Kinetic behavior of aggregation processes with complete annihilation

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The kinetic behavior of an aggregation-annihilation process of an n-species ($n \geqslant 2$) system is studied. In this model, an irreversible aggregation reaction occurs between any two clusters of the same species and an irreversible complete annihilation reaction occurs between any two different species. Based on the mean-field theory, we investigate the rate equations of the process with constant reaction rates to obtain the asymptotic solutions for the cluster-mass distributions. We find that the cluster-mass distribution of each species satisfies a modified scaling law, which reduces to the standard scaling law in some special cases. The scaling exponents of the system may strongly depend on the reaction rates for most cases; however, for the case with all the aggregation rates twice the annihilation rate, these exponents depend only on the initial concentrations. All the species annihilate each other completely except in the case in which at least one aggregation rate is less than twice the annihilation rate.

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I. INTRODUCTION

The phenomenon of aggregation and annihilation is central to a wide range of fields, such as physics, chemistry, and biology. Considerable interest has been focused on aggregation and annihilation processes since the 1970s [1–13]. It was found that the cluster-mass distribution in aggregation systems possesses scaling behavior in some particular cases [11–19]. Krapivsky studied an irreversible aggregation-annihilation process of a two-species system and found that the cluster-mass distribution obeys a scaling law in the long-time limit [11]. Zhang and Yang generalized the two-species model to the process in an n-species system and analyzed the scaling properties of the cluster-mass distribution in some symmetrical cases [12]. Most of this research was devoted only to partial annihilation, where the larger cluster is conserved after the reaction with the number of monomers equal to the difference between the two clusters [11,12]. Meanwhile, few studies were concerned with irreversible aggregation processes with complete annihilation, where the annihilation reaction between two clusters of different species always results in inertness independent of their masses. Ben-Naim and Krapivsky investigated the kinetics of a two-species aggregation process with complete annihilation in the special case where all the reaction rates are equal to 2 and found scaling descriptions of their mass distributions in the long-time limit [13]. In fact, these irreversible aggregation processes with complete annihilation are of great practical significance. For example, in a two-species chemical system with constituent high polymers A and B, aggregation of the same species can produce energetic open chains, while two clusters of different species can bond into an inert closed chain. The open chains continue to participate in the reaction process, but the closed chains will withdraw from the reaction process because of their lower energy.

In this work, we investigate the competition between the aggregation and annihilation processes of n species A^l ($l=1,2,\ldots,n,\ \ n\geqslant 2$). We assume that irreversible aggregation occurs only between two clusters of the same species, $A_i^l + A_j^l \xrightarrow{K_l(i,j)} A_{i+j}^l$, and the irreversible complete annihilation reaction occurs simultaneously between two clusters of different species, $A_i^l + A_j^m \xrightarrow{J_{lm}(i,j)}$ inert, where A_i^l denotes a cluster consisting of i-mers of species A^l ($l,m=1,2,\ldots,n,\ \ l\neq m$). The rate of the aggregation reaction between A_i^l and A_j^l clusters is equal to $K_l(i,j)$, and that of annihilation between A_i^l and A_j^m clusters is $J_{lm}(i,j)$.

The present investigation is based on the mean-field theory, which assumes that the reaction proceeds with a rate proportional to the reactant concentrations. Thus the mean-field approximation neglects spatial fluctuation of the reactant densities and therefore applies when the spatial dimension d of the system is greater than or equal to the critical dimension d_c [11,13]. When $d<d_c$, fluctuations in the densities of reactants may lead to dimension-dependent kinetic behavior in the long-time limit; however, the mean-field prediction may provide a useful description of the kinetic behavior for moderate times [11]. Numerical simulations have confirmed the mean-field predictions above the critical dimension [13]. The investigation of aggregation process can also be based on the particle coalescence model (PCM) in the diffusion-controlled limit [13,20,21]. For the PCM, it is found that $d_c=2$ [20]. Hence, it was suggested by Ben-Naim and Krapivsky that $d_c=2$ for the aggregation-annihilation model [13]. In this paper, we assume that in our system the spatial dimension d is greater than 2. Thus we have derived the asymptotic solutions for the cluster-mass distributions based on the mean-field assumption. The results show that the evolution behaviors of n types of cluster satisfy the standard scaling or modified scaling laws, and their exponents are strongly dependent on the reaction rates. The initial concentrations also play important roles in some special cases.

The paper is organized as follows. In Sec. II, we describe an irreversible aggregation-annihilation model with n ($n\geqslant 2$) species, and give the corresponding rate equations

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TABLE I. Organization of Sec. II.

Case	Title of subsection
A	All aggregation rates greater than $2J$ (twice the annihilation rate)
B	All aggregation rates equal to $2J$
C	All aggregation rates less than $2J$
D	Some aggregation rates less than $2J$ with others equal to or greater than $2J$
E	Some aggregation rates equal to $2J$ with others greater than $2J$

with constant reaction rates on the basis of mean-field theory. Then we determine the asymptotic solutions of the cluster-mass distributions in the cases illustrated in Table I. A brief summary is given in Sec. III.

II. MODEL OF *n*-SPECIES AGGREGATION PROCESS WITH COMPLETE ANNIHILATION

In our investigation, the theoretical approach to the aggregation process is based on the mean-field rate equations. There are n types of cluster in the system, A^l clusters, $l=1,2,\ldots,n$. The concentration of A^l clusters of k-mers is denoted as a_{lk}. Here, we consider a model with constant reaction rates. In order to investigate thoroughly the evolution behavior of the irreversible aggregation-annihilation system, we assume that the reaction rates of aggregation and annihilation have different constant values. All the annihilation reaction rates are equal to J and the aggregation rates of A^l clusters are different constants I_l. We generalize the rate equations of the aggregation-annihilation process given by Ben-Naim and Krapivsky [13] and write out the corresponding rate equations for this system as follows:

$$\frac{da_{lk}}{dt}=I_l\left(\frac{1}{2}\sum_{i+j=k}a_{li}a_{lj}-a_{lk}\sum_{j=1}^{\infty}a_{lj}\right)-Ja_{lk}\sum_{1\leqslant m\leqslant n,m\neq l}\sum_{j=1}^{\infty}a_{mj},\quad l=1,2,\ldots,n. \tag{1}$$

As we aim to find the analytical solutions of the evolution behavior of the clusters and investigate their long-time scaling properties, we assume that there exist only monomer clusters at $t=0$ and the cluster concentration of A^l species equals A_{l0}. Then the monodisperse initial conditions are

$$a_{lk}(0)=A_{l0}\delta_{k1},\quad l=1,2,\ldots,n. \tag{2}$$

In the above case, the set of rate equations can be solved with the help of the ansatz in Ref. [11]. We assume a_{lk} has the form

$$a_{lk}(t)=A_l(t)[a_l(t)]^{k-1},\quad l=1,2,\ldots,n. \tag{3}$$

Substituting Eq. (3) into Eqs. (1), we can transform it into the following differential equations:

$$\frac{da_l}{dt}=\frac{1}{2}I_lA_l,$$

$$\frac{dA_l}{dt}=-\frac{I_lA_l^2}{1-a_l}-JA_l\sum_{1\leqslant m\leqslant n,m\neq l}\frac{A_m}{1-a_m},\quad l=1,2,\ldots,n. \tag{4}$$

