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Pattern formation of spiral waves in an inhomogeneous medium with small-world connections

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Pattern formation of spiral waves in an inhomogeneous excitable medium with small-world connections is investigated. In both cases, with completely local regular connections and with completely random connections, spiral waves cannot survive in the given inhomogeneous medium. It is found that with small-world connections (basically local connections plus a small number of random connections) a well behaved spiral wave can be formed against the destructive role of the inhomogeneous medium. There also exists an optimal fraction of random connections which can greatly enhance the probability of spiral wave formation; this resembles the behavior of stochastic resonance.

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A coupled system can be regarded as a network consisting of a number of sites connected with certain topology. The topology of the connections can considerably influence the cooperative behavior of the system. For many decades two distinctive kinds of connections had been considered separately for different networks, one completely random connections and the other completely local regular connections. Very recently Watts and Strogatz proposed so-called "small-world" connections which are neither completely random nor completely local, but somewhere in between [1]. In particular, they studied a basically locally connected network with a small fraction of connections randomly rewired. The concept of small-world connections has quickly attracted much attention because of that: on one hand this kind of connection exists commonly in the real life of a human being, such as in social systems [2], neural networks [3], and epidemic problem [4]; on the other hand, the existence of a small fraction of random connections can essentially change the features of the given media, and plays very important role in determining the system behavior. *Nevertheless, a very important aspect, the effect of such small-world connections on spatial coherence and on pattern formations has not been investigated so far, to our knowledge.*

The topic of pattern formations is related to the problem of structures, functions, and various coherent behaviors of spatiotemporal systems, and is continually active in many decades in nonlinear science and in diverse fields [5–7]. Spiral waves are one of the most common and important patterns in nature, they appear in a hydrodynamic system, chemical reactions, and in a large variety of biological and physical systems [7,8]. In many cases the features of spiral waves can essentially determine the destination of the related systems. Previous experimental studies have demonstrated the involvement of spiral waves in both atrial and ventricular fibrillation [9]. Sudden cardiac death resulting from ventricular fibrillation is generated from the fragmenting or breakup of spiral waves [10]. In order to theoretically model the behavior of spiral wave formation, scientists have so far always used homogeneous media with completely local connections because it is naively expected that adding random connections in locally connected networks may be harmful for forming coherent patterns and increasing the fraction of random connections may decrease the possibility of forming spiral waves. *However, many media supporting spiral waves are, actually, inhomogeneous and random long-range couplings exist besides the basic local regular couplings.*

In this paper we will study the formation of spiral waves in an inhomogeneous medium with small-world connections. We find, to our great surprise, that a small fraction of random couplings can effectively enhance the possibility of spiral wave formation, when both completely local couplings and completely random couplings do not support any spiral wave in our inhomogeneous medium. This observation is strongly against the intuition since it is usually accepted that random connections are not favorable to the formation of any regular spatial patterns. We further find that there exists an optimal fraction of random connections for the small-world networks at which pattern formation can be conducted most effectively; this feature resembles the well known behavior of stochastic resonance (SR) [11], though here we do not have temporarily random forces, but have spatially random connections.

Let us start with an excitable medium [5,12] with the completely local nearest-neighbor couplings, the coupled FitzHugh-Nagumo (FN) model [16],

$$\dot{u}_{i,j}=\varepsilon_{i,j}^{-1}(u_{i,j}-u_{i,j}^{3}/3-v_{i,j})+D_{i,j},$$
$$\dot{v}_{i,j}=\varepsilon_{i,j}(u_{i,j}-\gamma v_{i,j}+\beta), \qquad (1)$$
$$D_{i,j}=D_{u}(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}),$$

where $i=1,\ldots,N_1$ and $j=1,\ldots,N_2$. The variables $u_{i,j}$ and $v_{i,j}$ in Eqs. (1) are the fast and slow variables, respectively, for small relaxation parameter $\varepsilon_{i,j}$, which controls the spatiotemporal scale separation. Item $\varepsilon_{i,j}$ can take constant or varies with *i,j* for homogeneous or inhomogeneous medium, respectively. Parameter β defines the asymmetry between the excitation and recovery for each element, and parameter γ

