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Internal modes of sine-Gordon solitons in the presence of spatiotemporal perturbations

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We investigate the dynamics of the sine-Gordon solitons perturbed by spatiotemporal external forces. We prove the existence of internal (shape) modes of sine-Gordon solitons when they are in the presence of inhomogeneous space-dependent external forces, provided some conditions (for these forces) hold. Additional periodic time-dependent forces can sustain oscillations of the soliton width. We show that, in some cases, the internal mode even can become unstable, causing the soliton to decay into an antisoliton and two solitons. In general, in the presence of spatiotemporal forces the soliton behaves as a deformable (nonrigid) object. A soliton moving in an array of inhomogeneities can also present sustained oscillations of its width. There are very important phenomena (like the soliton-antisoliton collisions) where the existence of internal modes plays a crucial role.

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The sine-Gordon solitons are very important in physics. They possess crucial applications in both particle physics and condensed matter theory. For instance, in solid state physics, they describe domain walls in ferromagnets, dislocations in crystals, charge density waves, fluxons in long Josephson junctions and Josephson transmission lines, etc. [1–6].

In general, nonintegrable soliton equations (e.g., the φ^4 equation and the double sine-Gordon) may possess internal degrees of freedom that are crucial in many phenomena [7–10]. A recent discussion of internal modes of solitary waves can be found in Ref. [11]. However, it is well known that the unperturbed (''pure'') sine-Gordon equation does not have internal modes.

A very remarkable question is the following: *can external forces create internal modes in the sine-Gordon equation?*

Recently there has been a hot debate in the scientific literature about the existence of internal modes of sine-Gordon solitons. Some authors [12–18] have claimed that they have found an internal quasimode described as a long-lived oscillation of the width of the sine-Gordon soliton.

On the other hand, a very recent and interesting paper is contradicting all these reports [19]. By considering the response of the soliton to ac forces and initial distortions, Quintero *et al.* show that neither intrinsic internal modes nor ''quasimodes'' exist in contrast to previous reports. We should stress that they use only time-dependent perturbations in their work.

In the present Rapid Communication we study the sine-Gordon equation perturbed by spatiotemporal external forces:

$$\phi_{tt}+\gamma\phi_t-\phi_{xx}+\sin\phi=F(x,t). \tag{1}$$

We will show that with some spatially inhomogeneous forces, the internal modes can exist for the sine-Gordon equation.

We have shown in previous papers [20–26] that in equations as the following:

$$\phi_{tt}+\gamma\phi_t-\phi_{xx}+\sin\phi=F_1(x), \tag{2}$$

if the force $F_1(x)$ possesses a zero x^* $[F_1(x^*)=0]$, this can be an equilibrium position for the soliton. If there is only one zero, this is a stable equilibrium position for the soliton if $[\partial F_1(x)/\partial x]_{x*}>0$. For the antisoliton, it is stable if $[\partial F_1(x)/\partial x]_{x*}<0$.

Let us suppose that $F_1(x)$ is defined as

$$F_1(x)=2(B^2-1)\sinh(Bx)/\cosh^2(Bx). \tag{3}$$

This is a function with a zero in the point $x^*=0$.

We have chosen this function because of the following properties: (i) the exact solution for the soliton resting on the equilibrium position can be obtained, and (ii) the stability problem of this soliton can be solved exactly. The results obtained with this function can be generalized qualitatively to other systems topologically equivalent to this one. Besides, real physical systems are related to this example [3]. For instance, in a Josephson junction a perturbation that can be described by a function of type $F(x)=dR(x)/dx$, where $R(x)$ is a bell-shaped function, is an Abrikosov vortex lying in the junction's plane perpendicular to its local dimension [27].

A similar function can describe a local deformation of a charge density wave system [28].

Usually, function $R(x)$ is taken as the Dirac's δ function. However, if we wish to model a finite-width inhomogeneity, function (3) is a better choice.

The exact stationary solution of Eq. (2), with $F_1(x)$ as defined in Eq. (3), is $\phi_k = 4\arctan[\exp(Bx)]$.

The stability analysis, which considers small amplitude oscillations around $\phi_k\,[\phi(k,x)=\phi_k(x)+f(x)e^{\lambda t}]$, leads to the eigenvalue problem [20–26]: $\hat{L}f=\Gamma f$, where $\hat{L}=-\partial_x^2+[1-2\cosh^{-2}(Bx)]$ and $\Gamma=-\lambda^2-\gamma\lambda$.

