Features of the extension of a statistical measure of complexity to continuous systems

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We discuss some aspects of the extension to continuous systems of a statistical measure of complexity introduced by López-Ruiz, Mancini, and Calbet [Phys. Lett. A **209**, 321 (1995)]. In general, the extension of a magnitude from the discrete to the continuous case is not a trivial process and requires some kind of choice. In the present study, several possibilities appear available. One of them is examined in detail. Some interesting properties desirable for any magnitude of complexity are discovered on this particular extension.

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I. INTRODUCTION

In recent years, many *complexity measures* have been proposed as indicators of the complex behavior found in different systems scattered in a broad spectrum of fields. Some of them come from physics such as the effective measure of complexity [1], the thermodynamical depth [2], and the simple measure of complexity [3]. Other attempts arise from the field of computational sciences such as algorithmic complexity [4,5], Lempel-Ziv complexity [6], and ϵ-machine complexity [7]. Other works try to illuminate this question in many other contexts: ecology, genetics, economy, etc., for instance, the complexity of a system based on its diversity [8] and the physical complexity of genomes [9].

Most of these proposals coincide in using concepts such as entropy (in physics) or information (in computational sciences) as a basic ingredient for quantifying the complexity of a phenomenon. There is also a general belief that the notion of complexity in physics must start by considering the perfect crystal and the isolated ideal gas as examples of simple models with zero complexity. Both systems are the extremes in an *entropy/information scale* and, therefore, some fundamental ingredient would be missing if one insisted on describing complexity with only the ordinary information or entropy.

It seems reasonable to adopt some kind of distance to the equipartition, or the disequilibrium of the system, as a new ingredient for defining an indicator of complexity. Going back to the two former examples, it is readily seen that they are extremes in a *disequilibrium scale* and, therefore, disequilibrium cannot be directly associated with complexity.

The recently introduced López-Ruiz–Mancini–Calbet (LMC) statistical measure of complexity [10] identifies the entropy or information stored in a system and its distance to the equilibrium probability distribution (the disequilibrium) as the two ingredients giving the correct asymptotic properties of a well-behaved measure of complexity. In fact, it vanishes for both completely ordered and completely random systems. Besides giving the main features of an intuitive notion of complexity, it has been shown that LMC complexity successfully enables us to discern situations regarded as complex in discrete systems out of equilibrium: one instance of a local transition to chaos via intermittency in the logistic map [10], the dynamical behavior of this quantity in a simplified isolated gas [11], and another example of classical statistical mechanics [12].

A possible formula of LMC complexity for continuous systems was suggested by López-Ruiz *et al.* [10]. Anteneodo and Plastino [13] pointed out some peculiarities concerning such an extension for continuous probability distributions. It is the aim of this work to offer a discussion of the extension of LMC complexity for continuous systems. A slightly modified extension generates interesting and very striking properties, and some of the Anteneodo and Plastino questions are resolved with the proposed definition.

In Sec. II, the extension of information and disequilibrium concepts for the continuous case are discussed. In Sec. III, the LMC measure of complexity is reviewed and possible extensions for continuous systems are suggested. We the present some properties of one of these extensions in Sec. IV. Finally, we establish our conclusions.

II. ENTROPY OR INFORMATION AND DISEQUILIBRIUM

Depending on the necessary conditions to fulfill, the extension of an established formula from the discrete to the continuous case always requires a careful study and, in many situations, some kind of choice between several possibilities. Next, we carry out this process for the entropy and disequilibrium formulas.