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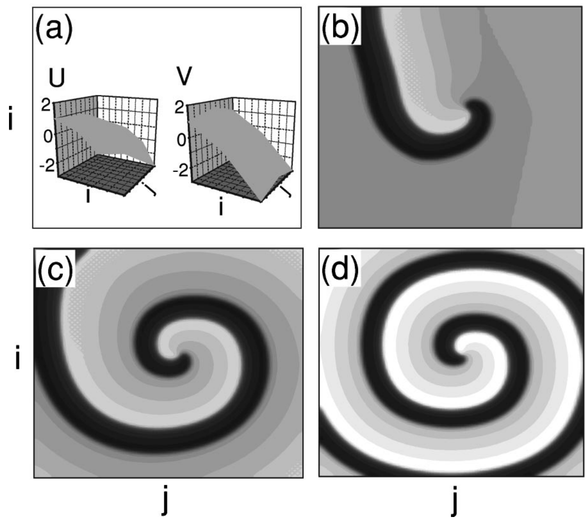

FIG. 1. The snapshot of the spiral wave formation in the homogeneous excitable medium with completely local couplings [Eqs. (1), $\varepsilon_{i,j}=0.2$]. The initial condition is given in (a). (b) $t=15$, (c) $t=35$, and (d) $t=300$. The spiral wave is well formed for $t\gg1$ in (d).

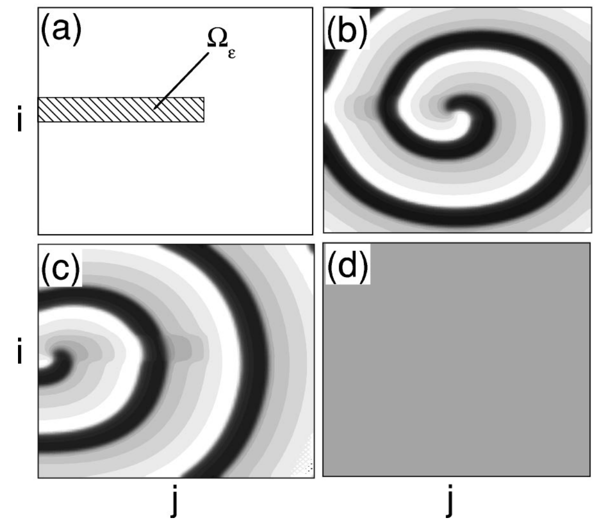

FIG. 2. (a) The inhomogeneous distribution of the control parameter $\varepsilon_{i,j}$, $\varepsilon_{i,j}=0.3$ in the shaded region Ω_ε, and $\varepsilon_{i,j}=0.2$ otherwise. (b)–(d) The same as Figs. 1(b)–1(d) with the inhomogeneous medium Eqs. (2) and Fig. 2(a) used. The spiral wave cannot survive in the given inhomogeneous medium. (b) $t=100$, (c) $t=300$, and (d) $t=500$.

>0 characterizes the dissipation of the slow variable $v_{i,j}$. $D_{i,j}$ represents the nearest local couplings on the fast variable of site (i,j). D_u is the coupling coefficient. It is worthwhile remarking that Eqs. (1) could be taken as a simplified model for cardiac tissue [12]. In the following discussion we will use free boundary conditions. In this paper, we will consider a single spiral wave formation by adopting a convenient initial condition [see Ref. [5] and Fig. 1(a)], and all the system parameters are fixed as $N_1=N_2=100$, $\beta=0.7$, $\gamma=0.5$, and $D_u=2.0$ (see Ref. [12]). In Fig. 1, we show the evolution of the spiral wave formation of the homogeneous excitable medium $\varepsilon_{i,j}=0.2$ with completely local coupling, starting from the initial condition Fig. 1(a).

