the membrane dynamics of competing neurons, have found successful applications in many ''winner-take-all'' types of problems. This paper studies the dynamic stability properties of general Lotka-Volterra recurrent neural networks with delays. Conditions for nondivergence of the neural networks are derived. These conditions are based on local inhibition of networks, thereby allowing these networks to possess a multistability property. Multistability is a necessary property of a network that will enable important neural computations such as those governing the decision making process. Under these nondivergence conditions, a compact set that globally attracts all the trajectories of a network can be computed explicitly. If the connection weight matrix of a network is symmetric in some sense, and the delays of the network are in L^2 space, we can prove that the network will have the property of complete stability.

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I. INTRODUCTION

The Lotka-Volterra model of neural networks was first proposed in Ref. [1]. Derived from conventional membrane dynamics of competing neurons, it has found successful applications in many ''winner-take-all'' types of problems, see Refs. [1–3]. Due to the application potential of this class of neural networks, it is necessary and useful to study its general dynamic properties. From an engineering point of view, the dynamic stability properties of neural networks are prerequisites towards effective applications. It is also important to be able to choose effective parameters that control the network's functions.

In this paper, we will study the dynamic stability properties of general Lotka-Volterra recurrent neural networks with delays. This model can be described by the following nonlinear differential equation with delays:

$$\dot{x}_i(t)=x_i(t)\left[h_i-x_i(t)+\sum_{j=1}^{n}\{a_{ij}x_j(t)+b_{ij}x_j[t-\tau_{ij}(t)]\}\right] \quad (i=1,\ldots,n) \quad (1)$$

for $t\geqslant 0$, where each x_i denotes the activity of neuron i, $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$ are real $n\times n$ matrices, each of their elements denotes the synaptic weights and represents the strength of the synaptic connection from neuron j to neuron i, $x=(x_1,\ldots,x_n)^T\in R^n$, $h_i\in R^n$ denotes external inputs, the delays $\tau_{ij}(t)$ $(i,j=1,\ldots,n)$ are non-negative continuous functions satisfying $0\leqslant\tau_{ij}(t)\leqslant\tau$ for $t\geqslant 0$, where $\tau\geqslant 0$ is a constant.

Incorporating delays in neural network models is important both in theory and applications. In Ref. [4], Hopfield realized that in hardware implementations, time delays occur due to the finite switching speed of the amplifiers. It is known that delays can affect the dynamic behavior of neural networks [5,6]. Delayed neural networks have found many applications in the processing of moving images, image compression [7], etc. In recent years, neural networks with delays have been widely studied, see for example, [5–20]. Today, delays have been widely accepted as important parameters associated with neural network models.

We will address three important properties of the network (1): nondivergence, global attractivity, and complete stability. These are important dynamic properties of a neural network, necessary for effective applications. We are interested to derive nondivergence conditions that will allow multistability, which is an essential property for certain neural computations [21,22]. Global attractivity is very useful for determining the final behavior of network's trajectories. We will give explicit expressions for calculating the global attractive compact sets of the networks. Complete stability describes a kind of convergence characteristics of networks. A completely stable network may also possess the multistability property. We will prove the complete stability of the networks by assuming the delays are in L^2 space.

This paper is organized as follows. Preliminaries will be given in Sec. II. Nondivergence and global attractivity will be studied in Sec. III In Sec. IV, we will study the complete stability of the networks. Examples will be given in Sec. V, and finally, conclusions will be drawn in Sec. VI

II. PRELIMINARIES

In this section, we will give preliminaries for analysis of the network (1).

Definition 1. The network (1) is said to be bounded if each of its trajectories is bounded.

Definition 2. Let S be a compact subset of R^n. We denote the ϵ *neighborhood* of S by S_ϵ. The compact set S is said to globally attract the network (1), if for any $\epsilon>0$, all trajectories of Eq. (1) ultimately enter and remain in S_ϵ.

A bounded network does not imply it will have global attractive sets. For example, consider a simple system $\dot{x}(t)=0$ for $t\geqslant 0$. Clearly, it is bounded but there does not exist any compact set to globally attract its trajectories. However, if a network possesses global attractive sets, then it must be bounded.

