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Class of self-limiting growth models in the presence of nonlinear diffusion

Sandip Kar, Suman Kumar Banik, and Deb Shankar Ray*
Indian Association for the Cultivation of Science, Jadavpur, Calcutta 700 032, India

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The source term in a reaction-diffusion system, in general, does not involve explicit time dependence. A class of self-limiting growth models dealing with animal and tumor growth and bacterial population in a culture, on the other hand, are described by kinetics with explicit functions of time. We analyze a reaction-diffusion system to study the propagation of spatial front for these models.

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I. INTRODUCTION

Reaction-diffusion systems are ubiquitous in almost all branches of physics [1], chemistry [2], and biology [3–5] dealing with population growth, fluid dynamics, pulse propagation in nerves, chemical reactions, optical, and other processes. The basic equation describes the dynamics of a field variable $n(x,t)$, a function of space and time in terms of a source term (also known as reaction term) and a diffusion term. An important early endeavor in this direction is the study of self-limiting growth models of which the most well known is the Fisher equation [6,7] that takes into account a linear growth and a nonlinear decay. The model and many of its variants have found wide applications both from a theoretical and experimental point of view [4]. A notable feature of these models is that the source or the reaction terms do not involve any explicit time dependence. On the other hand, there are situations [8–10] where the source terms contain explicit functions of time that put a constraint on the growth process in the long time limit. For example, the Gompertz growth [8,9] is a model used for study of growth of animals and tumors, where the growth rate is proportional to the current value, but the proportionality factor decreases exponentially in time so that

$$\frac{dn}{dt}=rn\exp(-\alpha t), \tag{1a}$$

where r and α are positive experimentally determined constants. Similarly another type of model proposed to analyze the growth of bacterial population in culture [10] is described by

$$\frac{dn}{dt}=knt\exp(-\beta t^2). \tag{1b}$$

Again k and β are positive constants required to fit the experimental data. An important feature of these models is that unlike the logistic growth process the asymptotic value of the density function n depends on its initial population.

Keeping in view of these experimental observations it is, therefore, worthwhile to generalize the specific cases in terms of an explicit function of time $\phi(t)$ such that we write

$$\frac{dn}{dt}=rn\phi(t), \tag{2}$$

where r is a constant for the growth process and $\phi(t)$ may be of the type (i) $\phi(t)=1$ for exponential growth, (ii) $\phi(t)=\exp(-\alpha t)$ for Gompertz growth, (iii) $\phi(t)=t\exp(-\beta t^2)$ for bacterial growth, etc.

The object of the present paper is to study a reaction-diffusion system with a reaction term describing a class of self-limiting growth processes (2). Since in many living organisms concentration dependent diffusivity [4,5,11–15] has been found to be essential to the modeling of reaction-diffusion systems we investigate the interplay of this nonlinear diffusion and self-limiting growth process in the dynamics. We show that the model and its variant with a finite memory transport [16–25] admit of exact solutions. The dependence of the rate of spread of the wave front on various parameters is explored.

II. THE REACTION-DIFFUSION SYSTEM

We consider a reaction-diffusion system with a source term describing self-limiting growth and with a nonlinear diffusion term in the following form:

$$\frac{\partial n(x,t)}{\partial t}=rn\phi(t)+\frac{\partial}{\partial x}Dn\frac{\partial n}{\partial x}, \tag{3}$$

where D is the diffusion coefficient. Our primary aim in this section is to provide an exact solution of Eq. (3). To this end we first make use of the following transformation:

$$n(x,t)=\tilde{u}(x,t)\exp\left(r\int_0^t\phi(t')dt'\right) \tag{4}$$

in Eq. (3) to obtain

$$\frac{\partial\tilde{u}(x,t)}{\partial t}=D\exp\left(r\int_0^t\phi(t')dt'\right)\frac{\partial}{\partial x}\left\{\tilde{u}\frac{\partial\tilde{u}}{\partial x}\right\}. \tag{5}$$

We now introduce the scaled time variable τ as

$$\tau=D\int_0^t f(t')dt'\equiv G(t)\quad (say), \tag{6a}$$

where

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*Email address: pcdsr@mahendra.iacs.res.in

