Dissipation of kinetic energy in two-dimensional bounded flows

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The role of no-slip boundaries as an enstrophy source in two-dimensional (2D) flows has been investigated for high Reynolds numbers. Numerical simulations of normal and oblique dipole-wall collisions are performed to investigate the dissipation of the kinetic energy $E(t)$, and the evolution of the enstrophy $\Omega(t)$ and the palinstrophy $P(t)$. It is shown for large Reynolds numbers that $dE(t)/dt = -2\Omega(t)/\mathrm{Re} \propto 1/\sqrt{\mathrm{Re}}$ instead of the familiar relation $dE(t)/dt \propto 1/\mathrm{Re}$ as found for 2D unbounded flows.

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Theoretical and numerical studies of two-dimensional (2D) turbulence on unbounded domains have yielded many important results on energy spectra (both for decaying and forced 2D turbulence), on vortex statistics and quasistationary final states of decaying turbulence, on tracer transport in forced 2D turbulence, etc. (for an overview see Refs. [1,2]). Attempts to experimentally confirm the presence of the inverse energy cascade $[E(k) \propto k^{-5/3}]$ and the direct enstrophy cascade $[E(k) \propto k^{-3}]$, the decay properties of the vortex population in decaying turbulence, and the characterization of the quasistationary final states of the flow have been carried out recently by using different experimental setups (e.g., in thin, magnetically forced, fluid layers, in soap films, or in stratified fluids) [3–7]. Some of the experimental data obtained in these investigations revealed the special role of no-slip boundaries, and initiated numerical simulations of 2D turbulence on bounded domains with no-slip walls [8–10] (although the reversed order of events also occurred: the experiments in stratified fluids on circular domains by Maassen *et al.* [7] were initiated by numerical studies of decaying turbulence in a disk with stress-free or no-slip boundaries [11,12]). These simulations have shown that the production of small-scale vorticity in the boundary layers modifies the 1D energy spectra near no-slip boundaries and the evolution of vortex statistics differs considerably from the unbounded case [9,10].

The production of small-scale vorticity in the boundary layers immediately raises the question how the dissipation of kinetic energy of the flow, for the case of flow on (un)bounded domains expressed by the following dimensionless relation

$$\frac{dE(t)}{dt} = -\frac{2}{\mathrm{Re}}\Omega(t), \tag{1}$$

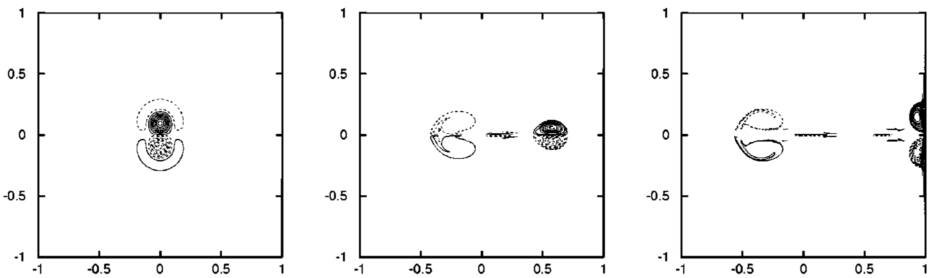
is modified by the presence of no-slip boundaries. It would be tempting to investigate the enstrophy production, and the dissipation of the kinetic energy of the turbulent flow, by performing 2D turbulence simulations on bounded domains with increasing Reynolds numbers. However, this approach will fail due to lack of suitable computer resources. The maximum integral-scale Reynolds number achievable is $\mathrm{Re} = UW/\nu \approx 20\,000$, with U the rms velocity of the flow field, W the half-width of the container, and ν the kinematic viscosity of the fluid. In order to be able to address this fundamental issue, an alternative numerical setup has been applied: the dipole-wall collision experiment as shown in Fig. 1. This setup enables the study of the interaction of two intense vortices with a no-slip wall, yielding a scaling relation for the amount of small-scale vorticity produced near rigid no-slip boundaries.