Correspondingly, the initial conditions of Eqs. (4) become

$$a_l=0,\quad A_l=A_{l0},\quad l=1,2,\ldots,n,\quad \text{at}\quad t=0. \tag{5}$$

Introducing new variables $\alpha_l(t)$,

$$\alpha_l=(1-a_l)^{-1},\quad l=1,2,\ldots,n, \tag{6}$$

we recast the differential equations (4) as the following equations:

$$\frac{d^2\alpha_l}{dt^2}=-\frac{d\alpha_l}{dt}\sum_{1\leqslant m\leqslant n,m\neq l}\frac{2J}{I_m\alpha_m}\frac{d\alpha_m}{dt},\quad l=1,2,\ldots,n, \tag{7}$$

and the initial conditions of Eqs. (7) are

$$\alpha_l=1,\quad \frac{d\alpha_l}{dt}=\frac{1}{2}I_lA_{l0},\quad l=1,2,\ldots,n,\quad \text{at}\quad t=0. \tag{8}$$

Equations (7) can be integrated as follows:

$$\frac{d\alpha_l}{dt}=\frac{1}{2}I_lA_{l0}\prod_{1\leqslant m\leqslant n,m\neq l}\alpha_m^{-2J/I_m},\quad l=1,2,\ldots,n. \tag{9}$$

Then we can derive the following equations from Eqs. (9)

$$I_mA_{m0}\alpha_l^{-2J/I_l}\frac{d\alpha_l}{dt}=I_lA_{l0}\alpha_m^{-2J/I_m}\frac{d\alpha_m}{dt},$$

$$l=1,2,\ldots,n,\quad m=1,2,\ldots,n. \tag{10}$$

In order to thoroughly investigate the kinetic behavior of the system, we discuss the solutions of Eqs. (9) and (10) in several different cases.

A. All aggregation rates greater than 2*J*

In this case, one can derive the following equations from Eqs. (10):

$$\frac{\alpha_m^{1-2J/I_m}-1}{A_{m0}(I_m-2J)}=\frac{\alpha_l^{1-2J/I_l}-1}{A_{l0}(I_l-2J)},$$

$$l=1,2,\ldots,n,\quad m=1,2,\ldots. \tag{11}$$

Substituting Eqs. (11) into Eqs. (9), we obtain

$$\frac{d\alpha_l}{dt}=\frac{I_lA_{l0}}{2}\prod_{1\leqslant m\leqslant n,m\neq l}\left[1-\frac{A_{m0}(I_m-2J)}{A_{l0}(I_l-2J)}+\frac{A_{m0}(I_m-2J)}{A_{l0}(I_l-2J)}\alpha_l^{1-2J/I_l}\right]^{2J/(2J-I_m)},$$

$$l=1,2,\ldots,n. \tag{12}$$

The system is assumed to reach its steady state at $t\to\infty$ and its steady conditions are then given as follows:

$$\frac{da_l}{dt}=\frac{1}{\alpha_l^2}\frac{d\alpha_l}{dt}=0,$$

$$\frac{dA_l}{dt}=\frac{2}{I_l\alpha_l^2}\frac{d^2\alpha_l}{dt^2}-\frac{4}{I_l\alpha_l^3}\left(\frac{d\alpha_l}{dt}\right)^2=0,\quad l=1,2,\ldots,n. \tag{13}$$

Thus we can conclude that either $\alpha_l\to\infty$ or $d\alpha_l/dt\to 0$ at $t\to\infty$. Further, from Eqs. (9) and (11) we know that $\alpha_l\to\infty$ at $t\to\infty$ for this case. Hence, $\alpha_l\gg 1$ at $t\gg 1$. In the long-time limit, Eqs. (12) can be rewritten as the following asymptotic equations:

$$\frac{d\alpha_l}{dt}\simeq\frac{I_lA_{l0}}{2}\prod_{1\leqslant m\leqslant n,m\neq l}\left[\frac{A_{m0}(I_m-2J)}{A_{l0}(I_l-2J)}\right]^{2J/(2J-I_m)}\times\alpha_l^{(I_l-2J)/I_l\Sigma_{1\leqslant m\leqslant n,m\neq l}\,2J/(2J-I_m)},$$

$$l=1,2,\ldots,n. \tag{14}$$

The asymptotic solutions of α_l in the long-time limit are directly given as

$$\alpha_l\simeq C_{1l}t^{R_{1l}},\quad l=1,2,\ldots,n, \tag{15}$$

where

$$R_{1l}=I_l\left[I_l-2J+2J\sum_{m=1}^{n}(I_l-2J)/(I_m-2J)\right]^{-1}$$

and

$$C_{1l}=\left\{\frac{I_lA_{l0}}{2R_{1l}}\prod_{m=1}^{n}\left[\frac{A_{m0}(I_m-2J)}{A_{l0}(I_l-2J)}\right]^{2J/(2J-I_m)}\right\}^{R_{1l}}.$$

We then obtain the following asymptotic solutions for the cluster-mass distributions:

$$a_{lk}(t)\simeq C'_{1l}t^{-1-R_{1l}}[1-C_{1l}^{-1}t^{-R_{1l}}]^{k-1},\quad l=1,2,\ldots,n, \tag{16}$$

where $C'_{1l}=2R_{1l}/I_lC_{1l}$. Further, Eqs. (16) can be rewritten as

$$a_{lk}(t)\simeq C'_{1l}t^{-1-R_{1l}}\exp(-x_l),\quad l=1,2,\ldots,n, \tag{17}$$

which are valid in the regions $k\gg 1$, $t\gg 1$, $x_l=(k/C_{1l})t^{-R_{1l}}=$ finite.