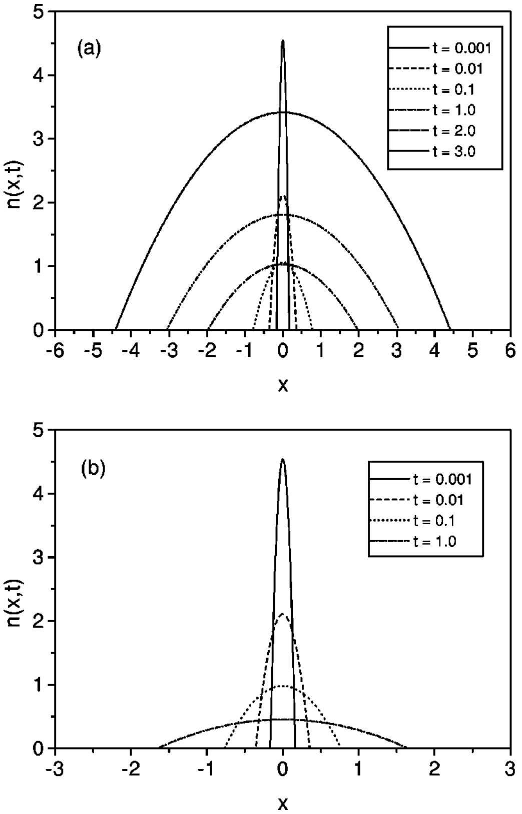

FIG. 1. Evolution of spatial front in time for the model with $\phi(t)=1$. (a) The population $n(x,t)$ is plotted against x for different times using $r=1.0$ and $D=1.0$. (b) The same as in (a) but for $r=0.001$ (arbitrary units).

$$f(t)=\exp\left[r\int_0^t \phi(t')\,dt'\right]. \tag{6b}$$

This reduces Eq. (5) to the following form:

$$\frac{\partial u(x,\tau)}{\partial \tau}=\frac{\partial}{\partial x}\left\{u(x,\tau)\frac{\partial u(x,\tau)}{\partial x}\right\} \tag{7}$$

with $\tilde{u}(x,t)\equiv\tilde{u}[x,G^{-1}(\tau)]=u(x,\tau)$ where time t has been expressed as an inverse function $G^{-1}(\tau)$ according to Eqs. (6a) and (6b).

Equation (7) is the well-known Boltzmann nonlinear diffusion equation [1,26]. Now subject to the initial condition of a unit point source at the origin,

$$n(x,0)=\delta(x)=\tilde{u}(x,0)=u(x,0), \tag{8}$$

we solve Eq. (7) under the following boundary conditions:

$$\lim_{x\to\pm\infty} u(x,\tau)=0, \quad \forall\tau>0 \tag{9}$$

and

$$\int_{-\infty}^{+\infty} u(x,\tau)\,dx=1, \quad \forall\tau>0. \tag{10}$$

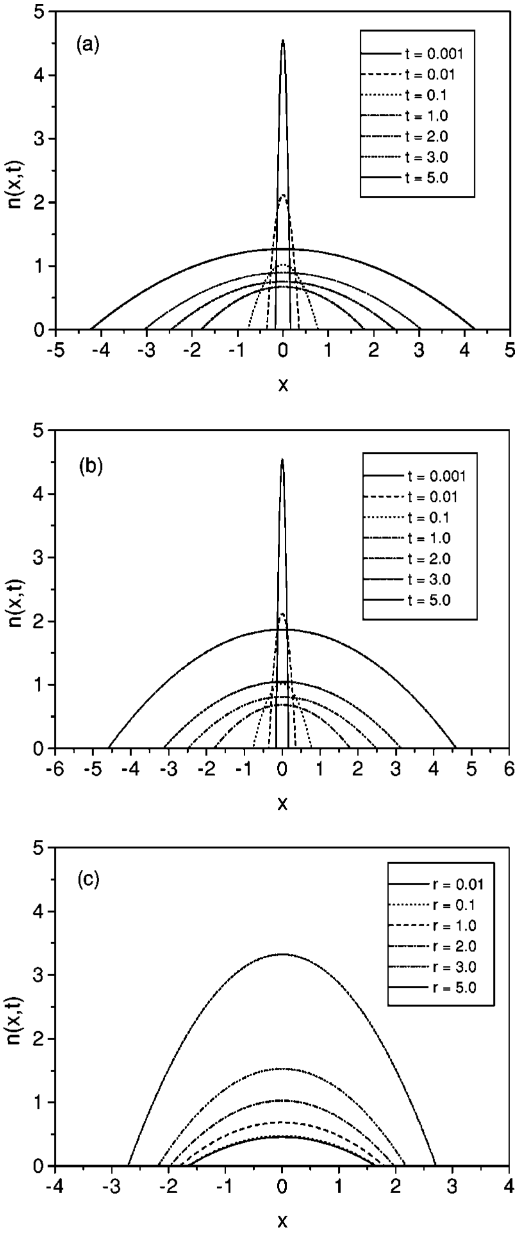

FIG. 2. Evolution of spatial front in time for the model with $\phi(t)=t\exp(-\beta t^2)$. (a) The population $n(x,t)$ is plotted against x for different times using $r=1.0$, $D=1.0$, and $\beta=0.1$. (b) The same as in (a) but for $\beta=0.01$. (c) The population $n(x,t)$ is plotted against x at $t=1.0$ for different r using $D=1.0$ and $\beta=0.01$ (arbitrary units).