Two different dipole-wall collision experiments are considered: a normal collision, i.e., the translation of the dipole being perpendicular to the no-slip wall, and a collision with an angle of incidence of 30°. The numerical simulations of the 2D Navier-Stokes equations on a 2D bounded square cavity with size $[-1,1]\times[-1,1]$ were performed with a 2D dealiased Chebyshev pseudospectral method [13], with a maximum of 601 Chebyshev modes in each direction. The integral-scale Reynolds number of the flow is $\mathrm{Re} = UW/\nu$, with U, W, and ν as defined above. This integral-scale Reynolds number is a well-defined number for our simulations, in contrast with Re_d, the Reynolds number based on the characteristic velocity and length scale of the dipole, which can only be estimated after the dipole has been formed (at $t \approx 0.1$). As will be shown later on, for the present runs $\mathrm{Re} \approx \mathrm{Re}_d$. Numerical experiments with the same initial conditions as the normal collision experiment are also carried out for flows with periodic boundary conditions. This enables separation of the dissipation of kinetic energy of the flow due to the vortex-wall interaction and the slow dissipation of the traveling dipole due to diffusion. This latter effect accounts for a small decrease of the kinetic energy of the flow with approximately 1% for $\mathrm{Re} = 20\,000$ (and $\approx 0.1\%$ for $\mathrm{Re} = 160\,000$) during the time needed for the dipole to travel to the wall.

The initial (scalar) vorticity field $\omega = (\partial v/\partial x) - (\partial u/\partial y)$, with u and v the velocity components in the x and y directions, respectively, should vanish at the boundary. This constraint guarantees absence of artificial boundary layers due to enforcing the no-slip condition at $t = 0$. In order to satisfy this constraint, two equally strong, oppositely signed, isolated monopoles are put close to each other near the center of the container. The vorticity distribution of the isolated monopoles is chosen as

$$\omega(r,t=0) = \omega_0[1-(r/r_0)^2]\exp[-(r/r_0)^2], \tag{2}$$

with r the distance from the center of the monopole, r_0 its dimensionless "radius" (at which the vorticity changes sign),



(a) t = 0.00 **(b) t = 0.20** **(c) t = 0.33**

FIG. 1. Vorticity contour plots of the dipole-wall collision simulations with no-slip boundary conditions (Re=40 000). The contour level increment in units of ω_0 is 40. Drawn and dashed contours represent positive and negative vorticity, respectively.

and ω_0 its dimensionless extremum vorticity value (in $r=0$). In present simulations $r_0=0.1$ and $\omega_0\approx\pm 320$. This particular value of ω_0 is determined by the condition that the total kinetic energy of the dipolar flow field,

$$E(t)=\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\mathbf{u}^2(\mathbf{r},t)dxdy, \qquad (3)$$

with $\mathbf{r}=(x,y)$ and $\mathbf{u}=(u,v)$, is normalized to $E(t=0)=2$, for all runs (or, alternatively, $U=1$). As a consequence, both U and W are fixed and increasing the Reynolds number is achieved by decreasing the kinematic viscosity ν only. The initial total enstrophy of the dipolar flow field,

$$\Omega(t)=\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\omega^2(\mathbf{r},t)dxdy, \qquad (4)$$

is $\Omega(t=0)\approx 800$. The initial position of the two isolated monopoles is $\{(x_1,y_1),(x_2,y_2)\}=\{(0,0.1),(0,-0.1)\}$ for the normal collision experiment, and $\{(0.084,0.087),(0.184,-0.087)\}$ for the oblique collision experiment. This particular choice of initial positions yields similar collision times of the dipole with the wall ($t\approx 0.32$) for both sets of numerical experiments. An impression of the flow evolution is presented in Fig. 1 where the vorticity contour plots of a run with Re=40 000 are shown at three instants of time: $t=0$, 0.20, and 0.33. The initial vorticity field (case 1) is shown in Fig. 1(a). After release of the two isolated monopoles at $t=0$ the rings of opposite vorticity, which are clearly visible in Fig. 1(a), are removed due to mutual interaction of the vortices, and form a weak dipolar structure that slowly moves in the negative x direction. This (relatively) weak coherent structure will be ignored in further discussions although its remainings are still visible in Figs. 1(b)–1(c). When the rings of opposite vorticity are removed, the vortex cores move closer together and form a strong dipole that moves with a large velocity in the positive x direction [see Fig. 1(b)]. A snapshot of the collision is shown in Fig. 1(c). The time at which the enstrophy reaches a maximum is defined as the collision time. The dipole-wall collision then takes place at $t\approx 0.32$. After formation of the boundary layers and subsequent detachment a complicated sequence of vortex-wall interactions take place, which will not be discussed here (see Ref. [14] for Re$\leqslant$5000).

