Exact calculation of the angular momentum loss, recoil force, and radiation intensity for an arbitrary source in terms of electric, magnetic, and toroid multipoles

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An exact calculation of the radiation intensity, angular momentum loss, and the recoil force for the most general type of source, characterized by electric, magnetic, and toroid multipole moments and radii of any multipolarity and an arbitrary time dependence, is presented. The results are expressed in terms of time derivatives of the multipole moments and mean radii of the corresponding distributions. Although quite cumbersome, the formulas found by us represent exact results in the correct multipole analysis of configurations of charges and currents that contain toroidal sources. So the longstanding problem in classical electrodynamics of relating the radiation properties of a system to quantities completely describing its internal electromagnetic structure is thereby exactly solved. By particularizations to the first multipole contributions, corrections to the familiar formulas from books are found, mostly on account of the toroid moments and their interference with the usual electric and magnetic ones.

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I. INTRODUCTION

While in the radiation emitted by a system there are two families of waves (El and Ml), the system itself is described by three families of multipoles (electric, magnetic, and toroid). The subject of toroid moments may look somewhat unusual to some readers, although Blatt and Weisskopf [1], for instance, have made some remarks about multipole contributions coming from the induction currents (the curl of magnetization) in the situation when multipole expansions are carried out in connection with a magnetically polarizable medium. Zeldovich [2] was the first to note that a closed toroidal current (which cannot be reduced to a usual charge and magnetic multipole moment) represents, in fact, a certain new kind of dipole (it selects a direction in space as the dipoles do). He did that in the context of the violations of discrete space-time symmetries, when he observed that a spin 1/2 particle, for instance, might possess, besides the usual electric and magnetic dipole moments, a third kind of dipole characteristic, to which he gave the interpretation in terms of a toroidal current. Although by the Glashow-Salam-Weinberg electroweak theory every lepton and quark must possess one such toroid dipole, the first experimental evidence of a nuclear spin-dependent contribution to atomic parity nonconservation arising on account of a nuclear toroidal dipole, came thirty years later and is actively investigated nowadays [3]. Theoretically, in work summarized in the review papers [4–6], by clarifying and generalizing Zeldovich's original idea, an entire class of toroidal multipoles was shown to be needed in order to achieve a correct and complete multipole parametrization of the most general type of source in both classical and quantum electrodynamics. Toroidal moments were investigated in various contexts and research areas ranging from classical electrodynamics to elementary particles, nuclear, atomic, molecular, and solid state physics. References to previous work may be found in the above mentioned reviews. We particularly note the following: work done in connection with parity nonconservation in atoms [7], the calculation done by Ginzburg and Tsytovich [8] for the Cherenkov radiation emitted by a classical pointlike toroidal dipole, electromagnetic properties of toroidal solenoids [9], toroidal electromagnetic structure of Majorana fermions [10,11], induced toroidal moments and toroidal polarizabilities [12,13], intrinsic toroidal moments of certain molecules arising even in the framework of the usual parity conserving electromagnetic interaction on account of the intricate internal structure of the molecule [14], work in condensed matter physics by Dubovik and collaborators [15], study of the fields of moving toroid dipoles [16], etc. We note also our own work [17] on the toroidal moments of the knotted linear currents.

If a system of charges and currents, besides the usual electric and magnetic (time dependent) multipole moments does possess also time varying toroid moments and distributions, there will be, in general, additional contributions to the radiation intensity, the angular momentum loss and the recoil force due to the radiation of electromagnetic waves by the toroidal sources. In a recent paper [18] the classical electrodynamics formula for the rate of angular momentum loss by a time-dependent toroid dipole has been derived and discussed in connection with a forced precession of the toroid dipole around a given axis. Actually the problem of the new (with respect to the usual electric and magnetic multipole moments and distributions) toroid contributions to the radiation intensity, momentum, and angular momentum loss, within classical electrodynamics, can be solved exactly for any multipole order l, not only in the dipole case ($l=1$), and we do this in the present paper. Indeed, the complete multipole analysis of Dubovik and Tscheshkov [4,6] is used in the present work to achieve an exact calculation of the radiation intensity, angular momentum loss, and recoil force for the most general source that includes all types of electric, magnetic, and toroid moments and distributions of any multipolarity order and an arbitrary time dependence. The results are expressed in terms of time derivatives of the mean-square radii of any order n ($n=0$ order means the corresponding multipole moment itself). Although quite long, the formulas found by us are exact results in the correct multipole analysis in the most general situation. By retaining from them only contributions of the first multipoles, we find corrections to the familiar formulas from books, mostly on account of the toroid moments and their interference with the usual electric and magnetic ones.