A. Entropy or information

Given a discrete probability distribution $\{p_i\}_{i=1,2,\cdots,N}$ satisfying $p_i \geqslant 0$ and $\Sigma_{i=1}^{N} p_i = 1$, the Boltzmann-Gibss-Shannon formula [14] that accounts for the entropy or information H stored in a system is defined by

$$H(\{p_i\}) = -k \sum_{i=1}^{N} p_i \log p_i , \tag{1}$$

where k is a positive constant. Some properties of this quantity are (i) *positivity*: $H \geqslant 0$ for any arbitrary set $\{p_i\}$; (ii) *concavity*: H is concave for arbitrary $\{p_i\}$ and reaches the extremal value for equiprobability ($p_i = 1/N \; \forall i$); (iii) *additivity*: $H(A \cup B) = H(A) + H(B)$, where A and B are two independent systems; and (iv) *continuity*: H is continuous for each of its arguments. Conversely, it has been shown that the only function of $\{p_i\}$ verifying the latter properties is given by Eq. (1) [14,15]. For an isolated system, the *irreversibility* property is also verified, that is, the time derivative of H is positive, $dH/dt \geqslant 0$, reaching equality only for equilibrium.

Calculation of H for a continuous probability distribution $p(x)$, with support on $[-L,L]$ and $\int_{-L}^{L} p(x)dx = 1$, can be performed by dividing the interval $[-L,L]$ into small equal-length pieces—$\Delta x = x_i - x_{i-1}$, $i=1,...,n$, with $x_0 = -L$ and $x_n = L$—and by considering the approximated discrete distribution $\{p_i\} = \{p(\overline{x_i})\Delta x\}$, $i=1,...,n$, with $\overline{x_i}$ a point in the segment $[x_{i-1}, x_i]$. This gives us

$$H^* = H(\{p_i\})$$
$$= -k\sum_{i=1}^{n} p(\overline{x_i})\log p(\overline{x_i})\Delta x - k\sum_{i=1}^{n} p(\overline{x_i})\log(\Delta x)\Delta x. \tag{2}$$

The second summation term of H^* in expression (2) grows as $\log n$ when n goes to infinity. Therefore, it seems reasonable simply to take the first and finite summation term of H^* as the extension of H to the continuous case: $H(p(x))$. It characterizes, with a finite number, the information contained in a continuous distribution $p(x)$. In the limit $n \to \infty$, we obtain

$$H(p(x)) = \lim_{n\to\infty}\left[-k\sum_{i=1}^{n} p(\overline{x_i})\log p(\overline{x_i})\Delta x\right]$$
$$= -k\int_{-L}^{L} p(x)\log p(x)dx. \tag{3}$$

If $p(x) \geqslant 1$ in some regions, the entropy defined by Eq. (3) can become negative. Although this situation is mathematically possible and coherent, it is unfounded from a physical point of view. See [16] for a discussion on this point. Let $f(p,q)$ be a probability distribution in phase space with coordinates (p,q), $f \geqslant 0$, and $dp\ dq$ having the dimension of an action. In this case, the volume element is $dp\ dq/h$, with h the Planck constant. Suppose that $H(f) < 0$. Because $\int (dp\ dq/h) f = 1$, the extent of the region where $f > 1$ must be smaller than h. Hence, a negative classical entropy arises if one tries to localize a particle in phase space in a region smaller than h, that is, if the uncertainty relation is violated. Consequently, not every classical probability distribution can be observed in nature. The condition $H(f) = 0$ could give us the minimal width that is physically allowed for the distribution, hence the maximal localization of the system under study. This *cutting* property has been used in the calculations performed in Ref. [12].

B. Disequilibrium

Given a discrete probability distribution $\{p_i\}_{i=1,2,...,N}$ satisfying $p_i \geqslant 0$ and $\Sigma_{i=1}^{N} p_i = 1$, its disequilibrium D is the quadratic distance of the actual probability distribution $\{p_i\}$ to equiprobability:

$$D(\{p_i\}) = \sum_{i=1}^{N}\left(p_i - \frac{1}{N}\right)^2. \tag{4}$$

D is maximal for fully regular systems and vanishes for completely random ones.