Krapivsky used a function $S(t)$ to denote the characteristic cluster mass in the long-time limit of such an aggregation system and wrote the concentration $c_k(t)$ of k-mer aggregates in scaling form as [11]

$$c_k(t)\simeq t^{-w}\Phi[k/S(t)],\quad S(t)\propto t^z. \tag{18}$$

The total number $N(t)$ and the total mass $M(t)$ of the clusters of one species in the long-time limit can then be expressed in power-law forms as [11]

$$N(t)=\sum_{k=1}^{\infty}c_k(t)\propto t^{-\lambda},\quad M(t)=\sum_{k=1}^{\infty}kc_k(t)\propto t^{-\mu}. \tag{19}$$

The four exponents (w,z,λ,μ) are universally used to describe the scaling nature of aggregation-annihilation processes in the long-time limit. From Eqs. (18) and (19), one can find the following exponent relations:

$$\lambda=w-z,\quad \mu=w-2z. \tag{20}$$

In this case, the asymptotic solutions of $a_{lk}(t)$ show that the cluster-mass distribution of each species satisfies the standard scaling law (18) in the long-time limit. From Eqs. (17), we obtain the four exponents as follows:

$$w=\frac{2I_l-2J+2J\sum_{m=1}^{n}(I_l-2J)/(I_m-2J)}{I_l-2J+2J\sum_{m=1}^{n}(I_l-2J)/(I_m-2J)},$$

$$z=\frac{I_l}{I_l-2J+2J\sum_{m=1}^{n}(I_l-2J)/(I_m-2J)},\quad \lambda=1, \tag{21}$$

$$\mu=\frac{2J\sum_{m=1}^{n}(I_l-2J)/(I_m-2J)-2J}{I_l-2J+2J\sum_{m=1}^{n}(I_l-2J)/(I_m-2J)}$$

$$\text{for } A^l \text{ clusters},\quad l=1,2,\ldots,n.$$

It is shown that the exponents w, z, and μ depend on the values of the reaction rates I_l and J. The total numbers and the total mass of all the species are found to decrease with time because $\lambda,\mu>0$, and no species remains at $t\to\infty$, independent of the initial data A_{l0} $(l=1,2,\ldots,n)$.

B. All aggregation rates equal to $2J$

In this case, we determine the following integrals of Eqs. (10):

$$\alpha_m^{A_{l0}}=\alpha_l^{A_{m0}},\quad l=1,2,\ldots,n,\quad m=1,2,\ldots,n. \tag{22}$$

Substituting Eqs. (22) into Eqs. (9), one can obtain

$$\frac{d\alpha_l}{dt}=JA_{l0}\alpha_l^{R_{2l}},\quad l=1,2,\ldots,n, \tag{23}$$

where $R_{2l}=1-\Sigma_{m=1}^{n}A_{m0}/A_{l0}$. The exact solutions of α_l can easily be derived from Eqs. (23):

$$\alpha_l = \left(J \sum_{m=1}^{n} A_{m0} t + 1 \right)^{R_{3l}}, \quad l = 1,2, \ldots, n, \tag{24}$$

where $R_{3l} = A_{l0} / \Sigma_{m=1}^{n} A_{m0}$. Then we obtain the scaling solutions of $a_{lk}(t)$ in the long-time limit as follows:

$$a_{lk}(t) \simeq A_{l0} \left(J \sum_{m=1}^{n} A_{m0} t \right)^{-1-R_{3l}} \exp(-x_l),$$

$$l = 1,2, \ldots, n, \tag{25}$$

and the different scaling variables for A^l clusters are

$$x_l = k \left(J \sum_{m=1}^{n} A_{m0} t \right)^{-R_{3l}}. \tag{26}$$

According to the standard scaling form (18), we determine the scaling exponents

$$w = \frac{A_{l0} + \sum_{m=1}^{n} A_{m0}}{\sum_{m=1}^{n} A_{m0}}, \quad z = \frac{A_{l0}}{\sum_{m=1}^{n} A_{m0}},$$

$$\lambda = 1, \quad \mu = \frac{\sum_{m=1}^{n} A_{m0} - A_{l0}}{\sum_{m=1}^{n} A_{m0}}$$

$$\text{for } A^l \text{ clusters}, \quad l = 1,2, \ldots, n. \tag{27}$$

The results show that all the species obey the standard scaling law in this case. The exponents w, z, and μ are related to the initial data A_{l0} and are independent of the reaction rates I_l and J. It is obvious that the heavy species with the larger initial concentration has the smaller value of μ, and dominates over the light one in the long-time limit. The results also indicate that both the total number and the total mass of each species decrease with time and all the species annihilate each other completely in the end.

C. All aggregation rates less than $2J$

We can also obtain Eqs. (11) and (12) for this case. The solutions of α_l are dependent on the values of $A_{l0}(2J - I_l)$. Without any loss of generality, one can assume that $A_{10}(2J - I_1) \geqslant A_{20}(2J - I_2) \geqslant \cdots \geqslant A_{n0}(2J - I_n)$.

When $A_{10}(2J - I_1) > A_{l'0}(2J - I_{l'})$, $l' = 2,3, \ldots, n$, it can be found from Eq. (12) that $\alpha_1 \to \infty$ at $t \to \infty$. So, $\alpha_1^{1-2J/I_1} \ll 1$ at $t \gg 1$. Thus we obtain the asymptotic differential equation for α_1 in the long-time limit from Eqs. (12):

$$\frac{d\alpha_1}{dt} \simeq C_{21}, \tag{28}$$

where $C_{21} = (I_1 A_{10}/2) \Pi_{l'=2}^{n} [1 - A_{l'0}(2J - I_{l'})/A_{10}(2J - I_1)]^{2J/(2J - I_{l'})}$. The asymptotic solution of α_1 can be obtained as

$$\alpha_1 \simeq C_{21} t. \tag{29}$$

Substituting Eq. (29) into Eqs. (11), we derive the solutions for $\alpha_{l'}$ as follows:

$$\alpha_{l'} \simeq C_{2l'} - C_{3l'} t^{1-2J/I_1}, \quad l' = 2,3, \ldots, n, \tag{30}$$

where $C_{2l'} = [1 - A_{l'0}(2J - I_{l'})/A_{10}(2J - I_1)]^{I_{l'}/(I_{l'} - 2J)}$ and $C_{3l'} = I_{l'} A_{l'0} C_{2l'} C_{21}^{1-2J/I_1} / [A_{10}(2J - I_1) - A_{l'0}(2J - I_{l'})]$. We then obtain the standard scaling solution for the cluster-mass distribution of A^1 species, which has the maximum value of $A_{l0}(2J - I_l)$

$$a_{1k}(t) \simeq \frac{2}{I_1 C_{21}} t^{-2} \exp(-x_1), \quad x_1 = k(C_{21} t)^{-1}, \tag{31}$$

and the exponents for the A^1 species are

$$w = 2, \quad z = 1, \quad \lambda = 1, \quad \mu = 0. \tag{32}$$

One can determine the asymptotic behaviors for the cluster-mass distributions of $A^{l'}$ species as

$$a_{l'k}(t) \simeq \frac{2(2J - I_1) C_{3l'}}{I_1 I_{l'} C_{2l'}^2} \left(\frac{C_{2l'} - 1}{C_{2l'}} \right)^k t^{-2J/I_1}$$

$$\times \exp(-x_{l'}), \quad l' = 2,3, \ldots, n, \tag{33}$$

where $x_{l'} = [C_{3l'} / C_{2l'}(C_{2l'} - 1)] k t^{-(2J - I_1)/I_1}$. Equations (33) are valid in the region $k t^{-(2J - I_1)/I_1} = \text{finite}$. The result implies that the standard scaling description (18) of the cluster-mass distribution breaks down for $A^{l'}$ species with smaller values of $A_{l'0}(2J - I_l')$.