This problem can be solved exactly [29]. The eigenvalues of the discrete spectrum [20–26] are given by the formula

$$\Gamma_n = B^2(\Lambda+2\Lambda n - n^2)-1, \tag{4}$$

where $\Lambda(\Lambda+1)=2/B^2$.

The integer part of Λ, i.e., $[\Lambda]$, yields the number of eigenvalues in the discrete spectrum, which correspond to the soliton modes (this includes the translational mode Γ_0, and the internal or shape modes Γ_n with $n>0$ [20–26]).

All this theoretical investigation produces the following results (note that parameter B will be our control parameter): For $B^2>1$, the translational mode is stable and there are no internal modes. If $\frac{1}{3}<B^2<1$, then the translational mode is unstable. However, still there are no internal modes. When $\frac{1}{6}<B^2<\frac{1}{3}$, apart from the translational mode, there is one internal mode. This internal mode is stable. In the case that $B^2<\frac{1}{6}$ there can appear many other internal modes. The exact number is $[\Lambda]-1$, where $\Lambda(\Lambda+1)=2/B^2$. For $B^2<2/[\Lambda_*(\Lambda_*+1)]$, where $\Lambda_*=(5+\sqrt{17})/2$, the first internal mode becomes unstable.

What happens when we shift the soliton center of mass away from the equilibrium position?

We have the following initial problem:

$$\phi(x,0)=4\arctan\{\exp[B(x-x_0)]\}, \tag{5}$$

$$\phi_t(x,0)=0. \tag{6}$$

In the stable case ($B^2>1$) the center of mass of the soliton will make damped oscillations (for $x_0\neq 0$) around the equilibrium point $x=0$.

In the case that the translational mode is unstable ($\frac{1}{3}<B^2<1$), the soliton will move away indefinitely from the equilibrium position.

Consider the next initial problem:

$$\phi(x,0)=4\arctan[\exp(Bx)]+C\sinh(Bx)\cosh^{-\Lambda}(Bx), \tag{7}$$

$$\phi_t(x,0)=0. \tag{8}$$

In this initial problem the initial soliton is deformed.

For $\frac{1}{6}<B^2<\frac{1}{3}$ we will observe oscillations of the soliton width (see Fig. 1). This is due to the fact that an internal mode has been excited. Eventually, due to unavoidable errors in the initial conditions or to energy exchange between the internal mode and the translational mode, the soliton will move away from the equilibrium position (remember that the equilibrium position is unstable for the soliton center of mass).

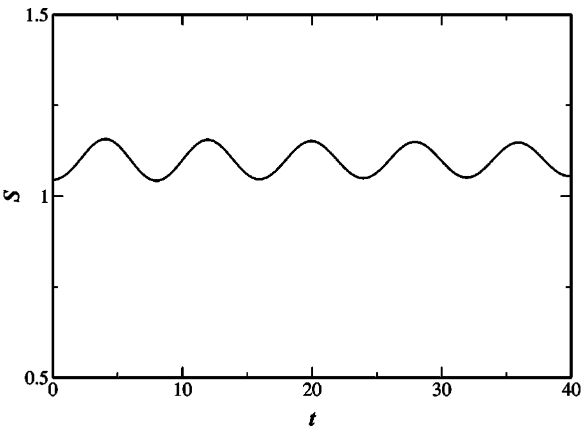

FIG. 1. Soliton's width oscillations when the internal mode can be excited and it is stable, $\frac{1}{6}<B^2<\frac{1}{3}$.

It is important to note that the instability of the translational mode does not mean instability of the soliton structure.

We have to say that the frequency of the oscillations observed in the numerical simulations coincides with the one obtained theoretically using Eq. (4). The frequency of the width oscillations can be obtained using the equations $\omega_1=\sqrt{\Gamma_1}$, where $\Gamma_1=B^2(3\Lambda-1)-1$. All our experiments confirm the prediction about the frequency of the shape oscillations.

The most spectacular phenomenon occurs for $B^2<2/[\Lambda_*(\Lambda_*+1)]$, $\Lambda_*=(5+\sqrt{17})/2$. In this case, the first internal mode is unstable. If we study the evolution of the soliton from the initial conditions (7) and (8) we will observe the destruction of the soliton (see Fig. 2). Two solitons move away (in different directions) to "infinity" (or to the boundaries of the system) and an antisoliton is formed in the place of the original soliton remaining there stabilized. In fact, the condition $B^2<1$ implies stability for the center of mass of an antisoliton.