Denote

$$R_+^n=\{x|x\in R^n,x_i\geqslant 0 \quad (i=1,\ldots,n)\}.$$

Definition 3. A vector $x^*=(x_1^*,\ldots,x_n^*)^T\in R_+^n$ is called an equilibrium point of Eq. (1) in R_+^n, if

$$x_i^*\left[h_i-x_i^*+\sum_{j=1}^n (a_{ij}+b_{ij})x_j^*\right]\equiv 0 \quad (i=1,\ldots,n).$$

In this paper, we are interested in the equilibrium points which are located in R_+^n.

Definition 4. The network (1) is said to be completely stable, if each of its trajectories converges to an equilibrium point.

We assume the initial condition as

$$\begin{aligned} &x_i(t)=\phi_i(t)\geqslant 0, \quad t\in[-\tau,0],\\ &\phi_i(0)>0 \quad (i=1,\ldots,n), \end{aligned} \tag{2}$$

where each ϕ_i is a continuous function defined on $[-\tau,0]$.

Lemma 1. Each solution $x(t)$ of Eq. (1) with the initial condition (2) satisfies

$$x_i(t)>0 \quad (i=1,\ldots,n)$$

for all $t\geqslant 0$.

Proof. Denote

$$r_i(t)=h_i-x_i(t)+\sum_{j=1}^n a_{ij}x_j(t)+\sum_{j=1}^n b_{ij}x_j[t-\tau_{ij}(t)]$$

for $t\geqslant 0$ and $(i=1,\ldots,n)$. Then, from Eq. (1), we have

$$x_i(t)=\phi_i(0)\exp\int_0^t r_i(s)ds>0 \quad (i=1,\ldots,n)$$

for $t\geqslant 0$. This completes the proof.

Lemma 1 shows an interesting property; if the initial condition satisfies Eq. (2), then the corresponding trajectory stays in positive domain of R^n. This property will allow us, in the following section, to use local inhibitions to guarantee nondivergence of the network (1).

Throughout this paper, for any constant $c\in R$, we denote

$$c^+=\max(0,c).$$

A continuous function $f(t)$ defined on $[0,+\infty)$ is said to be in L^2 space, if

$$\int_0^{+\infty} f^2(s)ds<+\infty.$$

III. NONDIVERGENCE AND GLOBAL ATTRACTIVITY

The trajectories of the Lotka-Volterra neural network (1) may diverge. For example, consider the simple one-dimensional network

$$\dot{x}(t)=x(t)[1+2x(t)]$$

for $t\geqslant 0$. Given any $x(0)>0$, it follows that

$$\frac{x(t)}{1+2x(t)}=\frac{x(0)}{1+2x(0)}e^t$$

for $t\geqslant 0$. Taking $x(0)=1$ in particular, it is easy to see that $x(t)$ diverges at $t=\ln(3/2)$.

Nondivergence is a basic enabling property for recurrent neural networks in practical applications. It is necessary to derive conditions for nondivergence of the network (1). The classical method to obtain conditions of nondivergence is to restrict the weights to be sufficiently small, see for example [20,23]. However, these conditions imply global convergence, that is all the trajectories of a network converge to an equilibrium point. Thus, the network is actually monostable. In Ref. [21], it is pointed out that monostable networks are computationally restrictive: they cannot deal with important neural computations such as those governing the decision making process. There are many applications in which a multistability property is a necessary condition. In Refs. [21,22], multistability for neural networks with unsaturating piecewise linear transfer functions are studied. The nondivergence conditions that we will present in the following allow for such multistable behavior. This will be achieved by applying local inhibition to the network's weights.

Theorem 1. If there exist constants $\gamma_i>0 (i=1,\ldots,n)$ such that

$$\beta_i\overset{\Delta}{=}1-a_{ii}-\frac{1}{\gamma_i}\sum_{j=1}^n \gamma_j[a_{ij}^+(1-\delta_{ij})+b_{ij}^+]>0 \quad (i=1,\ldots,n),$$

where

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j,\end{cases}$$

then the network (1) is bounded. Moreover, the compact set

$$S=\left\{x\left|0\leqslant x_i\leqslant \gamma_i \max_{1\leqslant j\leqslant n}\left\{\frac{h_j}{\beta_j},0\right\} \quad (i=1,\ldots,n)\right.\right\}$$

globally attracts the network (1).

Proof. We will first show that the network (1) is bounded. Clearly,

$$x_i(t)\begin{cases}\geqslant 0, & -\tau\leqslant t<0\\ >0, & t\geqslant 0\end{cases} \quad (i=1,\cdots,n).$$

Then, from Eq. (1), we have

$$\dot{x}_i(t)\leq x_i(t)\left[h_i-(1-a_{ii})x_i(t)+\sum_{j=1}^{n}\{a_{ij}^{+}(1-\delta_{ij})x_j(t)+b_{ij}^{+}x_j[t-\tau_{ij}(t)]\}\right] \tag{3}$$

for $t\geq 0$ and $(i=1,\ldots,n)$.