Realistic systems may often have inhomogeneous media (real heart with ischemia). Thus, it is important to study the effect of inhomogeneity of the medium on the spiral wave formation. Recently, there has been considerable investigation of spiral wave annihilation through various inhomogeneities in excitable media [12,13]. In our system we model the inhomogeneity of the medium by setting inhomogeneous $\varepsilon_{i,j}$ in Eqs. (1) as

$$\varepsilon_{i,j}=\begin{cases}\varepsilon_1 & \text{for } (i,j)\in\Omega_\varepsilon\\ \varepsilon_2, & \text{otherwise,}\end{cases}$$

$$\Omega_\varepsilon=\{i,j|i_1<i<i_2,\ \ j_1<j<j_2\}, \tag{2}$$

where Ω_ε is called the inhomogeneous domain, and $\Delta\varepsilon=\varepsilon_1-\varepsilon_2$ is an inhomogeneous deviation caused by some practical reason. A significant observation is that certain inhomogeneity may definitely destroy the spiral wave structure. For instance, in Figs. 2(b)–2(d) we plot the evolution of the cardiac tissue model Eqs. (1) with the inhomogeneity of $\varepsilon_1=0.3$, $\varepsilon_2=0.2$, and Ω_ε being given in Fig. 2(a) as $i_1=1$, $i_2=60$, $j_1=50$, and $j_2=60$. From the same initial condition Fig. 1(a), a spiral-wave-like structure can appear in the transient process [see Fig. 2(b)]. However, no spiral wave can survive asymptotically [see Figs. 2(c) and 2(d)].

In both Figs. 1 and 2 we take the completely local nearest-neighbor couplings to connect the system sites. Applying the idea of small-world connections, we now include partial random couplings as

$$\dot{u}_{i,j}=\varepsilon_{i,j}^{-1}(u_{i,j}-u_{i,j}^3/3-v_{i,j})+D_{i,j}+D'_{i,j},$$

$$\dot{v}_{i,j}=\varepsilon_{i,j}(u_{i,j}-\gamma v_{i,j}+\beta),$$

$$D_{i,j}=D_u(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}), \tag{3}$$

$$D'_{i,j}=\begin{cases}D_u(u_{i',j'}-u_{i,j}) & \text{for } (i,j)\sim(i',j')\in\Omega_R\\ 0, & \text{otherwise,}\end{cases}$$

where Ω_R is a set of M pairs of non-neighbor sites randomly chosen, $D'_{i,j}$ represents the effect of random couplings on site (i,j), $\varepsilon_{i,j}$ take the form of Eq. (2) and Fig. 2(a). The number of total random connections is M, and $q=M/(2\times100^2)$ is the ratio of random connections to local connections, called fraction of random connections. It is naively anticipated that random connections would make the spiral wave formation even worse, since any randomness may not be favorable to a regular pattern like the spiral wave. In Fig. 3 we do exactly the same as Fig. 2 with small-world connections [Eqs. (3)] of very small q. Two striking features are observed in Fig. 3. First, including random connections in Fig. 3 does not spoil spiral wave. In contract, these random connections make the spiral wave surviving, and the destructive role of inhomogeneity is satisfactorily overcome. Second, the effect of small-world connections is very high, i.e., an extremely small fraction of random connections ($q=0.001\ll1$) can successfully