Note that in these situations, the sine-Gordon solitons do not behave as rigid objects, which is what is expected from them in general [30]. The initial distortions of the width of the soliton will eventually be damped due to dissipation.

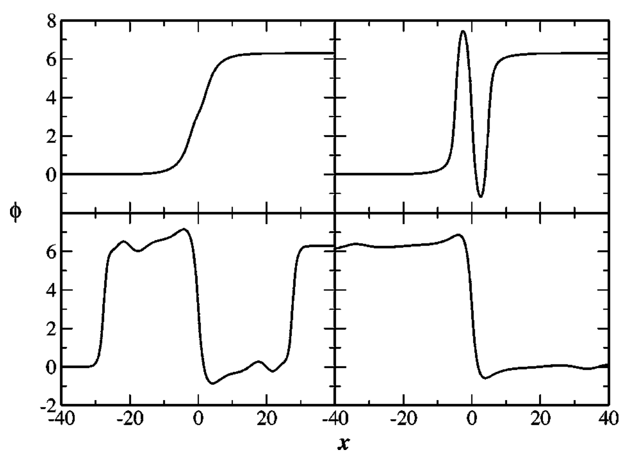

FIG. 2. Soliton's destruction when the internal mode is unstable, $B^2<2/[\Lambda_*(\Lambda_*+1)]$, where $\Lambda_*=(5+\sqrt{17})/2$.

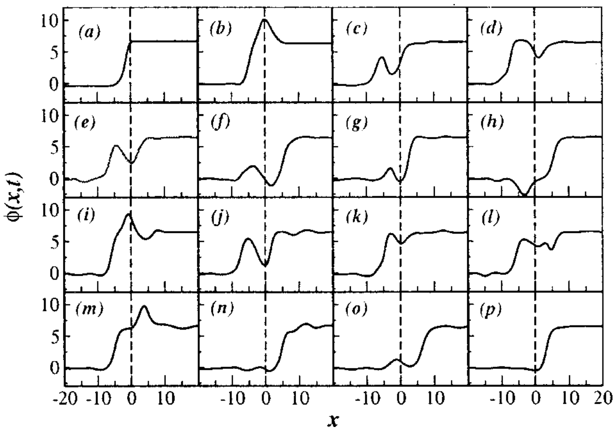

FIG. 3. Soliton profiles for different time instants governed by Eq. (9) ($B=0.5, D=0.2, \gamma=0.1, f_0=0.7, \omega=0.55, E=0.7$).

Once the internal modes are possible, as in Eqs. (2) and (3) with $\frac{1}{6}<B^2<\frac{1}{3}$, we need time-dependent external forces to sustain the oscillations of the soliton width.

On the other hand, if we wish the soliton to remain in some spatially localized zone, we need stable equilibrium positions for the center of mass of the soliton.

Let us consider the following spatiotemporal perturbation:

$$\phi_{tt}+\gamma\phi_t-\phi_{xx}+\sin\phi=F_2(x)+F_3(x,t), \tag{9}$$

where

$$F_2(x)=\begin{cases} F_1(x) & \text{if } -x_1\leqslant x\leqslant x_1, \\ \dfrac{A}{\cosh[B(x+x_1)]}-D & \text{if } x<-x_1, \\ D-\dfrac{A}{\cosh[B(x-x_1)]} & \text{if } x>x_1, \end{cases}$$

and $F_3(x,t)=f_0\cos(\omega t)\{1/\cosh^2[E(x+x_1)]+1/\cosh^2[E(x-x_1)]\}$. The space-dependent force $F_2(x)$ creates a double-well potential for the soliton. At the same time, in the interval $-x_1<x<x_1$, we have the same force $F_1(x)$, which was sufficient for the existence of the internal mode.

Actually, other forces can be used to excite the internal mode. In fact, if we have a function $F(x)$ that can mimic approximately the behavior of function $F_1(x)$ [specially in the interval $-x_1<x<x_1$, where $-x_1$ and x_1 are the extrema of function $F_1(x)$] when B satisfies the condition $\frac{1}{6}<B^2<\frac{1}{3}$, then this function is good for exciting the internal mode. And note that the behavior of function $F_1(x)$ in the interval $-x_1<x<x_1$ is a very common behavior for a function in an interval where there is a zero and two extrema.