Next we seek the similarity solution of the nonlinear diffusion Eq. (7). We make use of the well-known similarity transformation [1,5,26,27]:

$$u=\tau^{-1/3}v(z) \quad \text{and} \quad z=x\tau^{-1/3} \tag{11}$$

in Eq. (7) to obtain

$$3\frac{d}{dz}\left(v\frac{dv}{dz}\right)+v+z\frac{dv}{dz}=0. \tag{12}$$

On integration, Eq. (12) yields

$$3\left(v\frac{dv}{dz}\right)+zv=0. \tag{13}$$

Since we are interested in the symmetric solutions with $v'(0)=0$, we have put the integration constant zero in going from Eq. (12) to Eq. (13). On further integration Eq. (13) results in the solution

$$\begin{aligned} v(z)&=(A^2-z^2)/6, \quad |z|<A \\ &=0, \quad |z|>A, \end{aligned} \tag{14a}$$

where A is a constant that can be determined from the condition (10) to obtain

$$A=(9/2)^{1/3}. \tag{14b}$$

Therefore the solution of Eq. (7) in x and τ is given by

$$\begin{aligned} u(x,\tau)&=\frac{1}{6\tau}[A^2\tau^{2/3}-x^2], \quad |x|<A\tau^{1/3} \\ &=0, \quad |x|>A\tau^{1/3}. \end{aligned} \tag{15}$$

It is interesting to note that by virtue of the relations (6a) and (6b) τ is dependent on r and $\phi(t)$ that control the growth and self-limiting factors, respectively, of the source term. This implies that the shock-wave-like behavior with propagating wave front at $x=x_f=A\tau^{1/3}$ as evident from the similarity solutions (15) critically depends on the reaction terms. Specifically, the wave front propagates in the medium with speed

$$\frac{dx_f}{dt}=\frac{1}{3}\left(\frac{9D}{2}\right)^{1/3}f(t)\left[\int_0^t f(t')\,dt'\right]^{-2/3}, \tag{16}$$

where $f(t)$ is given by Eq. (6a) and in turn depends on the functional form of $\phi(t)$.

We now consider two specific cases to illustrate the spatial propagation patterns.

(i) $\phi(t)=1$. For a constant value of ϕ the model suggests an exponential growth. The relation (6a) in this case can then be utilized to obtain $f(t)=\exp(rt)$ so that $\tau=(D/r)[\exp(rt)-1]$. Putting this expression for τ in the solution (15) we have after using Eq. (4)

$$n(x,t)=\frac{[A^2\{(D/r)[\exp(rt)-1]\}^{2/3}]-x^2}{(6D/r)[\exp(rt)-1]\exp(-rt)}. \tag{17}$$

This solution clearly has a sharp wave front at $x_f=A\tau^{1/3}$, which propagates at a speed

$$\frac{dx_f}{dt}=\frac{1}{3}A(Dr^2)^{1/3}\exp(rt)[\exp(rt)-1]^{-2/3}. \tag{18}$$

To illustrate the spatial propagation of the population $n(x,t)$ in time we plot in Fig. 1 the spatial shock-wave-like patterns for $r=1.0$ and $D=1.0$. It is apparent that the sharp peaked distribution at $t=0$ starts spreading relatively slowly with peak at $x=0$ diminishing with time up to a period $t=0.1$. Beyond this time the spatial growth of population becomes comparatively large and it diverges due to the combined effect of exponential growth and nonlinear diffusion. For a much lower growth rate ($r=0.001$), however, the population spreads monotonically due to the nonlinear diffusion that overwhelms the effect of growth process. This is evident in Fig. 1(b).