Define

$$z_i(t)=\frac{x_i(t)}{\gamma_i}\quad (i=1,\ldots,n),$$

for $t\geq -\tau$. Obviously,

$$z_i(t)\begin{cases}\geq 0, & -\tau\leq t<0\\ >0, & t\geq 0\end{cases}\qquad (i=1,\ldots,n).$$

Moreover, from Eq. (3), we have

$$\dot{z}_i(t)\leq x_i(t)\left[h_i-(1-a_{ii})z_i(t)+\frac{1}{\gamma_i}\sum_{j=1}^{n}\gamma_j\{a_{ij}^{+}(1-\delta_{ij})z_j(t)+b_{ij}^{+}z_j[t-\tau_{ij}(t)]\}\right] \tag{4}$$

for $t\geq 0$.

Denote

$$|\phi|=\max_{1\leq i\leq n}[\sup_{-\tau\leq s\leq 0}\phi_i(s)],$$

and

$$\Pi=\max_{1\leq i\leq n}\left\{\frac{1}{\gamma_i}\right\}\max_{1\leq i\leq n}\left\{\left|\phi\right|+1,\frac{h_i}{\beta_i}\right\}.$$

We will prove that

$$z_i(t)<\Pi\quad (i=1,\ldots,n), \tag{5}$$

for all $t\geq 0$. Otherwise, if Eq. (5) is not true, since $z_i(t)\leq|\phi|/\gamma_i<\Pi$, $(i=1,\ldots,n)$ for $t\in[-\tau,0]$, there must exist a $t_1>0$ such that $\dot{z}_i(t_1)\geq 0$ and

$$z_i(t_1)=\Pi;\quad z_j(t)\begin{cases}<\Pi, & -\tau\leq t<t_1, & i=j\\ \leq\Pi, & -\tau\leq t\leq t_1, & i\neq j.\end{cases}$$

However, from Eq. (4), we have

$$\begin{aligned}\dot{z}_i(t_1)&\leq x_i(t_1)\left[h_i-(1-a_{ij})\Pi+\frac{1}{\gamma_i}\times\sum_{j=1}^{n}\gamma_j[a_{ij}^{+}(1-\delta_{ij})\Pi+b_{ij}^{+}\Pi]\right]\\&=x_i(t_1)(h_i-\beta_i\Pi)\\&<0.\end{aligned}$$

This poses a contradiction and it implies that Eq. (5) is true. Thus, the network (1) is bounded.

Next, we will prove the global attractivity of the network. Denote

$$m=\max_{1\leq i\leq n}\left\{\frac{h_i}{\beta_i},0\right\}.$$

Given any $\epsilon>0$, clearly, we can choose a constant $\eta>0$ such that

$$\frac{1-a_{ii}-\beta_i}{\beta_i}\eta\leq\frac{\epsilon}{2}\quad (i=1,\ldots,n). \tag{6}$$

Let N be the first non-negative integer such that $m+\epsilon+N\eta\geq\Pi$.

Define

$$t_k=kT^*,\quad k=1,2,\ldots,N,$$

where

$$T^*=\tau+\frac{2}{\beta\gamma\epsilon}\ln\left(\frac{2\Pi}{m+\epsilon}\right),\quad \beta=\min_{1\leq i\leq n}\{\beta_i\},\quad \gamma=\min_{1\leq i\leq n}\{\gamma_i\}.$$

We will prove that

$$z_i(t)\leq m+\epsilon+(N-k)\eta,\quad t\geq t_k, \tag{7}$$

for $k=0,1,2,\ldots,N$ by mathematical induction.

By Eq. (5), clearly, Eq. (7) holds for $k=0$. Suppose Eq. (7) holds for some $k(0\leq k<N)$, i.e.,

$$z_i(t)\leq m+\epsilon+(N-k)\eta\quad (i=1,\ldots,n) \tag{8}$$

for $t\geq t_k$. We will prove

$$z_i(t)\leq m+\epsilon+(N-k-1)\eta\quad (i=1,\ldots,n) \tag{9}$$

for $t\geq t_{k+1}$.