One can modify the standard scaling description (18) as follows [11]:

$$c_k(t) \simeq b^k t^{-w} \Phi[k/S(t)], \quad S(t) \propto t^z, \tag{34}$$

where b is a constant and $0 < b < 1$. In this case, the scaling exponent relations become

$$\lambda = w, \quad \mu = w. \tag{35}$$

Thus the exponents for the $A^{l'}$ species ($l' = 2,3, \ldots, n$) are given as

$$w = \lambda = \mu = 2J/I_1, \quad z = (2J - I_1)/I_1. \tag{36}$$

This shows that all the $A^{l'}$ species ($l' = 2,3, \ldots, n$) have the same scaling exponents, which depend on the aggregation rate of A^1 species and the annihilation rate J. In this case, $A^{l'}$ clusters have $\mu = 2J/I_1 > 1$ in contrast to $\mu = 0$ for A^1 clusters, which implies that only A^1 species remain at the end.

Now let us turn to the general case in which there are n' kinds of species ($1 < n' \leqslant n$) with the same largest values of

$A_{l0}(2J-I_l)$. When the system consists of only two species, we have $n'=n=2$. Thus we can obtain the following equations from Eqs. (11):

$$\alpha_1^{1-2J/I_1}=\alpha_2^{1-2J/I_2}. \tag{37}$$

In this case, we find that $\alpha_1,\alpha_2\to\infty$ at $t\to\infty$. Hence, $\alpha_1,\alpha_2\gg 1$ in the long-time limit. Substituting Eq. (37) into Eqs. (12), we obtain the differential equation for $\alpha_1(t)$ as follows:

$$\frac{d\alpha_1}{dt}=\frac{1}{2}I_1A_{10}\alpha_1^{2J(I_1-2J)/I_1(2J-I_2)}. \tag{38}$$

One can determine the exact solution of α_1 from Eq. (38):

$$\alpha_1=\left[\frac{(4J^2-I_1I_2)A_{10}}{2(2J-I_2)}t+1\right]^{(2I_1J-I_1I_2)/(4J^2-I_1I_2)}, \tag{39}$$

and the solution for α_2 can then be obtained as

$$\alpha_2=\left[\frac{(4J^2-I_1I_2)A_{20}}{2(2J-I_1)}t+1\right]^{(2I_2J-I_1I_2)/(4J^2-I_1I_2)}. \tag{40}$$

Thus we obtain the standard scaling descriptions for both A^1 and A^2 clusters in the long-time limit as

$$a_{1k}(t)\simeq\frac{2(2J-I_2)}{B_1(4J^2-I_1I_2)}t^{-(4J^2+2I_1J-2I_1I_2)/(4J^2-I_1I_2)}\times\exp(-x_1),$$
$$x_1=\frac{k}{B_1}t^{-(2I_1J-I_1I_2)/(4J^2-I_1I_2)}, \tag{41}$$

$$a_{2k}(t)\simeq\frac{2(2J-I_1)}{B_2(4J^2-I_1I_2)}t^{-(4J^2+2I_2J-2I_1I_2)/(4J^2-I_1I_2)}\times\exp(-x_2),$$
$$x_2=\frac{k}{B_2}t^{-(2I_2J-I_1I_2)/(4J^2-I_1I_2)},$$

where $B_1=[(4J^2-I_1I_2)A_{10}/2(2J-I_2)]^{(2I_1J-I_1I_2)/(4J^2-I_1I_2)}$ and $B_2=[(4J^2-I_1I_2)A_{20}/2(2J-I_1)]^{(2I_2J-I_1I_2)/(4J^2-I_1I_2)}$. The scaling exponents are

$$w=\frac{4J^2+2I_1J-2I_1I_2}{4J^2-I_1I_2},\quad z=\frac{2I_1J-I_1I_2}{4J^2-I_1I_2},$$
$$\lambda=1,\quad \mu=\frac{4J^2-2I_1J}{4J^2-I_1I_2}\quad \text{for } A^1 \text{ clusters}, \tag{42}$$
$$w=\frac{4J^2+2I_2J-2I_1I_2}{4J^2-I_1I_2},\quad z=\frac{2I_2J-I_1I_2}{4J^2-I_1I_2},$$
$$\lambda=1,\quad \mu=\frac{4J^2-2I_2J}{4J^2-I_1I_2}\quad \text{for } A^2 \text{ clusters}.$$

The results show that the exponents w, z, and μ are dependent on the values of the reaction rates I_1, I_2, and J. In this case, both the total number and total mass of either species decrease with time. Neither A^1 species nor A^2 species remains at $t\to\infty$, independent of the initial data A_{10} and A_{20}.

In the n-species system ($n>2$), we assume that $A_{l0}(2J-I_l)=A_{10}(2J-I_1)$ for $1<l\leqslant n'$ and $A_{l'0}(2J-I_{l'})<A_{10}(2J-I_1)$ for $n'<l'\leqslant n$. Thus we obtain the following equations from Eqs. (11):

$$\alpha_l^{1-2J/I_l}=\alpha_1^{1-2J/I_1},\quad l=1,2,\ldots,n',$$
$$\frac{\alpha_1^{1-2J/I_1}-1}{A_{10}(I_1-2J)}=\frac{\alpha_{l'}^{1-2J/I_{l'}}-1}{A_{l'0}(I_{l'}-2J)},\quad l'=n'+1,n'+2,\ldots,n. \tag{43}$$

In this case, we find that $\alpha_l\gg 1$ $(l=1,2,\ldots,n')$ in the long-time limit. Substituting Eqs. (43) into Eqs. (12), we obtain the asymptotic differential equation for $\alpha_1(t)$:

$$\frac{d\alpha_1}{dt}\simeq\frac{I_1A_{10}}{2}\prod_{l'=n'+1}^{n}\left[1-\frac{A_{l'0}(2J-I_{l'})}{A_{10}(2J-I_1)}\right]^{2J/(2J-I_{l'})}\alpha_1^{\beta}, \tag{44}$$

where $\beta=[(I_1-2J)/I_1]\Sigma_{m=2}^{n'}[2J/(2J-I_m)]$. One can determine the asymptotic solution of α_1 from Eq. (44) as

$$\alpha_1\simeq C_{41}t^{R_{41}}, \tag{45}$$

where

$$R_{41}=I_1\bigg/\left[I_1-2J+2J\sum_{l=1}^{n'}(2J-I_1)/(2J-I_l)\right]$$

and

$$C_{41}=\prod_{l'=n'+1}^{n}\left[1-\frac{A_{l'0}(2J-I_{l'})}{A_{10}(2J-I_1)}\right]^{2JR_{41}/(2J-I_{l'})}\left(\frac{I_1A_{10}}{2R_{41}}\right)^{R_{41}}.$$

