

***N*-dimensional nonlinear Fokker-Planck equation with time-dependent coefficients**

L. C. Malacarne, R. S. Mendes, and I. T. Pedron*

Departamento de Física, Universidade Estadual de Maringá, Avenida Colombo 5790, 87020-900 Maringá, PR, Brazil

E. K. Lenzi

Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brazil

(Received 19 November 2001; published 29 April 2002)

An *N*-dimensional nonlinear Fokker-Planck equation is investigated here by considering the time dependence of the coefficients, where drift-controlled and source terms are present. We exhibit the exact solution based on the generalized Gaussian function related to the Tsallis statistics. Furthermore, we show that a rich class of diffusive processes, including normal and anomalous ones, can be obtained by changing the time dependence of the coefficients.

DOI: 10.1103/PhysRevE.65.052101

PACS number(s): 05.20.-y, 05.40.Fb, 05.40.Jc

Anomalous diffusion processes appear in a large class of systems in several contexts. Some illustrative examples are diffusion in plasmas [1], relative diffusion in turbulent media [2], cetyltrimethylammonium bromide (CTAB) micelles dissolved in salted water [3], surface growth and transport of fluid in porous media [4], two-dimensional rotating flow [5], subrecoil laser cooling [6], diffusion on fractals [7], anomalous diffusion at liquid surfaces [8], diffusion in linear shear flows [9], enhanced diffusion in active intracellular transport [10], particle diffusion in a quasi-two-dimensional bacterial bath [11], and spatiotemporal scaling of solar surface flows [12], among the others.

The existence of the anomalous diffusion and its ubiquity has motivated, in particular, the analytical study by considering nonlinear [13–19] and fractional [20,21] Fokker-Planck equations, spatial dependence of the diffusion coefficient [17,19,22,23], and temporal dependence of the drift term [18,24].

In this paper, we consider an *N*-dimensional nonlinear Fokker-Planck equation incorporating the time dependence in every coefficient, where drift-controlled (external force) and source terms are present. In our exact solution, a rich class of anomalous behaviors can arise by choosing appropriate time dependence of the coefficients. These results indicate that the possible anomalies in diffusive processes can appear as a consequence of different causes. This paper contains, essentially, the main results obtained in Refs. [13,14,18,24] as special cases.

In order to investigate, in an exact way, a large class of anomalous diffusions, let us consider the *N*-dimensional nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial t} \hat{\rho}(\mathbf{r}, t) = D(t) \nabla^2 [\hat{\rho}(\mathbf{r}, t)]^\nu - \nabla \cdot [\mathbf{F}(\mathbf{r}, t) \hat{\rho}(\mathbf{r}, t)] - \alpha(t) \hat{\rho}(\mathbf{r}, t), \quad (1)$$

where we incorporate the time dependence in the external force,

$$\mathbf{F}(\mathbf{r}, t) = \mathbf{k}_1(t) + k_2(t) \mathbf{r}, \quad (2)$$

in the diffusion coefficient $\mathcal{D}(t)$ and in the source term $\alpha(t)$. In particular, we consider the case where the space constant term in $\mathbf{F}(\mathbf{r}, t)$ can be taken with different coefficients. This situation is useful, for example, to study diffusion in the gravitational field [$\mathbf{k}_1 = (0, 0, k_{1z})$]. The harmonic potential is considered isotropic.

The source term in Eq. (1) can be removed by an appropriate change in the solution

$$\hat{\rho}(\mathbf{r}, t) = \exp \left[- \int_0^t \alpha(t') dt' \right] \rho(\mathbf{r}, t). \quad (3)$$

This way, $\rho(\mathbf{r}, t)$ obeys the equation

$$\frac{\partial}{\partial t} \rho(\mathbf{r}, t) = D(t) \nabla^2 [\rho(\mathbf{r}, t)]^\nu - \nabla \cdot [\mathbf{F}(\mathbf{r}, t) \rho(\mathbf{r}, t)], \quad (4)$$

with $D(t) = \mathcal{D}(t) \exp[(1-\nu) \int_0^t \alpha(t') dt']$. Thus, Eq. (4) has the same structure of Eq. (1) without the source term, but with an additional time dependence of the diffusion coefficient. Observe that this additional time dependence of $D(t)$ is induced by the nonlinear term ρ^ν , disappearing when $\nu = 1$.