We will use two steps to prove Eq. (9). In the first step, let us prove that there exists a $\overline{t}\in[t_k+\tau,t_{k+1}]$ such that

$$z_i(\overline{t})\leq m+\epsilon+(N-k-1)\eta\quad (i=1,\ldots,n). \tag{10}$$

Suppose Eq. (10) is not true, then there must exist some i such that

$$z_i(t)>m+\epsilon+(N-k-1)\eta>m+\epsilon, \tag{11}$$

for all $t\in[t_k+\tau,t_{k+1}]$. From Eq. (8) and (11), we have

$$\sup_{t-\tau\leq\theta\leq t}z_j(\theta)\leq m+\epsilon+(N-k)\eta<z_i(t)+\eta\quad (j=1,\ldots,n) \tag{12}$$

for $t\in[t_k+\tau,t_{k+1}]$. Then, for $t\in[t_k+\tau,t_{k+1}]$, we have from Eqs. (4), (12), (11), and (6) that

$$\begin{aligned}\frac{d\ln z_i(t)}{dt} &\leqslant \gamma_i\left[h_i-(1-a_{ii})z_i(t)+\sum_{j=1}^n\{a_{ij}^+(1-\delta_{ij})z_j(t)+b_{ij}^+z_j[t-\tau_{ij}(t)]\}\right]\\ &\leqslant \gamma_i\left[h_i-(1-a_{ii})z_i(t)+\sum_{j=1}^n[a_{ij}^+(1-\delta_{ij})+b_{ij}^+][z_i(t)+\eta]\right]\\ &\leqslant \gamma_i\{h_i-(1-a_{ii})z_i(t)+(1-a_{ii}-\beta_i)[z_i(t)+\eta]\}\leqslant\gamma_i[h_i-\beta_i z_i(t)+(1-a_{ii}-\beta_i)\eta]\\ &\leqslant \gamma_i\beta_i\left[\frac{h_i}{\beta_i}-(m+\epsilon)+\frac{1-a_{ii}-\beta_i}{\beta_i}\eta\right]\\ &\leqslant \gamma_i\beta_i\left[\frac{h_i}{\beta_i}-(m+\epsilon)+\frac{\epsilon}{2}\right]\\ &\leqslant -\frac{\epsilon\beta_i\gamma_i}{2}\\ &\leqslant -\frac{\epsilon\beta\gamma}{2}.\end{aligned}$$

Thus,

$$\ln z_i(t_{k+1})-\ln z_i(t_k+\tau)\leqslant-\frac{\epsilon\beta\gamma}{2}(T^*-\tau),$$

and so

$$\begin{aligned}z_i(t_{k+1}) &\leqslant z_i(t_k+\tau)e^{-(\epsilon\beta\gamma/2)(T^*-\tau)}\\ &\leqslant \Pi\exp\left[-\ln\left(\frac{2\Pi}{m+\epsilon}\right)\right]\\ &= \frac{m+\epsilon}{2}.\end{aligned}$$

This contradicts Eq. (11). Therefore, Eq. (10) is true.

In the second step, we will prove that

$$z_i(t)\leqslant m+\epsilon+(N-k-1)\eta \quad (i=1,\ldots,n) \tag{13}$$

for all $t\geqslant\overline{t}$. Otherwise, there must exist a $\hat{t}>\overline{t}$ and some i such that

$$z_i(\hat{t})>m+\epsilon+(N-k-1)\eta, \quad \dot{z}_i(\hat{t})>0. \tag{14}$$

From Eq. (8) and (14),

$$z_j(\theta)\leqslant m+\epsilon+(N-k)\eta<z_i(\hat{t})+\eta \quad (j=1,\ldots,n) \tag{15}$$

for $\theta\geqslant t_k$. Thus, from Eqs. (4), (15), and (6), it follows that

$$\begin{aligned}\dot{z}_i(\hat{t}) &\leqslant x_i(\hat{t})\left[h_i-(1-a_{ii})z_i(\hat{t})+\sum_{j=1}^n\{a_{ij}^+(1-\delta_{ij})z_j(\hat{t})+b_{ij}^+z_j[\hat{t}-\tau_{ij}(\hat{t})]\}\right]\\ &\leqslant x_i(\hat{t})\left[h_i-(1-a_{ii})z_i(\hat{t})+\sum_{j=1}^n[a_{ij}^+(1-\delta_{ij})+b_{ij}^+][z_i(\hat{t})+\eta]\right]\\ &\leqslant x_i(\hat{t})[h_i-(1-a_{ii})z_i(\hat{t})+(1-a_{ii}-\beta_i)[z_i(\hat{t})+\eta]]\\ &\leqslant x_i(\hat{t})[h_i-\beta_i z_i(\hat{t})+(1-a_{ii}-\beta_i)\eta]=x_i(\hat{t})\beta_i\left[\frac{h_i}{\beta_i}-(m+\epsilon)+\frac{1-a_{ii}-\beta_i}{\beta_i}\eta\right]\\ &\leqslant -x_i(\hat{t})\frac{\epsilon\beta_i}{2}\\ &<0,\end{aligned}$$