The solutions of α_l and $\alpha_{l'}$ can then be obtained as

$$\alpha_l\simeq C_{4l}t^{R_{4l}},\quad l=1,2,\ldots,n',$$
$$\alpha_{l'}\simeq C_{4l'}-C_{5l'}t^{-R_{5l'}},\quad l'=n'+1,n'+2,\ldots,n, \tag{46}$$

where

$$R_{4l}=I_l\left(I_l-2J+2J\sum_{m=1}^{n'}\frac{2J-I_l}{2J-I_m}\right)^{-1},$$
$$C_{4l}=C_{41}^{I_l(2J-I_1)/I_1(2J-I_l)},$$

$$C_{4l'}=\left[1-\frac{A_{l'0}(2J-I_{l'})}{A_{10}(2J-I_1)}\right]^{I_{l'}/(I_{l'}-2J)},$$

$$C_{5l'}=\frac{I_{l'}A_{l'0}C_{4l'}C_{41}^{1-2J/I_1}}{A_{10}(2J-I_1)-A_{l'0}(2J-I_{l'})},$$

$$R_{5l'}=\left(\sum_{m=1}^{n'}\frac{2J}{2J-I_m}-1\right)^{-1}.$$

Thus we determine the standard scaling description for A^l species in the long-time limit:

$$a_{lk}(t)\simeq\frac{2R_{4l}}{I_lC_{4l}}t^{-1-R_{4l}}\exp(-x_l),$$

$$x_l=\frac{k}{C_{4l}}t^{-R_{4l}},\quad l=1,2,\ldots,n', \tag{47}$$

and the scaling exponents for the A^l species are

$$w=\frac{2I_l-2J+2J\sum_{m=1}^{n'}(2J-I_l)/(2J-I_m)}{I_l-2J+2J\sum_{m=1}^{n'}(2J-I_l)/(2J-I_m)},$$

$$z=\frac{I_l}{I_l-2J+2J\sum_{m=1}^{n'}(2J-I_l)/(2J-I_m)},\quad \lambda=1, \tag{48}$$

$$\mu=\frac{2J\sum_{m=1}^{n'}(2J-I_l)/(2J-I_m)-2J}{I_l-2J+2J\sum_{m=1}^{n'}(2J-I_l)/(2J-I_m)},$$

which are similar to Eq. (21). Meanwhile, the standard scaling description (17) breaks down for $A^{l'}$ species and we obtain the modified scaling description for them:

$$a_{l'k}(t)\simeq\frac{2C_{5l'}R_{5l'}}{I_{l'}C_{4l'}^2}\left(\frac{C_{4l'}-1}{C_{4l'}}\right)^k t^{-1-R_{5l'}}\exp(-x_{l'}),$$

$$l'=n'+1,n'+2,\ldots,n \tag{49}$$

with the different scaling variables $x_{l'}=[C_{5l'}/C_{4l'}(C_{4l'}-1)]kt^{-R_{5l'}}$. The same exponents for all the $A^{l'}$ species are

$$w=\lambda=\mu=\frac{\sum_{m=1}^{n'}2J/(2J-I_m)}{\sum_{m=1}^{n'}2J/(2J-I_m)-1},$$

$$z=\frac{1}{\sum_{m=1}^{n'}2J/(2J-I_m)-1}. \tag{50}$$

The values of μ for A^l species ($l=1,2,\ldots,n'$), which have the largest value of $A_l(2J-I_l)$, are less than those of $A^{l'}$ species ($l'=n'+1,n'+2,\ldots,n$). This implies that A^l species dominate over $A^{l'}$ species at $t\gg1$. The results show that all the species annihilate each other completely and no species remains at last, independent of the initial data.

It can be concluded from the above analyses that the mass distribution of the species with the largest value of $A_{l0}(2J-I_l)$ satisfies the standard scaling law, while the standard scaling description breaks down for the species with smaller values of $A_{l0}(2J-I_l)$. If there is only a certain species that has the largest value of $A_{l0}(2J-I_l)$, it will remain in the end; meanwhile, all the other species will be annihilated completely. If there are more than two species having the same largest value of $A_{l0}(2J-I_l)$, all the species will be annihilated completely in the end.

D. Some aggregation rates less than 2J with others equal to or greater than 2J

Now we investigate the case of some aggregation rates being less than $2J$ while the others are equal to or greater than $2J$. Without any loss of generality, we assume $I_m<2J$ for $m=1,2,\ldots,n_0$ $(1\leqslant n_0<n)$ and $I_{m'}\geqslant 2J$ for $m'=n_0+1,n_0+2,\ldots,n$. This case is similar to that in Sec. II C. The characteristics of the cluster-mass distributions are concerned with the values of $A_{l0}(2J-I_l)$. For simplicity, we assume that $A_{10}(2J-I_1)\geqslant A_{l_10}(2J-I_{l_1})$ for $l_1=2,3,\ldots,n_0$. From Eqs. (10), one can obtain the following equations for this case:

$$\frac{\alpha_1^{1-2J/I_1}-1}{A_{10}(I_1-2J)}=\frac{\alpha_{l'}^{1-2J/I_{l'}}-1}{A_{l'0}(I_{l'}-2J)}$$

$$\text{for }I_{l'}\neq 2J,\quad l'=2,3,\ldots,n,$$

$$\alpha_{l'}=\exp\left[\frac{I_{l'}A_{l'0}}{A_{10}(I_1-2J)}(\alpha_1^{1-2J/I_1}-1)\right]\quad\text{for }I_{l'}=2J. \tag{51}$$

When $n_0=1$ or $A_{10}(2J-I_1)>A_{l'0}(2J-I_{l'})$, we find that $\alpha_1\to\infty$ and $\alpha_l\to b_{l'}$ ($b_{l'}$ are constants and $b_{l'}>1$, $l'=2,3,\ldots,n$) at $t\to\infty$. Thus one can obtain the asymptotic differential equation for α_l in the long-time limit,

$$\frac{d\alpha_1}{dt}\simeq\frac{I_1A_{10}}{2}\prod_{l'=2}^{n}C_{6l'}^{-2J/I_{l'}}, \tag{52}$$

where $C_{6l'}=[1-A_{l'0}(2J-I_{l'})/A_{10}(2J-I_1)]^{I_{l'}/(I_{l'}-2J)}$ for $I_{l'}\neq 2J$ and $C_{6l'}=\exp[A_{l'0}I_{l'}/A_{10}(2J-I_1)]$ for $I_{l'}=2J$. The solution for α_1 can then be given as

$$\alpha_1\simeq C_{61}t, \tag{53}$$

where $C_{61}=\frac{1}{2}I_1A_{10}\Pi_{l'=2}^{n}C_{6l'}^{-2J/I_{l'}}$. Substituting Eq. (53) into Eqs. (51), we obtain the asymptotic solutions of $\alpha_{l'}(t)$ as

$$\alpha_{l'}\simeq C_{6l'}-C_{7l'}t^{1-2J/I_1},\quad l'=2,3,\dots,n, \tag{54}$$

where $C_{7l'}=I_{l'}A_{l'0}C_{6l'}C_{61}^{1-2J/I_1}/[A_{10}(2J-I_1)-A_{l'0}(2J-I_{l'})]$. Equations (53) and (54) are similar to Eqs. (29) and (30). So this case has the same results as in Sec. II C. In the long-time limit, the cluster-mass distribution of A^1 species, which has the largest value of $A_{l0}(2J-I_l)$, has the same standard scaling description as Eq. (31), where C_{21} is substituted by C_{61}. Meanwhile, we also determine similar modified scaling descriptions as (33) for $A^{l'}$ species ($l'=2,3,\dots,n$), where $C_{2l'}$ and $C_{3l'}$ are substituted by $C_{6l'}$ and $C_{7l'}$, respectively. Hence, A^1 and $A^{l'}$ species have the same scaling exponents (32) and (36), respectively.