In order to obtain an exact solution for Eq. (4) with the external force term given by Eq. (2), we are going to employ the ansatz

$$\rho(\mathbf{r}, t) = \frac{1}{Z(t)} \{ 1 - (1-q)\beta(t)[\mathbf{r} - \mathbf{r}_0(t)]^2 \}^{1/(1-q)} \quad (5)$$

if $1 - (1-q)\beta(t)(\mathbf{r} - \mathbf{r}_0)^2 \geq 0$ and $\rho(\mathbf{r}, t) = 0$ if $1 - (1-q)\beta(t)(\mathbf{r} - \mathbf{r}_0)^2 < 0$ (cutoff condition). We would like to remark that Eq. (5) can be justified via dimensional analysis and related to the renormalization group theory [25]. Furthermore, this ansatz can also be connected with Tsallis statistics [26,27]. In addition, Eq. (5) reduces to the Gaussian when $q \rightarrow 1$. In fact, by defining the function $\exp_q(-x^2) \equiv [1 - (1-q)x^2]^{1/(1-q)}$ if $1 - (1-q)x^2 \geq 0$, and $\exp_q(-x^2) \equiv 0$ if $1 - (1-q)x^2 < 0$ as a *q* Gaussian, we obtain the usual Gaussian function by taking the limit $q \rightarrow 1$. Equation (5) is a so-

*Electronic address: lcmala@dfi.uem.br

lution of Eq. (4) when $\nu=2-q$ and the time dependence of $\beta(t)$, $Z(t)$, and $\mathbf{r}_0(t)=\sum_{i=1}^N x_{0i}(t)\mathbf{e}_i$ are ruled by the following system of equations:

$$\frac{1}{Z} \frac{dZ}{dt} = 2(2-q)ND\beta Z^{-1+q} - Nk_2, \quad (6)$$

$$\frac{1}{\beta} \frac{d\beta}{dt} = -4(2-q)DZ^{-1+q}\beta + 2k_2, \quad (7)$$

and

$$\frac{dx_{0i}}{dt} = k_{1i} - k_2 x_{0i}. \quad (8)$$

Note that Eqs. (6) and (7) are nonlinear, Eq. (8) is independent of $\beta(t)$ and $Z(t)$, and $k_{1i}(t)$ does not appear in the nonlinear coupled differential equations for $\beta(t)$ and $Z(t)$. Furthermore, the spatially independent term in the external force only affects the time dependence of x_{0i} . So, Eq. (8) leads to

$$x_{0i}(t) = e^{-\mu(t)} \left[x_{0i}(0) + \int_0^t k_{1i}(s) e^{-\mu(s)} ds \right], \quad (9)$$

where $\mu(t) = \int_0^t k_2(s) ds$. For example, in a three-dimensional space with the presence of an isotropic time-independent harmonic potential and the gravitational field, i.e., $k_2(t) = k_2 = \text{const}$, $k_{1x} = k_{1y} = 0$, and $k_{1z}(t) = k_1 = \text{const}$, we get $x_0(t) = x_0(0)e^{-k_2 t}$, $y_0(t) = y_0(0)e^{-k_2 t}$, and

$$z_0(t) = \left[z_0(0) + \frac{k_1}{k_2} (e^{k_2 t} - 1) \right] e^{-k_2 t}. \quad (10)$$

The solution for the nonlinear coupled equations for $Z(t)$ and $\beta(t)$, [Eqs. (6) and (7)] is given by

$$Z(t) = Z_0 \left[1 - \frac{c_1}{N} f(t) \right]^{N/c_1} \quad (11)$$

and

$$\beta(t) = \beta_0 \left[1 - \frac{c_1}{N} f(t) \right]^{-2/c_1}, \quad (12)$$

with $c_1 = 2 + N(1-q)$, $\beta_0 = \beta(t=0)$, $Z_0 = Z(t=0)$, and

$$f(t) = e^{-c_1 \mu(t)} \times \int_0^t [Nk_2(s) - 2N(2-q)\beta_0 Z_0^{q-1} D(s)] e^{c_1 \mu(s)} ds. \quad (13)$$

It is usual to identify a normal diffusion process by a linear growth in time of the variance $\sigma^2 \equiv \langle (\mathbf{r} - \mathbf{r}_0)^2 \rangle$. Other time dependences on σ^2 are commonly related as anomalous diffusion, for instance, superdiffusive, subdiffusive, exponentially diffusive, and localized. In our study, we obtain a large class of diffusive processes that include these examples. In addition, σ^2 can have different behaviors for small and large times, enabling the description of a rich