In the continuous case, with support on the interval $[-L,L]$, the rectangular function $p(x) = 1/(2L)$, with $-L < x < L$, is the natural extension of the equiprobability distribution of the discrete case. The disequilibrium could be defined as

$$D^* = \int_{-L}^{L}\left(p(x) - \frac{1}{2L}\right)^2 dx = \int_{-L}^{L} p^2(x)dx - \frac{1}{2L}.$$

If we redefine D, omitting the constant summing term in D^*, the disequilibrium reads

$$D(p(x)) = \int_{-L}^{L} p^2(x)dx. \tag{5}$$

$D > 0$ for every distribution and it is minimal for the rectangular function, which represents the equipartition. D also tends to infinity when the width of $p(x)$ narrows significantly and becomes extremely peaked.

III. STATISTICAL MEASURE OF COMPLEXITY

LMC complexity C has been defined [10] as the interplay between the information H stored in a system and its disequilibrium D. Calculation of C for a discrete distribution $\{p_i\}$, with $p_i \geqslant 0$ and $i=1,...,N$, is given by the formula

$$C(\{p_i\}) = H(\{p_i\})D(\{p_i\})$$
$$= -k\left(\sum_{i=1}^{N} p_i \log p_i\right)\left[\sum_{i=1}^{N}\left(p_i - \frac{1}{N}\right)^2\right]. \tag{6}$$

This definition fits the intuitive arguments and verifies the required asymptotic properties: it vanishes for both completely ordered systems and fully random systems. C has been successfully calculated in different systems out of equilibrium: one instance of a local transition to chaos in a unidimensional mapping [10], the time evolution of C for a simplified model of an isolated gas, the ''tetrahedral'' gas [11], some statistical features of the behavior of LMC complexity for DNA sequences [17], and a modification of C as an effective method by which to identify the complexity in hydrological systems [18].

Feldman and Cruchtfield [19] presented as a main drawback that C vanishes, and that it is not an extensive variable for finite-memory regular Markov chains when the system size increases. This is not the general behavior of C in the thermodynamic limit, as has been suggested by Calbet and López-Ruiz [11]. On the one hand, when $N \to \infty$ and $k = 1/\log N$, LMC complexity is not a trivial function of the entropy in the sense that for a given H there exists a range of complexities between 0 and $C_{\max}(H)$, where $C_{\max}$ is given by

$$[C_{\max}(H)]_{N\to\infty} = H(1-H)^2. \tag{7}$$

Observe that in this case H is normalized, $0<H<1$, because $k=1/\log N$. On the other hand, nonextensivity cannot be considered as an obstacle since today it is well known that there exists a variety of physical systems for which the classical statistical mechanics seems to be inadequate, and for which an alternative nonextensible thermodynamics is being hailed as a possible basis of a theoretical framework appropriate to deal with them [20].

According to the discussion in Sec. II, the expression of C for the case of a continuum number of states x with support on the interval $[-L,L]$ and $\int_{-L}^{L} p(x)\ dx=1$ is defined by

$$C(p(x))=H(p(x))D(p(x))$$
$$=\left(-k\int_{-L}^{L} p(x)\log p(x)dx\right)\left(\int_{-L}^{L} p^2(x)dx\right). \tag{8}$$

Anteneodo and Plastino [13] pointed out that C can become negative. Obviously, $C<0$ implies $H<0$. Although this situation is coherent from a mathematical point of view, it is not physically possible. Hence, a negative entropy means localizing a system in phase space into a region smaller than h (Planck constant). This would imply a violation of the uncertainty principle (see discussion in Sec. II A). Then, a distribution can broaden without any limit, but it cannot become extremely peaked. The condition $H=0$ could indicate the minimal width that $p(x)$ is allowed to have. Similarly to the discrete case, C is positive for any situation and vanishes both for an extreme localization and for the widest delocalization embodied by the equiprobability distribution. Thus, LMC complexity can be straightforwardly calculated for any continuous distribution by Eq. (8). It has been applied, for instance, for quantifying C in a simplified two-level laser model in Ref. [12].