In the general case of $n_0>1$, we assume that $A_{l0}(2J-I_l)=A_{10}(2J-I)$ for $l=2,3,\dots,n''$ ($1<n''\leqslant n_0$) and $A_{l_10}(2J-I_{l_1})<A_{10}(2J-I_1)$ for $l_1=n''+1,n''+2,\dots,n_0$. In this case, we have $\alpha_l\to\infty$ ($l=1,2,\dots,n''$) and $\alpha_{l'}\to c_{l'}$ ($c_{l'}$ are constants and $c_{l'}>1$, $l'=n''+1,n''+2,\dots,n$) at $t\to\infty$. Thus in the long-time limit, we obtain the asymptotic differential equations for α_l from Eqs. (9),

$$\frac{d\alpha_l}{dt}\simeq\frac{I_lA_{l0}}{2}\prod_{l'=n''+1}^{n}C_{8l'}^{-2J/I_{l'}}\alpha_l^{R_{6l}},\quad l=1,2,\dots,n'', \tag{55}$$

where

$$R_{6l}=\frac{2J}{I_l}\left(1-\sum_{m=1}^{n''}\frac{2J-I_l}{2J-I_m}\right),$$

$$C_{8l'}=[1-A_{l'0}(2J-I_{l'})/A_{10}(2J-I_1)]^{I_{l'}/(I_{l'}-2J)}$$

for $I_{l'}\neq 2J$,

$$C_{8l'}=\exp[A_{l'0}I_{l'}/A_{10}(2J-I_1)]\quad\text{for}\quad I_{l'}=2J.$$

The asymptotic solutions for α_l and $\alpha_{l'}$ can be determined as follows:

$$\alpha_l\simeq C_{8l}t^{R_{7l}},\quad l=1,2,\dots,n'',$$

$$\alpha_{l'}\simeq C_{8l'}-C_{9l'}t^{-R_{8l'}},\quad l'=n''+1,n''+2,\dots,n, \tag{56}$$

where

$$R_{7l}=I_l\left(I_l-2J+2J\sum_{m=1}^{n''}\frac{2J-I_l}{2J-I_m}\right)^{-1},$$

$$C_{8l}=\left(\frac{I_lA_{l0}}{2R_{7l}}\prod_{l'=n''+1}^{n}C_{8l'}^{-2J/I_{l'}}\right)^{R_{7l}},$$

$$R_{8l'}=\left(\sum_{m=1}^{n''}\frac{2J}{2J-I_m}-1\right)^{-1},$$

$$C_{9l'}=\frac{C_{8l'}A_{l'0}I_{l'}}{A_{10}(2J-I_1)-A_{l'0}(2J-I_{l'})}C_{81}^{1-2J/I_1}.$$

Equations (56) are similar to Eqs. (46). Substituting for the constants C_{4l}, C_{5l}, $C_{5l'}$, and n' in Sec. II C with C_{8l}, $C_{8l'}$, $C_{9l'}$, and n'', respectively, we can obtain the modified equations (47)–(50) of the cluster-mass distributions and scaling exponents for this case. The results show that the kinetic behavior of this system is similar to that in Sec. II C. Moreover, if we have $n''=n'$, the results of this case are just identical to those in Sec. II C.

E. Some aggregation rates equal to 2*J* with the others greater than 2*J*

For simplicity, we assume $I_m=2J$ for $m=1,2,\dots,n_1$ ($1\leqslant n_1<n$) and $I_{m'}>2J$ for $m'=n_1+1,n_1+2,\dots,n$. Then we can derive the following equations from Eqs. (10):

$$\alpha_1^{A_{m0}}=\alpha_m^{A_{10}},\quad m=2,3,\dots,n_1,$$

$$\frac{\ln\alpha_1}{I_1A_{10}}=\frac{\alpha_{m'}^{1-2J/I_{m'}}-1}{A_{m'0}(I_{m'}-2J)},\quad m'=n_1+1,n_1+2,\dots,n. \tag{57}$$

Substituting Eqs. (57) into Eqs. (9), we obtain the differential equation of α_1 as follows:

$$\alpha_1^{\Sigma_{m_1=2}^{n_1}A_{m0}/A_{10}}\prod_{m'=n_1+1}^{n}\left[\ln\alpha_1+\frac{A_{10}I_1}{A_{m'0}(I_{m'0}-2J)}\right]^{2J/(I_{m'}-2J)}\frac{d\alpha_1}{dt}=\frac{1}{2}I_1A_{10}\prod_{m'=n_1+1}^{n}\left[\frac{A_{m'0}(I_{m'0}-2J)}{A_{10}I_1}\right]^{2J/(2J-I_{m'})}. \tag{58}$$

It can be decisively concluded from Eqs. (9) and (57) that $\alpha_l\to\infty$ ($l=2,3,\dots,n$) at $t\to\infty$. We integrate Eq. (58) and then derive the solution of α_1 in the implicit form at $t\gg 1$:

$$\sum_{i=1}^{\infty}\left[\left(-\sum_{m=1}^{n_1}\frac{A_{m0}}{A_{10}}\right)^{-i}\prod_{j=0}^{i-1}\left(\sum_{m'=n_1+1}^{n}\frac{2J}{I_{m'}-2J}-j\right)^{i}\times\alpha_1^{\Sigma_{m_1=1}^{n_1}A_{m0}/A_{10}}(\ln\alpha_1)^{-i+\Sigma_{m'=n_1+1}^{n}2J/(I_{m'}-2J)}\right]+\alpha_1^{\Sigma_{m_1=1}^{n_1}A_{m0}/A_{10}}(\ln\alpha_1)^{\Sigma_{m'=n_1+1}^{n}2J/(I_{m'}-2J)}\simeq\frac{1}{2}I_1\sum_{m=1}^{n_1}A_{m0}\prod_{m'=n_1+1}^{n}\left[\frac{A_{m'0}(I_{m'}-2J)}{A_{10}I_1}\right]^{2J/(2J-I_{m'})}t. \tag{59}$$