TABLE I. Large time behavior of $\sigma^2 \sim f(t)^{2/c_1}$ for $\alpha(t)=0$, $D(t)=\mathcal{D}(t)=D_0$, and $k_2(t)=kt^{-b}$, where $c_1=2+N(1-q)>0$.

b	k	$\sigma^2(t)$	Description
b	$k=0$	t^{2/c_1}	c_1 Diffusive ^a
$b=0$	$k>0$	$(1 - e^{-c_1 k t})^{2/c_1}$	Ornstein-Uhlenbeck
$b=0$	$k<0$	$e^{2 k t}$	Exponentially diffusive
$0<b<1$	$k>0$	t^{2b/c_1}	c_1 Diffusive ^b
$0<b<1$	$k<0$	$e^{2 k t^{1-b}}$	Less than exponentially diffusive
$b<0$	$k>0$	$1/t^{2 b /c_1}$	Localized
$b<0$	$k<0$	$e^{2 k t^{1-b}}$	More than exponentially diffusive
$b=1$	$k>-1/2$	t^{2/c_1}	c_1 Diffusive ^a
$b=1$	$k=-1/2$	$(t \ln t)^{2/c_1}$	Log divergent
$b=1$	$k<-1/2$	$t^{2 k }$	Superdiffusive
$b>1$	k	t^{2/c_1}	c_1 Diffusive ^a

^aThe process is superdiffusive for $c_1 < 2$, normal for $c_1 = 2$, and subdiffusive for $c_1 > 2$.

^bThe process is superdiffusive for $c_1 < 2b$, normal for $c_1 = 2b$, and subdiffusive for $c_1 > 2b$.

structure of diffusion regime. Here we are going to analyze, as an illustration, a set of representative kinds of these anomalous behaviors in the asymptotic regime taking some specific dependence of the coefficients into account. In this direction, from Eqs. (3) and (5), we investigate the asymptotic temporal behavior of the variance

$$\sigma^2 = \frac{\int (\mathbf{r} - \mathbf{r}_0)^2 \hat{\rho}(\mathbf{r}, t) d^N \mathbf{r}}{\int \hat{\rho}(\mathbf{r}, t) d^N \mathbf{r}} = C(q, N) \beta^{-1}, \quad (14)$$

where $C(q, N)$ is a constant depending only on q and N . Note that the convergence of the integral for $q > 1$ in σ^2 imposes a restriction over the parameters: $2 + N(1-q) > 2(q-1)$. This implies that c_1 is a positive constant for all q values.

First, let us consider the case without the source term, with the diffusion coefficient constant, and with the time dependence of the harmonic external force given by $k_2(t) = kt^{-b}$. From Eq. (13), we obtain

$$f(t) \sim \{1 - \exp[-c_1 k t^{1-b}/(1-b)]\} + \{c_2 \exp[-c_1 k t^{1-b}/(1-b)] - c_3 t^b\} \quad (15)$$

for large t and $b < 1$, where c_2 and c_3 are constants that depend on N, k, q , and b . Again, for large t and $b = 1$, we have $f(t) \sim t^{-2k} - c_2 t^{-2k} (t^{1+2k} - 1)/(1+2k)$, and finally, for $b > 1$, we get $f(t) \sim t$. By using the fact that the mean square displacement is given by $\sigma^2 \sim f(t)^{2/c_1}$, several asymptotic behaviors can be obtained. We summarize in Table I the possible behaviors related to the above asymptotic results. When we restrict our analysis to the one-dimensional linear case ($\nu = 1$), this drift-controlled anoma-

TABLE II. Large time behavior of $\sigma^2 \sim f(t)^{2/c_1}$ for $\alpha(t)=0$, $D(t)=\mathcal{D}(t)=D_0 t^d$, and $k_2(t)=0$, where $c_1=2+N(1-q)>0$.