(ii) $\phi(t)=t\exp(-\beta t^2)$. With the above expression for $\phi(t)$ for bacterial self-limiting growth we obtain from Eqs. (6a) and (6b)

$$f(t)=\exp\{(-r/2\beta)[\exp(-\beta t)-1]\} \tag{19}$$

and

$$\tau=D\exp(r/2\beta)\int_0^t \exp[(-r/2\beta)\exp(-\beta t')]\,dt'. \tag{20}$$

By defining $z=(r/2\beta)\exp(-\beta t)$ the above expression can be reduced to the following form:

$$\tau=-D\frac{\exp(r/2\beta)}{\beta}\int_{(r/2\beta)}^{(r/2\beta)\exp(-\beta t)}\frac{\exp(-z)}{z}\,dz. \tag{21}$$

The integral in Eq. (21) can be put into a standard form with the help of Ei function [28] so that τ can be expressed as

$$\tau=D\frac{\exp(r/2\beta)}{\beta}[\mathrm{Ei}(-r/2\beta)-\mathrm{Ei}((-r/2\beta)\exp(-\beta t))]. \tag{22}$$

The corresponding density $n(x,t)$ and the speed of the wave front dx_f/dt at x_f are given by

$$n(x,t)=\frac{A^2[D\exp(r/2\beta)/\beta]^{2/3}[\mathrm{Ei}(-r/2\beta)-\mathrm{Ei}(-(r/2\beta)\exp(-\beta t))]^{2/3}-x^2}{(6D/\beta)[\mathrm{Ei}(-r/2\beta)-\mathrm{Ei}((-r/2\beta)\exp(-\beta t))]\exp[(r/2\beta)\exp(-\beta t)]} \tag{23}$$

and

$$\frac{dx_f}{dt}=A\left(D\frac{\exp(r/2\beta)}{\beta}\right)^{1/3}\frac{d}{dt}[\mathrm{Ei}(-r/2\beta)-\mathrm{Ei}((-r/2\beta)\exp(-\beta t))]^{1/3}, \tag{24}$$

respectively.

In Figs. 2(a) and 2(b) we show the shock-wave-like spread of population by plotting $n(x,t)$ vs x for several values of time for $D=1$ and $r=1$. Since β puts a limit to the growth at large time the peak of $n(x,t)$ at $x=0$ as shown in Fig. 2(a) ($\beta=0.1$) does not increase too much as compared to the earlier case considered in Fig. 1(a). It has been observed that for a unique value of $\beta \geqslant 1.0$ there is a monotonic decrease in the peak population $n(x,t)$ at $x=0$. For smaller values of β [Fig. 2(b)] the spread is similar to that in Fig. 1(a). In Fig. 2(c) we exhibit the spatial front propagation for several values of growth rate r at a time $t=1.0$ keeping $D=1$ and $\beta=0.01$. It is apparent that with increase of r the reaction dominates over diffusion so that the peak population at $x=0$ increases compared to spreading.

III. EFFECT OF FINITE MEMORY TRANSPORT

We now generalize the proposed reaction-diffusion model to include the effect of finite memory transport. It has been observed that an animal's movement at a particular instant of time often depends on its motion in the immediate past. This results in a delay in population flux, or a memory in the diffusion coefficient. A number of attempts have been made in the recent literature [16–25] to analyze the delayed population growth in several models and related context in heat conduction and transport processes. To consider a finite memory in the present model we modify the nonlinear diffusion term in Eq. (3) to the following form:

$$\frac{\partial n(x,t)}{\partial t} = r n(x,t)\phi(t) + \frac{\partial}{\partial x}\left[D\gamma \int_0^t \exp[-\gamma(t-\tau)] \times n(x,\tau)\frac{\partial n(x,\tau)}{\partial x}\, d\tau\right], \tag{25}$$

where γ refers to the inverse of relaxation time. The population flux takes into account the relaxation effect due to the delay of the particles in adopting a definite direction of propagation. Differentiating both sides of the above equation with respect to t and using it again we obtain

$$\frac{\partial^2 n}{\partial t^2} = (r\phi - \gamma)\frac{\partial n}{\partial t} + (r\dot{\phi} + r\phi\gamma)n + \frac{\partial}{\partial x}\left[D\gamma n \frac{\partial n}{\partial x}\right]. \tag{26}$$

In the limit of vanishing relaxation time, i.e., $1/\gamma \rightarrow 0$ Eq. (26) reduces to Eq. (3). When memory effects are taken into account, the dispersal of the organisms are not mutually independent. Hence the correlation between the successive movement of the diffusing particles results in a delay in the transport. Thus Eq. (26) is a typical form of a delayed transport equation.