which contradicts Eq. (14). This proves that Eq. (13) is true.

Since $t_{k+1} \geqslant \overline{t}$, then using Eq. (13), it follows that Eq. (9) holds. By mathematical induction, Eq. (7) must be true. This completes the proof of Eq. (7).

Taking $k=N$ in Eq. (7), we have

$$z_i(t) \leqslant m+\epsilon, \quad (i=1,\ldots,n)$$

for all $t \geqslant t_N$. Thus, it follows that

$$x_i(t) \leqslant \gamma_i \cdot (m+\epsilon) \quad (i=1,\ldots,n)$$

for all $t \geqslant 0$. This shows that the set S globally attracts the trajectories of Eq. (1). This completes the proof.

The conditions for nondivergence given in Theorem 1 use the local inhibition of the network (1), i.e., we only use part of the weights of the network to achieve nondivergence. Due to this fact, the network is allowed to have the property of multistability. Examples in Sec. V will further confirm this point. It is interesting that following Theorem 1, the global attractive compact set can be calculated explicitly.

If we let $\gamma_i = 1 (i=1,\ldots,n)$ in Theorem 1, then it gives an especially simple results.

Corollary 1. If

$$\overline{\beta}_i \overset{\Delta}{=} 1 - a_{ii} - \sum_{j=1}^{n} [a_{ij}^{+}(1-\delta_{ij}) + b_{ij}^{+}] > 0 \quad (i=1,\ldots,n),$$

where

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j, \end{cases}$$

then the network (1) is bounded. Moreover, the compact set

$$S = \left\{ x \,\middle|\, 0 \leqslant x_i \leqslant \max_{1 \leqslant j \leqslant n} \left\{ \frac{h_j}{\overline{\beta}_j}, 0 \right\} \quad (i=1,\ldots,n) \right\},$$

globally attracts the network (1).

IV. COMPLETE STABILITY

In certain applications, a network which possess the nondivergence property is not sufficient. More desirable dynamic properties are required to enable the network to be effectively used. Convergence is one of the most important properties of a recurrent neural network towards applications. In this section, we analyze the complete stability of the network (1). Complete stability requires that every trajectory of a network converges to an equilibrium point. This property guarantees a network to work well without exhibiting any oscillations or chaotic behavior. In addition, a complete stable network may possess multistability property. Stable and unstable equilibrium points may coexist in a completely stable network. This property has its important applications in certain networks [22]. Complete stability analysis for other models of neural networks could be found in [4,10,17,24].

Theorem 2. Suppose the network (1) is bounded. If there exists a diagonal matrix D with positive elements such that the matrix $D(A+B)$ is symmetric, and

$$\tau_{ij}(t) \in L^2 \quad (i,j=1,\ldots,n), \tag{16}$$

then the network (1) is completely stable.

Proof. Since the network is bounded, there exists a constant $c>0$ such that

$$0 < x_i(t) \leqslant c \quad (i=1,\ldots,n)$$

for $t \geqslant 0$. From Eq. (1), clearly, $\dot{x}_i(t)(i=1,\ldots,n)$ are bounded. Then, there exists a constant $m>0$ such that

$$|\dot{x}_i(t)| \leqslant m \quad (i=1,\ldots,n) \tag{17}$$

for $t \geqslant 0$.