At any rate, the positivity of C for every distribution in the continuous case can be recovered by taking the exponential of H. If we define $\hat{H}=e^H$, we obtain a new expression $\hat{C}$ of the statistical measure of complexity given by

$$\hat{C}(p(x))=\hat{H}(p(x))D(p(x))=e^{H(p(x))}D(p(x)). \tag{9}$$

In addition to the positivity, $\hat{C}$ encloses other interesting properties that we describe in the next section.

IV. PROPERTIES OF $\hat{C}$

The quantity $\hat{C}$, given by Eq. (9), has been presented as one of the possible extensions of the LMC complexity for continuous systems. We now proceed to present some of the properties that characterize such a complexity indicator.

A. Invariance under translations and rescaling transformations

If $p(x)$ is a density function defined on the real axis $\mathbf{R}$, $\int_{\mathbf{R}} p(x)\ dx=1$, and $\alpha>0$ and β are two real numbers, we denote by $p_{\alpha,\beta}(x)$ the new probability distribution obtained by the action of a β translation and an α-rescaling transformation on $p(x)$,

$$p_{\alpha,\beta}(x)=\alpha p[\alpha(x-\beta)]. \tag{10}$$

When $\alpha<1$, $p_{\alpha,\beta}(x)$ broadens, whereas if $\alpha>1$, it becomes more peaked. Observe that $p_{\alpha,\beta}(x)$ is also a density function. After making the change of variable $y=\alpha(x-\beta)$ we obtain

$$\int_{\mathbf{R}} p_{\alpha,\beta}(x)dx=\int_{\mathbf{R}} \alpha p[\alpha(x-\beta)]dx=\int_{\mathbf{R}} p(y)dy=1.$$

The behavior of H under the transformation given by Eq. (10) is the following:

$$H(p_{\alpha,\beta})=-\int_{\mathbf{R}} p_{\alpha,\beta}(x)\log p_{\alpha,\beta}(x)dx$$
$$=-\int_{\mathbf{R}} p(y)\log[\alpha p(y)]dy$$
$$=-\int_{\mathbf{R}} p(y)\log p(y)dy-\log\alpha\int_{\mathbf{R}} p(y)dy$$
$$=H(p)-\log\alpha.$$

Then,

$$\hat{H}(p_{\alpha,\beta})=e^{H(p_{\alpha,\beta})}=\frac{\hat{H}(p)}{\alpha}.$$

It is straightforward to see that $D(p_{\alpha,\beta})=\alpha D(p)$, and to conclude that

$$\hat{C}(p_{\alpha,\beta})=\hat{H}(p_{\alpha,\beta})D(p_{\alpha,\beta})=\frac{\hat{H}(p)}{\alpha}\alpha D(p)=\hat{C}(p). \tag{11}$$

Observe that translations and rescaling transformations also keep the shape of the distributions. Then it could be reasonable to denote the invariant quantity $\hat{C}$ as the *shape complexity* of the family formed by a distribution $p(x)$ and its transformed $p_{\alpha,\beta}(x)$. Hence, for instance, the rectangular $\Pi(x)$, the isosceles-triangle shaped $\Lambda(x)$, the Gaussian $\Gamma(x)$, or the exponential $\Xi(x)$ distributions continue to belong to the same Π, Λ, Γ, or Ξ family, respectively, after applying the transformations defined by Eq. (10). Calculation of $\hat{C}$ on these distribution families gives us

$$\hat{C}(\Pi)=1,$$

$$\hat{C}(\Lambda)=\frac{2}{3}\sqrt{e}\approx 1.0991,$$

$$\hat{C}(\Gamma)=\sqrt{\frac{e}{2}}\approx 1.1658,$$

$$\hat{C}(\Xi)=\frac{e}{2}\approx 1.3591.$$

Note that the family of rectangular distributions has a smaller $\hat{C}$ than the rest of the distributions. This fact is true for every distribution and it will be proved in Sec. IV D.