We now consider a specific case $\phi(t)=1$. Substitution of the traveling wave form $N(z)[=n(x,t)]$ with $z=x+ct$ satisfies

$$c^2\frac{\partial^2 N}{\partial z^2} = c(r-\gamma)\frac{\partial N}{\partial z} + r\gamma N + D\gamma \frac{\partial}{\partial z}\left(N\frac{\partial N}{\partial z}\right), \tag{27}$$

where c is the speed of the traveling wave to be determined.

We now consider the trial solution of Eq. (27) of the form $N(z)=N_0 \exp(sz^b)$ subject to the initial condition that at $z=0$, $N=N_0$, where s and b are positive constants to be determined. Substitution of this solution in Eq. (27) yields the following relation:

$$\begin{aligned}&[c^2s^2b^2z^{2(b-1)} + c^2sb(b-1)z^{(b-2)} - csb(r-\gamma)z^{(b-1)} \\ &\quad - r\gamma]\exp(sz^b) - D\gamma N_0 sb[2sbz^{2(b-1)} \\ &\quad + (b-1)z^{(b-2)}]\exp(2sz^b) \equiv L(z) = 0.\end{aligned} \tag{28}$$

For $L(z)=0$, for all z, the coefficients of $\exp(sz^b)$ and $\exp(2sz^b)$ within the square brackets must vanish identically. For this the only acceptable solution for b is $b=1$. We obtain

$$2s^2D\gamma N_0 = 0 \tag{29a}$$

and

$$c^2s^2 - cs(r-\gamma) - r\gamma = 0. \tag{29b}$$

From the above two equations the solution for s is given by

$$s = \frac{c[(1/\gamma)-(1/r)] + \{c^2[(1/\gamma)-(1/r)]^2 + 4/r[(c^2/\gamma)+2DN_0]\}^{1/2}}{2[(c^2/\gamma r)+(2DN_0/r)]}. \tag{30}$$

In the limit of instantaneous relaxation, i.e., $1/\gamma \rightarrow 0$ Eq. (30) yields

$$s = \frac{c\{-1+[1+(DN_0 r/c^2)]^{1/2}\}}{4DN_0}. \tag{31}$$

Furthermore the above expression in the limit of weak diffusion $D\rightarrow 0$ we obtain from Eq. (31) after a Taylor expansion

$$s = \frac{r}{c}. \tag{32}$$

To determine the speed of the propagation of the wave front we now rearrange the solution for s in Eq. (30) to obtain

$$c = \frac{(r-\gamma) + [(r-\gamma)^2 + 4(r\gamma - 2s^2D\gamma N_0)]^{1/2}}{2s}. \tag{33}$$

For real values of c, the quantity inside the square root must be positive, which determines the minimum value of c for $s=r/c$ [Eq. (32)] as

$$c_{min}=\frac{2r^2D\gamma N_0}{(r+\gamma)^2}. \tag{34}$$

Equation (27), therefore, admits of an exact traveling-wave-like solution:

$$N(z)$$

$$=N_0\exp\left[\frac{c(r-\gamma)+[c^2(r-\gamma)^2+4r\gamma(c^2+2D\gamma N_0)]^{1/2}}{2(c^2+2D\gamma N_0)}\right]z. \tag{35}$$

It is interesting to observe that the speed of the traveling wave front not only depends on nonlinear diffusion and growth rate but also on the initial concentration and memory. A comparison of the solutions in this section and in the preceding one shows that Eq. (35) does not reduce to Eq. (17) in the limit of vanishing relaxation time $(1/\gamma\rightarrow 0)$ although Eq. (26) goes over to Eq. (3) under this condition. This is because of the fact that the nature of the partial differential equation changes due to the inclusion of relaxation terms and also the boundary conditions for the shock-wave-like ‘‘diffusing solutions’’ (17) are different for the traveling wave front solution (35). The nature of the two solutions are thus generically different. We point out in passing that thedependence on initial concentration on speed as shown in Eq. (34) is rather an unusual feature in reaction-diffusion system.

IV. CONCLUSIONS

In this paper we have analyzed a class of reaction-diffusion systems in which the kinetic term describes the self-limiting growth processes of the Gompertz type and is an explicit function of time. We have shown that the model can be solved exactly to analyze the spatial front propagation problem. To make the model more realistic we have included the effect of finite relaxation to concentration-dependent diffusive processes. In view of the fact that the source terms have their direct relevance on experimental measurement on animal and tumor growth or bacterial culture we think that the solutions discussed in this paper will be pertinent in the context of reaction-diffusion systems, in general.

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