Constructing an energy function

$$E(t) = -H^T D x(t) + \frac{1}{2} x^T(t) D (I-A-B) x(t) \tag{18}$$

for $t \geqslant 0$, where I is the $n \times n$ identity matrix, and $H = (h_1,\ldots,h_n)^T$. Suppose $D = \mathrm{diag}(d_1,\ldots,d_n)$, clearly, $d_i > 0 (i=1,\cdots,n)$. Using the fact that $D(A+B)$ is symmetric, from Eqs. (18) and (1), we have

$$
\begin{aligned}
\dot{E}(t) &= -\sum_{i=1}^{n} d_i\left(h_i - x_i(t) + \sum_{j=1}^{n}(a_{ij}+b_{ij})x_j(t)\right)\dot{x}_i(t) \\
&= -\sum_{i=1}^{n} d_i\left(h_i - x_i(t) + \sum_{j=1}^{n}\{a_{ij}x_j(t) + b_{ij}x_j[t-\tau_{ij}(t)]\}\right)\dot{x}_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} d_i b_{ij}\left(\int_t^{t-\tau_{ij}(t)}\dot{x}_i(s)ds\right)\dot{x}_i(t) \\
&= -\sum_{i=1}^{n} d_i \frac{\dot{x}_i^2(t)}{x_i(t)} + \sum_{i=1}^{n}\sum_{j=1}^{n} d_i b_{ij}\left(\int_t^{t-\tau_{ij}(t)}\dot{x}_i(s)ds\right)\dot{x}_i(t) \\
&\leqslant -\sum_{i=1}^{n} d_i \frac{\dot{x}_i^2(t)}{x_i(t)} + \sum_{i=1}^{n} d_i \sum_{j=1}^{n}|b_{ij}|\left(\int_{t-\tau_{ij}(t)}^{t}\left|\dot{x}_i(s)\right|ds\right)|\dot{x}_i(t)| \\
&\leqslant -\sum_{i=1}^{n}\frac{d_i}{c}\dot{x}_i^2(t) + \sum_{i=1}^{n} d_i \sum_{j=1}^{n}|b_{ij}|m\tau_{ij}(t)|\dot{x}_i(t)| \\
&= -\sum_{i=1}^{n}\frac{d_i}{c}\dot{x}_i^2(t) + \sum_{i=1}^{n} d_i \sum_{j=1}^{n}\left(\frac{b_{ij}^2 m^2 nc}{2}\tau_{ij}^2(t) + \frac{\dot{x}_i^2(t)}{2nc}\right) \\
&= -\sum_{i=1}^{n}\frac{d_i}{2c}\dot{x}_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{d_i nc b_{ij}^2 m^2}{2}\tau_{ij}^2(t) \\
&\leqslant -\delta\|\dot{x}(t)\|^2 + \eta\sum_{i=1}^{n}\sum_{j=1}^{n}\tau_{ij}^2(t)
\end{aligned}
$$

for $t \geqslant 0$, where

$$\delta = \min_{1\leqslant i\leqslant n}\left\{\frac{d_i}{2c}\right\}, \quad \eta = \max_{1\leqslant i,j\leqslant n}\left\{\frac{d_i nc b_{ij}^2 m^2}{2}\right\}.$$

Denote

$$\tilde{E}(t) = E(t) - \eta\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t \tau_{ij}^2(s)ds, \quad t \geqslant 0.$$

Then,

$$\dot{\tilde{E}}(t) \leqslant -\delta\|\dot{x}(t)\|^2 \tag{19}$$

for $t \geqslant 0$. Clearly, $\tilde{E}(t)$ is monotone decreasing. Since $E(t)$ is bounded and following Eq. (16),

$$\int_0^{+\infty}\tau_{ij}^2(s)ds < +\infty \quad (i,j=1,\ldots,n),$$

then, $\tilde{E}(t)$ must be bounded. Thus, there must exist a constant $\tilde{E}_0$ such that

$$\lim_{t\to+\infty}\tilde{E}(t) = \tilde{E}_0.$$

From Eq. (19) we have,

$$\|\dot{x}(t)\|^2 \leqslant -\frac{1}{\delta}\dot{\tilde{E}}(t),$$

for $t \geqslant 0$. Then,

$$
\begin{aligned}
\int_0^{+\infty}\|\dot{x}(s)\|^2 ds &\leqslant \frac{1}{\delta}[\tilde{E}(0) - \lim_{t\to+\infty}\tilde{E}(t)] \\
&= \frac{1}{\delta}[E(0) - \tilde{E}_0] < +\infty.
\end{aligned} \tag{20}
$$

From Eq. (17), $\dot{x}_i(t)$ $(i=1,\ldots,n)$ are bounded, then each $x_i(t)$ is uniformly continuous on $[-\tau,+\infty)$. From Eq. (1), it follows that each $\dot{x}_i(t)$ and then $\|\dot{x}(t)\|^2$ is uniformly continuous on $[0,+\infty)$. Using Eq. (20), it must follow that

$$\lim_{t\to+\infty}\|\dot{x}(t)\|^2 = 0,$$

and so

$$\lim_{t\to+\infty}\dot{x}_i(t) = 0 \quad (i=1,\ldots,n).$$

Clearly, there must exist a constant vector $x^* \in R_+^n$ such that

$$\lim_{t\to+\infty}x(t) = x^*.$$

It is easy to check that x^* is an equilibrium point of Eq. (1) in R_+^n. This completes the proof.