An issue so far untouched is the relation between $\mathrm{Re}=UW/\nu$ and $\mathrm{Re}_d=U_dD/\nu$, the Reynolds number based on the dipole translation speed U_d and the diameter D of the dipole half. The dipole shown in Fig. 1(b) can be modeled by a Lamb dipole moving with a constant velocity U_d [15]. The stream function distribution ψ is given by $\psi=[2U_dJ_1(kr)/kJ_1'(kD)]\sin\theta$ for $r<D$ and $\psi=0$ for $r\geqslant D$, and $kD\approx 3.83$. Evaluation of the dimensionless energy and enstrophy yields: $E=\pi(U_dD/UW)^2$ and $\Omega=\pi(kD)^2(U_d/U)^2$ (using W and U as characteristic length and velocity scales). Assuming $E=2$ and $\Omega=800$ we obtain $D/W\approx 0.2$ and $U_d/U\approx 4.2$, which results in: $\mathrm{Re}_d=(U_dD/UW)\mathrm{Re}\approx 0.8\mathrm{Re}$. It is important to note that only an approximate value for Re_d can be found. Hence it is preferable to use the integral-scale Reynolds number Re.

The numerical experiments have been carried out for nine different Reynolds numbers: Re=625, 1250, 2500, 5000, 10 000, 20 000, 40 000, 80 000, and 160 000 (or $\mathrm{Re}_d\approx 500,\ldots,128\,000$). For these runs we have measured several integral quantities during the first, and most intense collision, such as the maximum enstrophy Ω_{max} and the maximum palinstrophy P_{max} (which is a measure of the vorticity gradients in the flow: $P(t)=\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}[\nabla\omega(\mathbf{r},t)]^2dxdy$). We have also computed the difference in dissipation Δ between the no-slip runs and runs with periodic boundary conditions at $t_e=0.5$, i.e., after the first dipole-wall collision. This difference, which is actually the difference in total enstrophy production in both kinds of runs, is defined as

$$\Delta=-\int_0^{t_e}[\Omega_p(\tau)-\Omega_{ns}(\tau)]d\tau, \qquad (5)$$

with the subscripts p and ns referring to the runs with periodic and no-slip boundary conditions, respectively. The computed values for Ω_{max},P_{max}, and Δ are plotted in Figs. 2(a)–2(c). An error margin of these data can be estimated by recomputing the runs shown in Fig. 2 at lower resolution. The error margin of the data obtained for runs with Re $\leqslant$20 000 is small. The estimated error for the runs with Re =40 000, 80 000, and 160 000 is larger and might increase up to 5% for Ω_{max} and 10% for P_{max} when Re=160 000. The accuracy of Δ, which is measured at $t_e=0.5$ (thus after the intense dipole-wall interaction), appears to be much higher and not very sensitive for the resolution of the simulation. Two regimes can be recognized: for Re$\leqslant$20 000 a sharp increase in the enstrophy and the palinstrophy is observed for increasing Reynolds number. In this regime it appears, both for the normal and the oblique angle of incidence, that

$$\Omega_{max}\propto\mathrm{Re}^{0.8}, \quad P_{max}\propto\mathrm{Re}^{2.25}, \quad \text{and} \quad \Delta\propto\mathrm{Re}^{0.8}. \qquad (6)$$

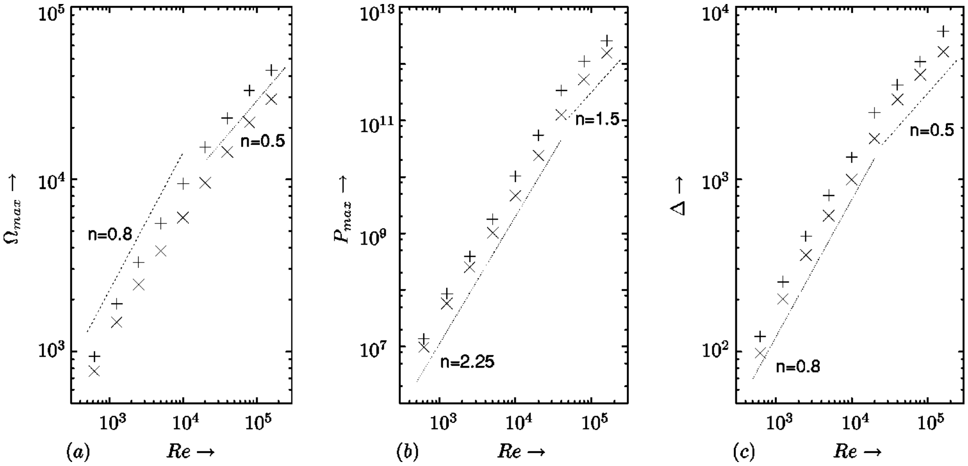


FIG. 2. (a) Ω_{max}, (b) P_{max}, and (c) Δ versus Re. The value of n indicates the scaling behavior of Ω_{max}, P_{max}, and Δ. The data from the normal and oblique dipole-wall collision are denoted by the $+$ and the $\times$, respectively.