d	q	$\sigma^2(t)$	Description
$d=0$	$q=1$	t	Normal
$d=0$	$q<1$	$t^{2/[2+N(1-q)]}$	Subdiffusive
$d=0$	$q>1$	$t^{2/[2+N(1-q)]}$	Superdiffusive
$d<0$	$q=1$	$t^{1- d }$	Subdiffusive
$d>0$	$q=1$	t^{1+d}	Superdiffusive
$d>N(1-q)/2$	q	$t^{2(1+d)/[2+N(1-q)]}$	Superdiffusive
$d<N(1-q)/2$	q	$t^{2(1+d)/[2+N(1-q)]}$	Subdiffusive

lous diffusion contains the results given in Ref. [24] as a particular case. Consider now the nonlinear diffusion equation with neither source nor linear external force, but with the time dependence of the diffusion coefficient given by $D(t)=\mathcal{D}(t)=D_0 t^d$. In this case, for large t , Eq. (13) leads to

$$f(t) \sim t^{d+1}. \quad (16)$$

Since $\sigma^2 \sim f(t)^{2/[2+N(1-q)]}$, there is a competition between the parameters q and d to define the diffusion regimes. Table II contains a summary of these regimes. Another possible situation is to take the nonlinear diffusion equation with a time-dependent source term, $\alpha(t)=\alpha_0 t^a$, with time-independent diffusion constant, $\mathcal{D}(t)=D_0$, and without linear external force, $k_2(t)=0$. For $\nu=1$ ($q=1$) the source term does not affect the diffusion regime. On the other hand, for $\nu \neq 1$ ($q \neq 1$) and large t , Eq. (13) gives

$$f(t) \sim t^a \exp\left[\frac{(q-1)\alpha_0 t^{1+a}}{1+a}\right], \quad (17)$$

when $a > -1$. For $a = -1$, we have $f(t) \sim t^{(q-1)\alpha_0+1}$, and for $a < -1$, Eq. (13) reduces to $f(t) \sim t$. Table III gives us a summary of the above behaviors. To conclude our observations about the anomalous diffusion induced by the time-dependent coefficients, we stress that the investigation of a more complex time dependence of the coefficients can be reduced to the analysis of Eq. (13).

Summing up, we have investigated an N -dimensional nonlinear Fokker-Planck equation by incorporating the time dependence of every coefficient, including those of the external force and the source term. An exact solution is obtained in the case of external force with isotropic spatial linear term and a possible anisotropic spatial constant term. This anisotropic constant term is useful, for instance, to ana-

TABLE III. Large time behavior of $\sigma^2 \sim f(t)^{2/c_1}$ for $\alpha(t)=\alpha_0 t^a$, $\mathcal{D}(t)=D_0$, and $k_2(t)=0$, where $c_1=2+N(1-q)>0$.

α_0	a	$\sigma^2(t)$	Description
$(q-1)\alpha_0 < 0$	$a=0$	$\left(\frac{e^{(q-1)\alpha_0 t}-1}{(q-1)\alpha_0}\right)^{2/c_1}$	Stationary
$(q-1)\alpha_0 > 0$	$a=0$	$\left(\frac{e^{(q-1)\alpha_0 t}-1}{(q-1)\alpha_0}\right)^{2/c_1}$	Exponentially
α_0	$a=-1$	$t^{2[(q-1)\alpha_0-1]/c_1}$	diffusive c_1 Diffusive Less than
$(q-1)\alpha_0 > 0$	$-1 < a < 0$	$\frac{2[(q-1)\alpha_0]t^{1+a}}{\exp\frac{c_1(1+a)}{c_1(1+a)}}$	exponentially diffusive c_1 Diffusive
$(q-1)\alpha_0 < 0$	$-1 < a < 0$	$t^{2 a /c_1}$	More than exponentially
$(q-1)\alpha_0 > 0$	$a > 0$	$\frac{2[(q-1)\alpha_0]t^{1+a}}{\exp\frac{c_1(1+a)}{c_1(1+a)}}$	diffusive Localized
$(q-1)\alpha_0 < 0$	$a > 0$	t^{-2a/c_1}	c_1 Diffusive
α_0	$a < -1$	t^{2/c_1}	c_1 Diffusive

^aThe process is superdiffusive for $c_1 < 2[(q-1)\alpha_0 - 1]$, normal for $c_1 = 2[(q-1)\alpha_0 - 1]$, and subdiffusive for $c_1 > 2[(q-1)\alpha_0 - 1]$.

^bThe process is superdiffusive for $c_1 < 2|a|$, normal for $c_1 = 2|a|$, and subdiffusive for $c_1 > 2|a|$.