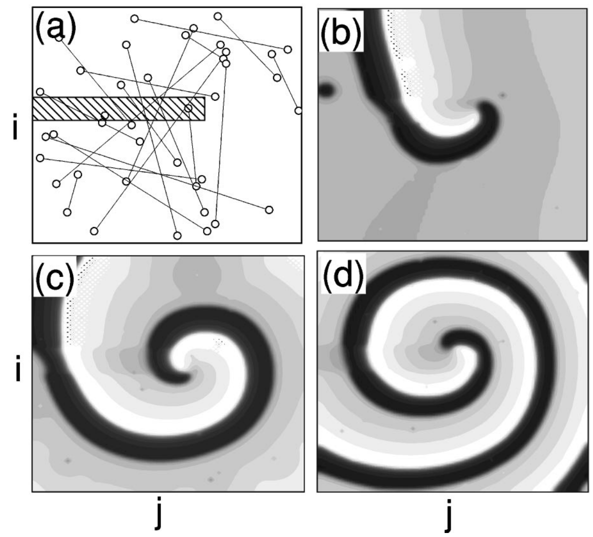

FIG. 3. The same as Fig. 2 with a fraction of random long-range connections, Eqs. (4), with $M=20$ ($q=0.001\ll 1$) applied. The barbells in (a) show the particular realization of random couplings. The small number of random connections can stabilize the spiral wave against the destruction of inhomogeneity of the medium. (b) $t=15$, (c) $t=35$, and (d) $t=2000$.

stabilize the spiral wave which is annihilated by the inhomogeneity for completely local connections.

Nevertheless, the behavior of Fig. 3 does not mean that random connections are always favorable to forming spiral wave. Actually, random connections have the nature to destroy any regular spatial pattern. In Fig. 4 we do the same as Fig. 3 with $M=1200$ ($q=0.06$), where the spiral wave is obviously spoiled by this large fraction of random connections. Actually, we found that, with $M=1200$ random connections, the spiral wave cannot exist even for homogeneous

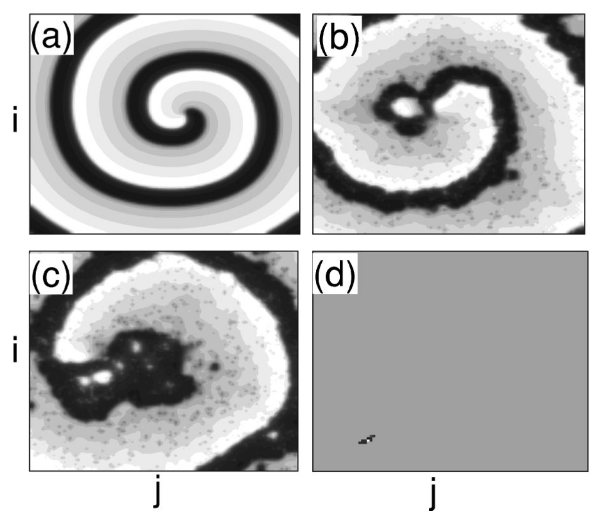

FIG. 4. A spiral wave cannot be formed from the initial condition Fig. 1(a) with large fractions ($q\gtrsim 0.06$) of random connections. Here the destruction process of an initial spiral wave by random connections with large q ($M=1200$, $q=0.06$) is presented. (a) $t=0$, an initial spiral wave state, (b) $t=100$, (c) $t=200$, and (d) $t=300$.

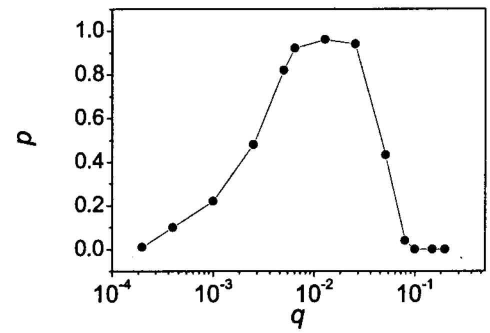

FIG. 5. The probability p of successfully forming spiral wave vs the fraction of random connections q. There is an optimal region $0.006\lesssim q\lesssim 0.025$, where p takes the largest value (very near to one).

medium, this confirms the destructive nature of random connections.