If there exists integral N satisfying the equation $\Sigma_{m=n_1+1}^{n}2J/(I_m-2J)-N=0$, the infinite terms in Eq. (59) will be simplified to finite terms of $i=1\sim N$. In the long-time limit, the value of the summation in Eq. (59) is far smaller than that of $\alpha_1^{\Sigma_{m_1=1}^{n_1}A_{m0}/A_{10}}(\ln\alpha_1)^{\Sigma_{m'=n_1+1}^{n}2J/(I_{m'}-2J)}$ and is negligible. Then Eq. (59) reduces to

$$\alpha_1^{\Sigma_{m_1=1}^{n_1}A_{m0}/A_{10}}(\ln\alpha_1)^{\Sigma_{m'=n_1+1}^{n}2J/(I_{m'}-2J)} \simeq \frac{1}{2}I_1\sum_{m=1}^{n_1}A_{m0}\prod_{m'=n_1+1}^{n}\left[\frac{A_{m'0}(I_{m'}-2J)}{A_{10}I_1}\right]^{2J/(2J-I_{m'})}t. \tag{60}$$

The asymptotic solution for α_1 in the long-time limit can be given as

$$\alpha_1\simeq C_{101}t^{R_{91}}(\ln t)^{R'_{91}}, \tag{61}$$

where $C_{101}=\{\frac{1}{2}I_1\Sigma_{m_1=1}^{n_1}A_{m0}\Pi_{m'=n_1+1}^{n}[A_{m'0}(I_{m'0}-2J)/A_{10}I_1]^{2J/(2J-I_{m'})}\}^{R_{91}}$, $R_{91}=(\Sigma_{m_1=1}^{n_1}A_{m0}/A_{10})^{-1}$, and $R'_{91}=\Sigma_{m'=n_1+1}^{n}2JR_{91}/(2J-I_{m'})$. Substituting Eq. (61) into Eqs. (57), one can derive the asymptotic solutions for α_m and $\alpha_{m'}$ as follows:

$$\alpha_m\simeq C_{10m}t^{R_{9m}}(\ln t)^{R'_{9m}}, \quad m=1,2,\ldots,n_1,$$

$$\alpha_{m'}\simeq C_{10m'}(\ln t)^{I_{m'}/(I_{m'}-2J)}, \quad m'=n_1+1,n_1+2,\ldots,n, \tag{62}$$

where $R_{9m}=(\Sigma_{m_1=1}^{n_1}A_{m_10}/A_{m0})^{-1}$, $R'_{9m}=\Sigma_{m'=n_1+1}^{n}\times 2JR_{9m}/(2J-I_{m'})$, $C_{10m}=C_{101}^{A_{m0}/A_{10}}$, and $C_{10m'}=[\{A_{m'0}(I_{m'}-2J)/2JA_{m0}\}R_{9m}]^{I_{m'}/(I_{m'}-2J)}$. The asymptotic solutions for the long-time mass distributions of A^m and $A^{m'}$ clusters are then obtained as follows:

$$a_{mk}(t)\simeq\frac{2R_{9m}}{I_mC_{10m}}t^{-1-R_{9m}}(\ln t)^{-R'_{9m}}\exp(-x_m), \quad m=1,2,\ldots,n_1, \tag{63}$$

$$a_{m'k}(t)\simeq\frac{2C_{10m'}^{-1}}{I_{m'}-2J}t^{-1}(\ln t)^{-(2I_{m'}-2J)/(I_{m'}-2J)}\times\exp(-x_{m'}), \quad m'=n_1+1,n_1+2,\ldots,n,$$

which are valid in the scaling regions $k\gg1$, $t\gg1$, $x_m=(k/C_{10m})t^{-R_{9m}}(\ln t)^{-R'_{9m}}=$ finite, and $x_{m'}=(k/C_{10m'})(\ln t)^{-I_{m'}/(I_{m'}-2J)}=$finite, respectively.

The solutions show that the standard scaling description (18) of the cluster-mass distribution breaks down for all the A^l clusters $(l=1,2,\ldots,n)$ in this case and they come into a rather peculiar scaling regime. We may modify the above scaling description (18) further into

$$c_k(t)\simeq C_0h^k[g(t)]^{-w_1}[f(t)]^{-w_2}\Phi[k/S(t)],$$

$$S(t)\propto[g(t)]^{z_1}[f(t)]^{z_2}, \quad g'(t),f'(t)>0, \tag{64}$$

where C_0 and h denote two constants, and $0<h\leqslant1$. $g(t)$ and $f(t)$ are unusual functions of time, such as e^t, $\ln t$, 2^t, and so on.

The total number and the total mass of the clusters can then be rewritten as

$$N(t)=\sum_{k=1}^{\infty}c_k(t)\propto[g(t)]^{-\lambda_1}[f(t)]^{-\lambda_2},$$

$$M(t)=\sum_{k=1}^{\infty}kc_k(t)\propto[g(t)]^{-\mu_1}[f(t)]^{-\mu_2}. \tag{65}$$

The exponent relations are derived from Eqs. (64) and (65) to be

$$\lambda_1=w_1-z_1, \quad \mu_1=w_1-2z_1, \quad \lambda_2=w_2-z_2,$$

$$\mu_2=w_2-2z_2 \quad \text{for } h=1, \tag{66}$$

$$\lambda_1=w_1, \quad \mu_1=w_1, \quad \lambda_2=w_2, \quad \mu_2=w_2 \quad \text{for } 0<h<1.$$

When $f(t)=g(t)\equiv t$, one can find the relations between (w,z,λ,μ) and $(w_1,w_2,z_1,z_2,\lambda_1,\lambda_2,\mu_1,\mu_2)$ as

$$w=w_1+w_2, \quad z=z_1+z_2,$$

$$\lambda=\lambda_1+\lambda_2, \quad \mu=\mu_1+\mu_2. \tag{67}$$

In this case, we find the scaling exponents by letting $g(t)\equiv t$ and $f(t)\equiv\ln t$ for A^m clusters $(m=1,2,\ldots,n_1)$

$$w_1=\frac{A_{m0}+\sum_{m_1=1}^{n_1}A_{m_10}}{\sum_{m_1=1}^{n_1}A_{m_10}},$$

$$w_2=z_2=-\mu_2=\frac{\sum_{m'=n_1+1}^{n}2J/(2J-I_{m'})}{\sum_{m_1=1}^{n_1}A_{m_10}/A_{m0}}, \tag{68}$$

$$z_1=\frac{A_{m0}}{\sum_{m_1=1}^{n_1}A_{m_10}}, \quad \lambda_1=1, \quad \lambda_2=0,$$

$$\mu_1=\frac{\sum_{m_1=1}^{n_1}A_{m_10}-A_{m0}}{\sum_{m_1=1}^{n_1}A_{m_10}},$$

and $A^{m'}$ clusters $(m'=n_1+1,n_1+2,\ldots,n)$

$$w_1=1,\quad w_2=\frac{2I_{m'}-2J}{I_{m'}-2J},\quad z_1=0,\ z_2=\frac{I_{m'}}{I_{m'}-2J},$$

$$\lambda_1=1,\ \lambda_2=1, \tag{69}$$

$$\mu_1=1,\quad \mu_2=\frac{2J}{2J-I_{m'}}\quad \text{for } A^{m'} \text{ clusters.}$$