^cThe process is superdiffusive for $c_1 < 2$, normal for $c_1 = 2$, and subdiffusive for $c_1 > 2$.

lyze diffusion when the gravitational field is relevant. We have showed that a rich class of anomalous behaviors arises by choosing appropriate time dependence of the coefficients. These results indicate that the anomaly in diffusive processes may appear as a consequence of different causes. In particular, the combination of nonlinearity and time dependence of the coefficients can lead to the normal diffusion (time linear growth of the mean square displacement). This fact implies that the normal diffusion, in general, can not be associated with the Gaussian shape of the distribution $\rho(\mathbf{r}, t)$, i.e., the linear time increase of variance does not necessarily mean that we are in the presence of ordinary diffusion. Finally, we hope that the results obtained here may be useful to clarify a possible origin of a large class of different anomalous diffusive processes in theoretical and experimental contexts.

We thank CAPES, CNPq, and PRONEX (Brazilian agencies) for partial financial support.

[1] J.G. Berryman, J. Math. Phys. **18**, 2108 (1977).
[2] H.G.E. Hentschel and I. Procaccia, Phys. Rev. A **29**, 1461 (1984).
[3] A. Ott, J.P. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990).
[4] H. Spohn, J. Phys. I **3**, 69 (1993).
[5] T.H. Solomon, E.R. Weeks, and H.L. Swinney, Phys. Rev. Lett. **71**, 3975 (1993).

[6] F. Bardou, J.P. Bouchaud, O. Emile, A. Aspect, and C. Cohen-Tannoudji, Phys. Rev. Lett. **72**, 203 (1994).
[7] J. Stephenson, Physica A **222**, 234 (1995).
[8] O.V. Bychuk and B. O'Shaughnessy, Phys. Rev. Lett. **74**, 1795 (1995); S.C. Venkataramani, T.M. Antonsen, and E. Ott, *ibid.* **78**, 3864 (1997).
[9] A. Compte, D. Jou, and Y. Katayama, J. Phys. A **30**, 1023 (1997).

- [10] A. Caspi, R. Granek, and M. Elbaum, *Phys. Rev. Lett.* **85**, 5655 (2000).
- [11] Xiao-Lun Wu and A. Libchaber, *Phys. Rev. Lett.* **84**, 3017 (2000).
- [12] J.K. Lawrence, A.C. Cadavid, A. Ruzmaikin, and T.E. Berger, *Phys. Rev. Lett.* **86**, 5894 (2001).
- [13] A.R. Plastino and A. Plastino, *Physica A* **222**, 347 (1995).
- [14] C. Tsallis and D.J. Bukman, *Phys. Rev. E* **54**, R2197 (1996).
- [15] A. Compte, D. Jou, and Y. Katayama, *J. Phys. A* **29**, 4321 (1996).
- [16] L. Borland, *Phys. Rev. E* **57**, 6634 (1998).
- [17] L. Borland, F. Pennini, A.R. Plastino, and A. Plastino, *Eur. Phys. J. B* **12**, 285 (1999).
- [18] C. Giordano, A.R. Plastino, M. Casas, and A. Plastino, *Eur. Phys. J. B* **22**, 361 (2001).
- [19] L.C. Malacarne, R.S. Mendes, I.T. Pedron, and E.K. Lenzi, *Phys. Rev. E* **63**, 030101(R) (2001).
- [20] R. Metzler and J. Klafter, *Phys. Rev. E* **61**, 6308 (2000); *Phys. Rep.* **339**, 1 (2000).
- [21] M. Bologna, C. Tsallis, and P. Grigolini, *Phys. Rev. E* **62**, 2213 (2000).
- [22] L.F. Richardson, *Proc. R. Soc. London, Ser. A* **110**, 709 (1926).
- [23] B. O'Shaughnessy and I. Procaccia, *Phys. Rev. Lett.* **54**, 455 (1985); *Phys. Rev. A* **32**, 3073 (1985).
- [24] F. Lillo and R.N. Mantegna, *Phys. Rev. E* **61**, R4675 (2000).
- [25] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, *Frontiers in Physics*, Vol. 85, (Perseus Books Publishing, Reading Massachusetts, 1992), pp. 287–332.
- [26] C. Tsallis, R.S. Mendes, and A.R. Plastino, *Physica A* **261**, 534 (1998).
- [27] The connection between renormalization group theory and Tsallis statistics, suggested in Eq. (5), is an interesting theme to be clarified. However, this particular subject seems nontrivial and deserves further investigations.