The condition (16) requires that the delays are in the L^2 space. Thus, the delays will decay as time approaches infinity. Decaying of delays in neural networks seems reasonable,

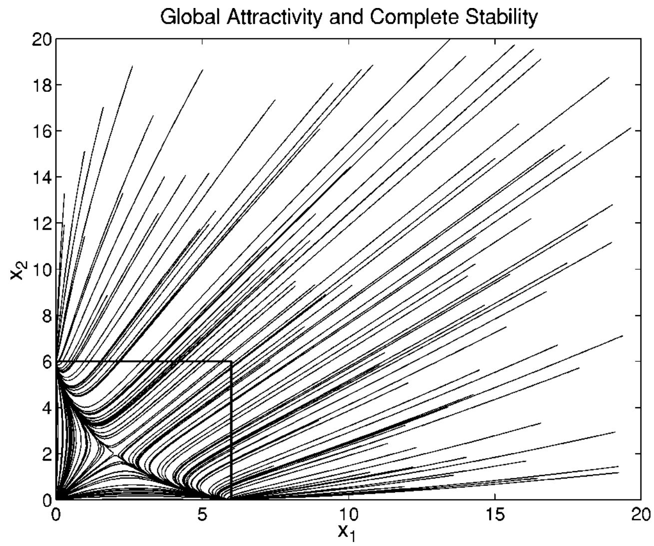

FIG. 1. Global attractivity and complete stability.

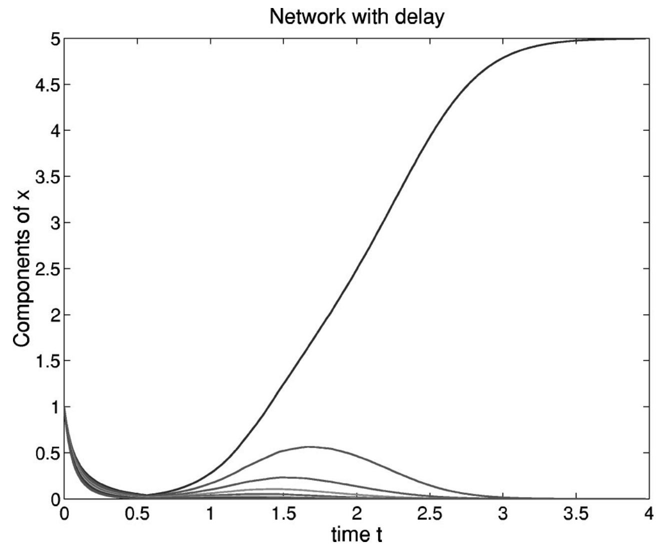

FIG. 2. Convergence of network with delay $\tau(t)=e^{-t}$.

since delays occur usually at the beginning of the operations of networks due to the hardware characteristics such as switching delays, parameter variability, parasitic capacitance, and inductance, etc.

If we let $D=I$ in Theorem 2, we have following simple result.

Corollary 2. Suppose the network (1) is bounded. If $(A+B)$ is a symmetrical matrix, and

$$\tau_{ij}(t)\in L^2, \quad (i,j=1,\ldots,n)$$

then the network (1) is completely stable.

V. EXAMPLES

In this section, we will employ some examples to further illustrate the above theory.

Example 1. Consider the two-dimensional network

$$\dot{x}_1(t)=x_1(t)\{3-x_1(t)+0.5x_1(t)-x_2[t-\tau_1(t)]\},$$

$$\dot{x}_2(t)=x_2(t)\{3-x_2(t)-x_1[t-\tau_2(t)]+0.5x_2(t)\} \quad (21)$$

for $t\geqslant 0$, where $\tau_i(t)$ $(i=1,2)$ are bounded non-negative continuous functions. It is easy to check that the conditions of Theorem 1 are satisfied. Thus, it is bounded. By Theorem 1, it can be calculated that there is a compact set

$$S=\{0\leqslant x_1\leqslant 6; \quad 0\leqslant x_2\leqslant 6\},$$

which globally attracts all the trajectories of the network. Clearly, all the equilibrium points in R_+^2 of the network must be located in the set S. In fact, we can calculate that this network has four equilibrium points in R_+^2. They are (0,0), (6,0), (0,6), and (2,2). Clearly, they are all located in S. If $\tau_i(t)\in L^2$ $(i=1,2)$, say $\tau_i(t)=e^{-t}$ $(i=1,2)$, then the conditions of Theorem 2 are satisfied and the network (21) is completely stable.