For the two-species system, it is obvious that $n_1=1$. Equations (68) and (69) are then simplified as

$$w_1=2,\quad w_2=\frac{2J}{2J-I_2},\quad z_1=1,\quad z_2=\frac{2J}{2J-I_2},$$

$$\lambda_1=1,\quad \lambda_2=0, \tag{70}$$

$$\mu_1=0,\quad \mu_2=\frac{2J}{I_2-2J}\quad \text{for } A^1 \text{ clusters,}$$

$$w_1=1,\quad w_2=\frac{2I_2-2J}{I_2-2J},\quad z_1=0,\quad z_2=\frac{I_2}{I_2-2J},$$

$$\lambda_1=1,\quad \lambda_2=1, \tag{71}$$

$$\mu_1=1,\quad \mu_2=\frac{2J}{2J-I_2}\quad \text{for } A^2 \text{ clusters.}$$

It is shown that the exponents are dependent only on the larger aggregation rate I_2 $(I_2>I_1=2J)$ and the annihilation rate J, and they are independent of the initial data. Comparison between the total masses $M_1(t)\propto(\ln t)^{-2J/(I_2-2J)}$ of A^1 clusters and $M_2(t)\propto t^{-1}(\ln t)^{2J/(I_2-2J)}$ of A^2 clusters shows that A^1 clusters dominate over A^2 clusters in the long-time limit. The results also show that both the total number and total mass of either species decrease with time. In this case, the two species annihilate each other completely and no clusters remain at the end.

When the system consists of n kinds of species $(n>2)$ and there is only one species A^1 whose aggregation rate is equal to $2J$, Eqs. (68) and (69) can be rewritten as

$$w_1=2,\quad w_2=\sum_{m'=2}^{n}\frac{2J}{2J-I_{m'}},\quad z_1=1,$$

$$z_2=\sum_{m'=2}^{n}\frac{2J}{2J-I_{m'}},\quad \lambda_1=1,\quad \lambda_2=0, \tag{72}$$

$$\mu_1=0,\quad \mu_2=\sum_{m'=2}^{n}\frac{2J}{I_{m'}-2J}\quad \text{for } A^1 \text{ clusters,}$$

$$w_1=1,\quad w_2=\frac{2I_{m'}-2J}{I_{m'}-2J},\quad z_1=0,$$

$$z_2=\frac{I_{m'}}{I_{m'}-2J},\quad \lambda_1=1,\quad \lambda_2=1, \tag{73}$$

$$\mu_1=1,\quad \mu_2=\frac{2J}{2J-I_{m'}}$$

$$\text{for } A^{m'} \text{ clusters,}\quad m'=2,3,\ldots,n.$$

The results show that the exponents also depend on the larger aggregation rates $I_{m'}$ $(I_{m'}>I_1=2J,\ m'=2,3,\ldots,n)$ and the annihilation rate J, and they are independent of the aggregation rate of A^1 species and all the initial concentrations. In this case, the large A^1 clusters dominate over the corresponding clusters of the other species in the long-time limit. For $A^{m'}$ species $(m'=2,3,\ldots,n)$, the species with lower aggregation rate has the minimum value of μ_2 and thus dominates over the others in the long-time limit, which is also independent of the initial data. The results imply that no clusters remain in the end.

In the general cases of $n>2$ and $n_1>1$, Eq. (70) indicates that the exponents of A^m clusters $(m=1,2,\ldots,n_1)$ depend both on the reaction rates of $A^{m'}$ species $(m'=n_1+1,n_1+2,\ldots,n)$ and on the initial concentrations of A^m species. But the exponents of $A^{m'}$ clusters depend only on their own reaction rates. It is obvious that in this case $\mu_1<1$ for A^m species $(m=1,2,\ldots,n_1)$ while $\mu_1=-1$ for all the $A^{m'}$ species $(m'=n_1+1,n_1+2,\ldots,n)$. Making a comparison between the total masses $M_m(t)\propto t^{-\mu_1}(\ln t)^{-\mu_2}$ of A^m species and $M_{m'}(t)\propto t^{-1}(\ln t)^{-\mu_2'}$ of $A^{m'}$ species, one finds that A^m clusters dominate over $A^{m'}$ clusters in the long-time limit, independent of the initial data. It is not surprising because we have assumed that A^m species have lower aggregation rates $(I_m=2J)$ than $A^{m'}$ species $(I_{m'}>2J)$. It is also shown that for all the A^m species $(m=1,2,\ldots,n_1)$, the species with largest initial concentration has the minimum value of μ_1 and thus dominates over the others. Moreover, all the kinds of species annihilate each other completely in the end.

III. SUMMARY

We studied an irreversible aggregation-annihilation system consisting of n kinds of distinct species on the basis of the mean-field theory. Considering the constant-reaction-rate model, we analyzed the kinetic behavior of the aggregation process with complete annihilation. In the first case of all the aggregation rates being greater than $2J$, we found that the cluster-mass distribution of each species obeys the standard scaling law in the long-time limit, and its exponents are dependent on the aggregation rates and the annihilation rate, but independent of all the initial concentrations A_{l0} $(l=1,2,\ldots,n)$. In the second case of all the aggregation rates being equal to $2J$, the system also has the standard scaling description for the mass distribution of each species, but the scaling exponents depend only on the initial concentrations. In the third case of all the aggregation rates being less than $2J$ and in the fourth case of some aggregation rates being less than $2J$, we have found that only a certain species which has the largest value of $A_{l0}(2J-I_l)$ satisfies the standard scaling law, and the exponents are dependent only on the aggregation rates and the annihilation rate. Meanwhile, the standard scaling description breaks down for the other species with the smaller values of $A_{l0}(2J-I_l)$, and they have other modified scaling behaviors. In the fifth case of some aggregation rates being equal to $2J$ and others greater than $2J$, we find that no species has standard scaling behavior, but they satisfy modified scaling laws.

In any case, the evolution behavior of the total number and total mass of the clusters of all the species strongly depends on the reaction rates and the initial concentrations. When all the aggregation rates are equal to or greater than $2J$, both the total number and total mass of each species decrease with time and no species remains at $t\rightarrow\infty$. In the case that there are more than one species whose aggregation rates are less than $2J$ while the others are equal to or greater than $2J$, whether a certain species still remains in the end or all the species annihilate completely is strongly dependent on both the reaction rates and the initial concentrations. If there is only one species which has the largest value of $A_{l0}(2J-I_l)$ among those species whose aggregation rates are less than $2J$, it dominates over the others in the long-time limit and finally remains alone. In short, the aggregation process with complete annihilation always satisfies a standard scaling or modified scaling law and its exponents strongly depend on the reaction rates. Of course, the initial concentrations also play an important role in the evolution behavior of the system.

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