For $\mathrm{Re} \geqslant 20\,000$ the rate of increase of Ω_{max}, P_{max}, and Δ with respect to Re slows down and the following relations seem to be valid:

$$\Omega_{max} \propto \mathrm{Re}^{0.5}, \quad P_{max} \propto \mathrm{Re}^{1.5}, \quad \text{and} \quad \Delta \propto \mathrm{Re}^{0.5}. \tag{7}$$

To enable simulations with Re as large as 160 000 (or Re_d up to 128 000), while minimizing the influence of the left, top, and bottom walls on the dipole evolution for the normal collision experiment (see Fig. 1), the ratio $W/D \approx 5$ should be used (increasing the ratio W/D enforces $\mathrm{Re}_d < 128\,000$, which is undesirable). Therefore, the present results can be interpreted as obtained from a dipole collision with an infinite, planar wall. To support this conjecture, numerical experiments have also been carried out in a square cavity with $W/D \approx 10$. Essential for this comparison is that Re_d should be the same as for the analogous run with $W/D \approx 5$, which necessitates a twice as large integral-scale Reynolds number $[\mathrm{Re}_d = (U_d D/UW)\mathrm{Re}$ with U_d/U constant]. The initial vortex positions are then $\{(x_1,y_1),(x_2,y_2)\} = \{(0.5,0.05),(0.5,-0.05)\}, r_0 = 0.05$, and $\omega_0 \approx \pm 640$. Furthermore, $E(t=0) = 0.5$ and $\Omega(t=0) = 800$. These simulations revealed that the scaling behavior of Ω_{max}, P_{max}, and Δ is indeed independent of the box size.

The scaling behavior for $\mathrm{Re} \geqslant 20\,000$ can be understood on basis of a simplified boundary layer theory. Consider the following schematic picture of a snapshot of the dipole-wall collision: a vortex with circulation Γ_v is situated near a no-slip wall where it induced a boundary layer with thickness δ and width D. The circulation in the boundary layer Γ_b is assumed to be independent of Re, but it is not necessary that $\Gamma_b = -\Gamma_v$. Assuming a finite pressure distribution along the boundary it can be shown with the momentum equations that the normal vorticity gradient at the boundary satisfies the scaling $\lim_{\mathrm{Re}\to\infty} \partial\omega/\partial n|_b \propto \mathrm{Re}$, with $\partial/\partial n$ representing the normal derivative with respect to the boundary. The boundary layer thickness scales like $\delta \propto \mathrm{Re}^{-1/2}$. Combination of the large Reynolds number scaling of $\partial\omega/\partial n$ and δ yields for the vorticity ω_b at the boundary: $\omega_b \propto \mathrm{Re}^{1/2}$ (consistent with the alternative estimate $\omega_b \propto \Gamma_b/D\delta$). The enstrophy and palinstrophy of the boundary layer induced by the dipole then scale like $\Omega \propto D\delta\omega_b^2 \propto D\mathrm{Re}^{1/2}$ and $P \propto D\delta(\partial\omega/\partial n|_b)^2 \propto D\mathrm{Re}^{3/2}$. The total dissipation Δ, as defined in Eq. (5), should thus scale as $\Delta \propto \mathrm{Re}^{1/2}$.

The results shown in Fig. 2 for $\mathrm{Re} < 20\,000$ cannot be understood with this simplified analysis. The numerical data reveal that the boundary layer thickness scales approximately with $\mathrm{Re}^{-1/2}$, in line with boundary layer theory, but ω_b and $\partial\omega/\partial n|_b$ show no clear scaling behavior. Computation of the circulation in the boundary layer for $t > 0.25$, thus during the dipole-wall interaction, revealed that Γ_b depends strongly on the Reynolds number for $\mathrm{Re} < 20\,000$ and becomes approximately Reynolds number independent for $\mathrm{Re} \geqslant 20\,000$. In Fig. 3(a) we have plotted the total circulation Γ of the flow in the first quadrant of the domain ($x \geqslant 0$ and $y \geqslant 0$) for a normal collision experiment with $\mathrm{Re} = 40\,000$. The circulation is computed as follows:

$$\Gamma = \int_0^1 v(x=0,y)\,dy + \int_0^1 u(x,y=0)\,dx \tag{8}$$

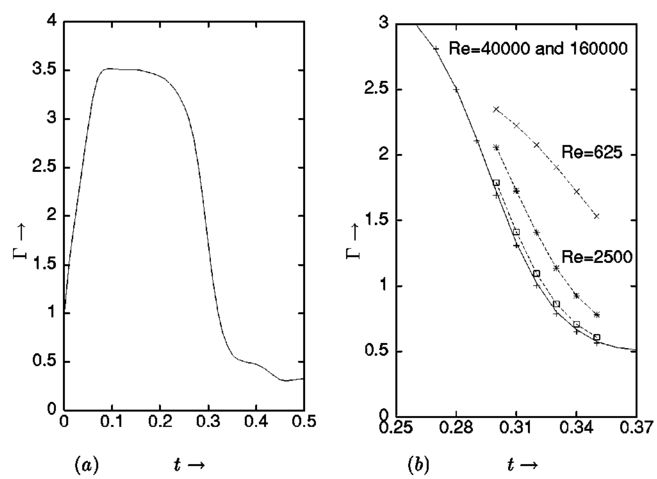


FIG. 3. The total circulation, with respect to the subdomain $(x,y) = [0,1] \times [0,1]$. (a) The normal dipole-wall collision with $\mathrm{Re} = 40\,000$ and (b) a comparison of Γ for $\mathrm{Re} = 625$ ($\times$), $\mathrm{Re} = 2\,500$ (*), $\mathrm{Re} = 10\,000$ ($\square$), and $\mathrm{Re} = 160\,000$ ($+$). The data for $\mathrm{Re} = 40\,000$ are represented by the solid line.

(contributions from the no-slip boundaries are zero). The circulation of a dipole half is $|\Gamma_v|\approx 3.5$, as can be concluded from Fig. 3(a) (the increase of Γ for $t<0.08$ is due to the shedding of the ring of opposite vorticity) and is close to the value of the vortex core as defined in Eq. (2) ($\Gamma_{core}=\pi\omega_0 r_0^2/e\approx 3.7$). At the moment of collision ($t\approx 0.32$)$|\Gamma_b|\approx 2.5$ when Re$\geqslant$20 000 [see Fig. 3(b)]. However, for Re$<$20 000 it is obvious that $|\Gamma_b(t\approx 0.32)|$ varies between 1.3 (Re$=$625)) and 2.4 (Re$=$10 000) and is thus an increasing function of Re. By using the relation $\omega_b \propto \Gamma_b/D\delta$ we can derive the following estimates for the enstrophy and the palinstrophy of the dipole-induced boundary layer:

$$\Omega\propto\frac{\Gamma_b^2}{D\delta},\qquad P\propto\frac{\Gamma_b^2}{D\delta^3}. \tag{9}$$

When Γ_b is an increasing function of Re both the enstrophy and the palinstrophy will increase faster than $\mathrm{Re}^{0.5}$ and $\mathrm{Re}^{1.5}$, respectively.

It would be tempting to estimate the ratio $P/\Omega\propto\delta^{-2}\propto\mathrm{Re}$, as found for 2D flows in a periodic box. For Re $\geqslant$20 000 this yields indeed the expected scaling, but for the other regime (Re$<$20 000) this scaling is obviously absent (see Figs. 2a and b). This is explained by considering the following expression for the enstrophy dissipation rate of 2D flows in a bounded domain:

$$\frac{d\Omega}{dt}=-\frac{2}{\mathrm{Re}}P+\frac{1}{\mathrm{Re}}\int_{\partial D}\omega\frac{\partial\omega}{\partial n}ds, \tag{10}$$

with ds an infinitesimal element of the boundary ∂D and $\partial/\partial m$ the normal outward derivative. For flows in bounded domains, P/Ω does not need to scale with Re.

The production of small-scale vorticity in the boundary layer has important implications. For unbounded flows the enstrophy is bounded by its initial value [16], thus the dissipation of kinetic energy of the flow scales like Re^{-1} [see Eq. 1]. For bounded flows (with no-slip boundaries) the present simulations indicate, by combining Eq. (1) with the observation $\Omega\propto\mathrm{Re}^{1/2}$, that the dissipation of kinetic energy scales like $\mathrm{Re}^{-1/2}$. This process is entirely due to the enstrophy production in the boundary layers. First attempts have shown that during the initial stage of 2D decaying turbulence (with a setup as in Refs. [9,10]) a scaling $dE/dt\propto\mathrm{Re}^{-1/2}$ is found in the range of Reynolds numbers (based on the vortex size and the rms velocity) 500$\leqslant$Re$\leqslant$10 000. This intriguing phenomenon should be explored in more detail in future direct numerical simulation studies of forced 2D turbulence in bounded domains when sufficient computer power is available.

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