Figure 1 shows the simulation results of the global attractivity and complete stability of the network. For the convenience of simulation, we assume $\tau_i(t)\equiv 0$ $(i=1,2)$. The part contained in the square shown in the figure is the set S. It contains all the equilibrium points of the network (21). Among these equilibrium points, (6,0) and (0,6) are stable, while (0,0) and (2,2) are unstable. This example also shows that in a completely stable neural network, stable and unstable equilibrium points can coexist in the network. Existing unstable directions in the state space of a neural networks is essential for certain neural computations [22].

Example 2. Consider the following ten-dimensional Lotka-Volterra neural network with one delay,

$$\dot{x}_i(t)=x_i(t)\left[h_i-x_i(t)-\sum_{j=1,j\neq i}^{n}\{x_j(t)+x_j[t-\tau(t)]\}\right],$$

$$(i=1,\ldots,10), \quad (22)$$

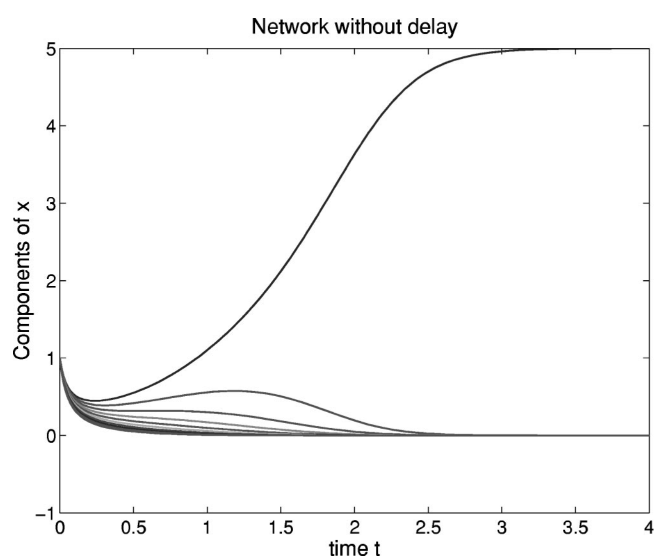

FIG. 3. Convergence of network without delay.

for $t \geqslant 0$, where $h_i = 0.5(11 - i)$ $(i = 1, \ldots, 10)$, and the delay $\tau(t) = e^{-t} \in L^2$. This network satisfies the conditions of Theorem 1 and Theorem 2. It is completely stable. Figure 2 shows the simulation results for the convergence of the trajectory with initial condition $\phi_i = 1$ $(i = 1, \ldots, 10)$ for $t \in [-1, 0]$.

If we let the delay $\tau(t) \equiv 0$ in the network (22), this network can perform as a ''winner-take-all'' function [1–3]. Figure 3 shows the corresponding delay-free simulation result to Fig. 2.

VI. CONCLUSIONS

In this paper, we have studied the dynamic stability of general recurrent Lotka-Volterra neural networks with delays. We have addressed three basic dynamical properties for this class of networks: nondivergence, attractivity, and complete stability. The class of networks, considered in this paper, possess the property that the trajectories will remain in the positive regime if the initial conditions are positive. Using this property together with the development of functional differential equation theory, we are able to establish Theorem 1 that provides conditions for nondivergence and global attractivity based on local inhibition of the networks. Under these nondivergence conditions, the global attractive sets of the networks can be explicitly calculated. The conditions given in Theorem 1 are sufficient conditions. It may be possible and will be interesting to further relax these conditions to be both necessary and sufficient.

In Theorem 2, with some sense of symmetry conditions attached to the networks connection matrix, complete stability of the network can be achieved, by assuming the delays are in the L^2 space. It requires the delays to decay along with the evolution of time. This decaying delay assumption is a reasonable assumption used in certain applications. An interesting area for further exploration is to establish if the network can still retain the complete stability property when the condition that the delays are in L^2 space is removed.

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