B. Invariance under replication

Lloyd and Pagels [2] recommend that a complexity measure should remain essentially unchanged under replication. We now show that $\hat{C}$ is a replicant invariant, that is, the shape complexity of m replicas of a given distribution is equal to the shape complexity of the original one.

Suppose that $p(x)$ is a compactly supported density function, $\int_{-\infty}^{\infty}p(x)\ dx=1$. Take n copies $p_m(x)$, $m=1,...,n$, of $p(x)$,

$$p_m(x)=\frac{1}{\sqrt{n}}p[\sqrt{n}(x-\lambda_m)],\quad 1\leqslant m\leqslant n,$$

where the supports of all the $p_m(x)$, centered at $\lambda_m' s$ points, $m=1,...,n$, are all disjoint. Observe that $\int_{-\infty}^{\infty}p_m(x)\ dx=1/n$, which make the union

$$q(x)=\sum_{i=1}^{n} p_m(x)$$

also a normalized probability distribution, $\int_{-\infty}^{\infty}q(x)\ dx=1$. For every $p_m(x)$, a straightforward calculation shows that

$$H(p_m)=\frac{1}{n}H(p)+\frac{1}{n}\log\sqrt{n},$$

$$D(p_m)=\frac{1}{n\sqrt{n}}D(p).$$

Taking into account that the m replicas are supported on disjoint intervals on $\mathbf{R}$, we obtain

$$H(q)=H(p)+\log\sqrt{n},$$

$$D(q)=\frac{1}{\sqrt{n}}D(p).$$

Then,

$$\hat{C}(q)=\hat{C}(p),\tag{12}$$

which completes the proof of the replicant invariance of $\hat{C}$.

C. Near-continuity

Continuity is a desirable property of an indicator of complexity. For a given scale of observation, similar systems should have a similar complexity. In the continuous case, similarity between density functions defined on a common support suggests that they take close values almost everywhere. More strictly speaking, let δ be a positive real number. It will be said that two density functions $f(x)$ and $g(x)$ defined on the interval $I\in\mathbf{R}$ are *δ-neighboring functions* on I if the Lebesgue measure of the points $x\in I$ verifying $|f(x)-g(x)|\geqslant\delta$ is zero. A real map T defined on density functions on I will be called *near-continuous* if for any $\epsilon>0$ there exists $\delta(\epsilon)>0$ such that if $f(x)$ and $g(x)$ are δ-neighboring functions on I then $|T(f)-T(g)|<\epsilon$.

It can be shown that the information H, the disequilibrium D, and the shape complexity $\hat{C}$ are near-continuous maps on the space of density functions defined on a compact support. At this point, we must stress the importance of the compactness condition of the support in order to have near-continuity. Take, for instance, the density function defined on the interval $[-1,L]$,

$$g_{\delta,L}(x)=\begin{cases}1-\delta & \text{if } -1\leqslant x\leqslant 0\\ \dfrac{\delta}{L} & \text{if } 0\leqslant x\leqslant L\\ 0 & \text{otherwise}\end{cases},\tag{13}$$

with $0<\delta<1$ and $L>1$. If we calculate H and D for this distribution we obtain

$$H(g_{\delta,L})=-(1-\delta)\log(1-\delta)-\delta\log\left(\frac{\delta}{L}\right),$$

$$D(g_{\delta,L})=(1-\delta)^2+\frac{\delta^2}{L}.$$

Also consider the rectangular density function

$$\chi_{[-1,0]}(x)=\begin{cases}1 & \text{if } -1\leqslant x\leqslant 0\\ 0 & \text{otherwise}\end{cases}.\tag{14}$$

If $0<\delta<\bar{\delta}<1$, then $g_{\delta,L}(x)$ and $\chi_{[-1,0]}(x)$ are $\bar{\delta}$-neighboring functions. When $\delta\to 0$, we have that $\lim_{\delta\to 0}g_{\delta,L}(x)=\chi_{[-1,0]}(x)$. In this limit process the support is maintained and near-continuity manifests itself as the following,

$$\left[\lim_{\delta\to 0}\hat{C}(g_{\delta,L})\right]=\hat{C}(\chi_{[-1,0]})=1.\tag{15}$$