In order to have a global view on the influence of small-world connections on spiral wave formation, we present Fig. 5 where the inhomogeneity is the same as Fig. 2(a). For each M (i.e., each q) we arbitrarily choose 50 realizations of random connections, and investigate how many realizations (say G realizations) produce asymptotically stable spiral waves, and then compute the probability p as $p=G/50$. Figure 5 shows p vs q. An interesting high p peak is observed around $0.006\lesssim q\lesssim 0.025$.

Now it is important to understand the mechanism underlying the behavior of Figs. 3 and 5. Naively speaking, there are two requirements necessary for the formation of spiral waves in excitable media, apart from the requirement for proper initial condition. First, the medium is homogeneous (or inhomogeneous slightly) and the control parameters of local dynamics should be given in regimes suitable for spiral wave formation. If the medium contains certain localized spatial inhomogeneities linking the boundary, the spiral wave generated can be led to the boundary along the inhomogeneous zone and annihilated. This is exactly what happens in Fig. 2. Second, local regular couplings should dominate in connecting various space units. Random couplings definitely intend to destroy spatial order including spiral waves. This point is clearly confirmed in Fig. 4. The key reason for the effect of small-world connections shown in Figs. 3 and 5 is that the random connections play two seemly opposite roles for the pattern formation in inhomogeneous systems. Apart from the destructive role annihilating spiral wave shown in Fig. 4, random connections can couple different space regions from far away, and this function makes the medium more ''homogeneous.'' When the spiral wave tip moves to the inhomogeneous zone, it is interacted by not only the local inhomogeneous couplings but also the random homogeneous nonlocal couplings, and the latter considerably reduce the destructive role of inhomogeneity in favor of stabilizing the wanted pattern. For effective spiral wave formation the small-world connections have to be made such that the fraction of random connections should be so small that they cannot considerably spoil the spiral wave pattern established by the local connections in the stable space regions on one hand, and this fraction should be sufficiently large for the random long-range couplings to well prevent the inhomogeneity destroying the spiral on the other hand. Thus the high peak in Fig. 5 indicates the optimal fraction of random connections for the best performance in the competition process of these two opposite roles, resembling the well known SR behavior.

In this paper we use a special system as our example and take the inhomogeneity for a particular parameter. However, the same phenomena can be observed for different systems and for different inhomogeneous parameters. For instance, we have examined the inhomogeneous media considered in all the cases of Refs. [12–15], and obtained very similar results. Moreover, the above observations are not limited to the free boundary condition used in this paper. We have tried fixed boundary condition numerically, and the results of Figs. 2–5 are well reproduced. The particular shape of the inhomogeneous region is not important. We have numerically computed the cases with inhomogeneous regions of straight bar, the sinuous strip, single bar, and multiple bars, and the results are again qualitatively the same as Figs. 2–5. Our method works even when the inhomogeneous bar of Fig. 2 itinerates slowly among the whole space, then no any spiral wave survives wherever the spiral core is located initially. In this case, a small fraction of random couplings can still maintain the spiral wave against the fluctuating and itinerating inhomogeneity.

In conclusion we have investigated the formation of spiral waves in an inhomogeneous excitable medium with small-world couplings. An interesting result is that including certain random connections between the sites can be favorable to the spiral wave formation. The effect of random connections is surprisingly high that a very small fraction of random connections can enormously increase the probability of successfully forming spiral wave. Since spiral wave cannot exist in our inhomogeneous medium for both completely local connections and completely random connections, there appears a new concept, the optimal fraction of random connections of the small-world system, at which spiral wave formation can be most effectively conducted. Finally we would like to remark that the cardiac tissue systems (which obviously have inhomogeneous excitable media and for which the pattern formation of spiral waves is an extremely important issue [17]) may contain both local short-range couplings (gap junction) and random long-range connections (specific conduction system). The pattern formation in small-world couplings discussed in this paper may find its applications for a better understanding the pattern dynamics of cardiac systems.

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