The time-dependent force $F_3(x,t)$ will cause the soliton width to oscillate. The center of mass of the soliton will also oscillate, jumping between the potential wells created by force $F_2(x)$.

Although the soliton is not always in the interval $-x_1<x<x_1$, it will return to this interval regularly. While the soliton is in this interval, all the conditions hold for the internal mode to be excited.

Figure 3 shows the extraordinary deformations suffered by the soliton in some cases.

However, we have corroborated that other spatiotemporal forces also can sustain the soliton width oscillations. This includes the "ubiquitous" force $F_3(t)=f_0\cos(\omega t)$.

The propagation of solitons in disordered media has been studied intensively in recent years [26,31].

Consider the equation

$$\phi_{tt}+\gamma\phi_t-\phi_{xx}+\sin\phi=F(x), \tag{10}$$

where $F(x)$ is defined in such a way that it possesses many zeros, maxima, and minima. This system describes an array of inhomogeneities.

For our study, we have defined $F(x)$ in the following way:

$$F(x)=\sum_{n=-q}^{q}4(1-B^2)\frac{e^{B(x+x_n)}-e^{3B(x+x_n)}}{(e^{2B(x+x_n)}+1)^2}, \tag{11}$$

where $x_n=(n+2)\ln(\sqrt{2}+1)/B$ ($n=-q,-q+1,\ldots,q-1,q$), and $q+2$ is the number of extrema points of $F(x)$.

In our array, there is a "superposition" of the "disorder" with a dc component, which will cause the soliton to move to the right all the time. When the soliton is moving over intervals where $dF(x)/dx<0$, the internal mode can be excited. In fact, the points x_i, where $F(x_i)=0$ and $dF(x_i)/dx<0$, are "barriers" that the soliton can overcome due to its kinetic energy. These "collisions" with the barriers will excite the internal modes if in these intervals the function $F(x)$ mimics the behavior of $F_1(x)$ when $\frac{1}{6}<B^2<\frac{1}{3}$. The simulations show that the width of the soliton will perform sustained oscillations during its motion in a disordered medium.

We have shown that in sine-Gordon equations perturbed by inhomogeneous (space-dependent) forces $F(x)$, the solitons can possess internal modes.

Some of our results are in agreement with previous works [19,30]. In fact, in Eq. (2), with $F_1(x)$ as in Eq. (3), if we put $B^2=1$, then there are no external forces, and from Eq. (4) we obtain that there are no internal modes in that case either.

Moreover, even when there is an inhomogeneous external force, not for every $F(x)$ we have internal modes.

For instance, if $F(x)$ has a zero that corresponds to a stable equilibrium position for the soliton, even then the internal modes are impossible. This explains why it has been so difficult to find sine-Gordon internal modes. For the existence of internal modes for sine-Gordon solitons we need zeros of function $F(x)$ that corresponds to unstable positions for the center of mass of the soliton. When the soliton center of mass is very close to an unstable equilibrium position, there is a pair of forces acting in opposite directions on the "body" of the soliton. This pair of forces should be sufficiently large to stretch the soliton "body," such that the soliton internal mode can be excited.

Function $F(x)$ can possess several zeros corresponding to unstable and stable equilibrium positions. For instance, we have studied a force $F(x)$ that creates a double-well potential for the soliton.

Periodic time-dependent forces (besides the spatially inhomogeneous forces) can sustain the oscillations of the soliton width.

A soliton moving in an array of inhomogeneities can also undergo sustained oscillations of its width.

All this is possible because the internal mode of the soliton can exist when it is moving in media where there are inhomogeneous space-dependent forces with unstable equilibrium positions.

Nonetheless, we have discovered another more remarkable phenomenon: The sine-Gordon internal mode not only can exist for some external forces, but (in some situations) it can become unstable. If we have an unstable equilibrium position for the soliton center of mass and the pair of forces acting on the soliton is too large, then the soliton can be destroyed.

When the soliton is destroyed, it can be transformed into an antisoliton and two new solitons. The topological charge is conserved. We had found this phenomenon before for the ϕ^4 equation [21,22]. However, here we have shown not only that the sine-Gordon soliton internal mode can exist, but that it can become unstable and destroy the soliton. This is a spectacular manifestation of the fact that the sine-Gordon soliton can behave as a deformable (nonrigid) object.

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