But if we allow the support L to become infinitely large, the compactness condition is not verified and, although $\lim_{L\to\infty}g_{\delta,L}(x)$ and $\chi_{[-1,0]}(x)$ are $\bar{\delta}$-neighboring distributions, we have that

$$\left[\left(\lim_{L\to\infty}\hat{C}(g_{\delta,L})\right)\to\infty\right]\neq\hat{C}(\chi_{[-1,0]})=1.\tag{16}$$

Then near-continuity in the map $\hat{C}$ is lost due to the noncompactness of the support when $L\to\infty$. This example suggests that the shape complexity $\hat{C}$ is near-continuous on compact supports. This property will be rigorously proved elsewhere.

D. The minimal shape complexity

If we calculate $\hat{C}$ on the example given by Eq. (13), we can verify that the shape complexity can be as large as we want. Take, for instance, $\delta=\frac{1}{2}$. The measure $\hat{C}$ now reads

$$\hat{C}(g_{\delta=1/2,L})=\frac{1}{2}\sqrt{L}\left(1+\frac{1}{L}\right). \tag{17}$$

Thus, $\hat{C}$ becomes infinitely large after taking the limits $L\rightarrow 0$ or $L\rightarrow\infty$. Note that even in the case where $g_{\delta,L}$ has a finite support, $\hat{C}$ has no upper bound. The density functions $g_{(\delta=1/2),(L\rightarrow 0)}$ and $g_{(\delta=1/2),(L\rightarrow\infty)}$ of infinitely increasing complexity have two zones with different probabilities. In the case $L\rightarrow 0$ there is a narrow zone where probability rises to infinity, and in the case $L\rightarrow\infty$ there exists an increasingly large zone where probability tends to zero. Both kinds of density functions show a similar pattern to distributions of maximal LMC complexity in the discrete case, where there is a state of dominating probability, and where the rest of the states have the same probability.

The minimal $\hat{C}$ given by Eq. (17) is found when $L=1$, that is, when $g_{\delta,L}$ becomes the rectangular density function $\chi_{[-1,1]}$. In fact, the value $\hat{C}=1$ is the minimum of the possible shape complexities and it is reached only on the rectangular distributions. We now show some steps that prove this result.

Suppose

$$f=\sum_{k=1}^{n}\lambda_k\chi_{E_k} \tag{18}$$

is a density function consisting of several rectangular pieces E_k, $k=1,...,n$, on disjoint intervals. If μ_k is the Lebesgue measure of E_k, calculation of $\hat{C}$ gives

$$\hat{C}(f)=\prod_{k=1}^{n}(\lambda_k^{-\lambda_k\mu_k})\left(\sum_{k=1}^{n}\lambda_k^2\mu_k\right).$$

The Lagrange multipliers method is used to find the real vector $(\mu_1,\mu_2,\ldots,\mu_n;\lambda_1,\lambda_2,\ldots,\lambda_n)$ that makes extremal the quantity $\hat{C}(f)$ under the condition $\Sigma_{k=1}^{n}\lambda_k\mu_k=1$. This is equivalent to studying the extrema of $\log\hat{C}(f)$. We define the function $z(\lambda_k,\mu_k)=\log\hat{C}(f)+\alpha(\Sigma_{k=1}^{n}\lambda_k\mu_k-1)$, then

$$z(\lambda_k,\mu_k)=-\sum_{k=1}^{n}\mu_k\lambda_k\log\lambda_k+\log\left(\sum_{k=1}^{n}\mu_k\lambda_k^2\right)+\alpha\left(\sum_{k=1}^{n}\lambda_k\mu_k-1\right).$$

Differentiating this expression and making the result equal to zero, we obtain

$$\frac{\partial z(\lambda_k,\mu_k)}{\partial\lambda_k}=-\mu_k\log\lambda_k-\mu_k+\frac{2\lambda_k\mu_k}{\Sigma_{j=1}^{n}\mu_j\lambda_j^2}+\alpha\mu_k=0 \tag{19}$$

$$\frac{\partial z(\lambda_k,\mu_k)}{\partial\mu_k}=-\lambda_k\log\lambda_k+\frac{\lambda_k^2}{\Sigma_{j=1}^{n}\mu_j\lambda_j^2}+\alpha\lambda_k=0. \tag{20}$$

Dividing Eq. (19) by μ_k and Eq. (20) by λ_k, we get

$$\frac{2\lambda_k}{\Sigma_{j=1}^{n}\mu_j\lambda_j^2}+\alpha-1=\log\lambda_k$$

$$\frac{\lambda_k}{\Sigma_{j=1}^{n}\mu_j\lambda_j^2}+\alpha=\log\lambda_k.$$

Solving these two equations for every λ_k, we have

$$\lambda_k=\sum_{j=1}^{n}\mu_j\lambda_j^2 \quad \text{for all } k.$$

Therefore, f is a rectangular function taking the same value λ for every interval E_k, that is, f is the rectangular density function

$$f=\lambda\chi_L \text{ with } \lambda=\frac{1}{\Sigma_{i=1}^{n}\mu_i}=\frac{1}{L},$$

where L is the Lebesgue measure of the support.

As a result, $\hat{C}(f)=1$ is the minimal value for a density function composed of several rectangular pieces because, as we know from the example given by Eq. (17), $\hat{C}(f)$ is not upper bounded for this kind of distribution.

Furthermore, for every compactly supported density function g and for every $\epsilon>0$, it can be shown that near-continuity of $\hat{C}$ allows us to find a δ-neighboring density function f of the type given by expression (18), verifying $|\hat{C}(f)-\hat{C}(g)|<\epsilon$. The arbitrariness of the election of ϵ leads us to conclude that $\hat{C}(g)\geqslant 1$ for every probability distribution g. Thus, we can conclude that the minimal value of $\hat{C}$ is 1, and it is reached only by the rectangular density functions.

V. CONCLUSIONS

Complexity theory of discrete systems has been equipped with a function that not only vanishes for perfectly ordered and disordered systems, but has also been helpful in detecting complexity in patterns produced by a process. Thus, LMC complexity has been shown to be very useful in quantifying complex behavior in local transitions to chaos in discrete mappings [10] and permitting us to advance the concept of *maximum complexity path* in the field of systems far from equilibrium [11]; furthermore, an attempt to quantify complexity in a model of a two-level laser system was performed in Ref. [12].

Another remarkable feature of LMC complexity is the extension $\hat{C}$ to the continuous case. Results found in both the discrete and continuous cases are consistent: extreme values of $\hat{C}$ are observed for distributions characterized by a peak superimposed onto a uniform sea. Other merits of this extension have been studied and explained in the present work.

First, we find that this quantity is invariant under translations and rescaling transformations. $\hat{C}$ does not change if the scale of the system is modified while its shape is maintained. It has been calculated on different families of distributions invariant under those transformations. The result allows us to consider $\hat{C}$ as a parameter that characterizes every family of distributions.

Second, it seems reasonable and intuitive that the complexity of m replicas of a given system should be the same as the original one. We show that $\hat{C}$ embodies this property and that it is invariant under replication.

Third, continuity is not an evident property for such a map $\hat{C}$. Thus, the compactness of the support of the distributions is an important requirement in order to have similar complexity for neighboring distributions. This condition has been strictly established and demonstrated with an example.

Finally, complexity should be minimal when the system has reached equipartition. We demonstrate that the minimum of $\hat{C}$ is found on the rectangular density functions. Its value is $\hat{C}=1$. Moreover, $\hat{C}$ is not an upper bounded function and it can become infinitely large.

We believe and hope that the present discussion on the extension of LMC complexity to the continuous case may trigger some practical future considerations in the area of complex systems theory.

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