Due to the complexity of the calculations and in order to fix notations and conventions, we have to go to much detail either in the presentation of the general formalism of Refs. [4], [6] or in the description of our own results obtained on its basis, so that this paper contains also some parts essentially methodological, aimed to achieve a better clarity.

The paper is organized as follows: we shall continue to present in this Introduction (Sec. I) the main formulas from Ref. [4], with minor corrections and changes of notations, in order to have at our disposal a sound basis for the next lengthy calculations. In Sec. II, the exact formulas for the potentials and fields, expressed either in terms of the electric, magnetic, and toroid multipole form factors or, alternatively, in terms of time derivatives of mean radii are presented. Section III contains the presentation of the expressions of the fields at large distances r, in order $O(1/r)$ [and $O(1/r^2)$ for those combinations needed in the course of the paper]. Sections IV, V, and VI contain, respectively, the calculations of the radiation intensity, angular momentum loss, and recoil force. Section VII is devoted to conclusions and a short discussion of the relation between our work and previous results given in the literature. A number of appendixies are included to help the reader as much as possible to follow and control the calculations.

In Ref. [4], it has been shown that the most general distribution of charges and currents expressed by the charge density $\rho(\vec{r},t)$ and current density $\vec{j}(\vec{r},t)$, related by the continuity relation

$$\frac{\partial\rho(\vec{r},t)}{\partial t}+\vec{\nabla}\cdot\vec{j}(\vec{r},t)=0, \tag{1.1}$$

can be completely parametrized in terms of three families of electric, magnetic, and toroid multipole form factors $Q_{lm}(-k^2,t),M_{lm}(-k^2,t),T_{lm}(-k^2,t)$ as follows:

$$\rho(\vec{r},t)=\frac{1}{(2\pi)^3}\sum_{l,m,k}(-ik)^l\frac{\sqrt{4\pi(2l+1)}}{(2l+1)!!}\times Q_{lm}(-k^2,t)\mathcal{F}_{lmk}(\vec{r}), \tag{1.2}$$

$$\begin{aligned}\vec{j}(\vec{r},t)=&\frac{c}{(2\pi)^3}\sum_{l,m,k}(-ik)^{l-1}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\\&\times\Bigg\{kM_{lm}(-k^2,t)\vec{\mathcal{F}}^{(0)}_{lmk}(\vec{r})\\&+\left[\frac{1}{c}\dot{Q}_{lm}(0,t)+k^2T_{lm}(-k^2,t)\right]\\&\times\vec{\mathcal{F}}^{(+)}_{lmk}+\frac{1}{c}\sqrt{\frac{l}{l+1}}\dot{Q}_{lm}(-k^2,t)\vec{\mathcal{F}}^{(-)}_{lmk}(\vec{r})\Bigg\},\end{aligned} \tag{1.3}$$

$$\sum_k=\int_0^\infty k^2dk,\quad m=-l,\dots l,\quad l=0,1,2\ \dots .$$

The sums over l in the equations above and throughout below starts at $l=0$ for the electric multipole form factors and at $l=1$ for the magnetic and toroid ones.

The dot over Q_{lm} means derivation with respect to t. $\mathcal{F}_{lmk}(\vec{r})$ is the system of regular solutions of the Helmholtz equation,

$$(\Delta+k^2)\mathcal{F}_{lmk}(\vec{r})=0, \tag{1.4}$$

$$\mathcal{F}_{lmk}(\vec{r})=j_l(kr)Y_{lm}(\vec{n}),\quad \vec{n}=\frac{\vec{r}}{r}, \tag{1.5}$$

$$j_l(kr)=(2\pi)^{3/2}i^lJ_{l+1/2}(kr)/\sqrt{kr},$$

where j_l and $J_{l+1/2}$ are spherical and cylindrical Bessel functions (see Appendix A for conventions and properties) while Y_{lm} are the usual spherical harmonics. The normalization and completeness conditions are

$$\int\mathcal{F}_{lmk}(\vec{r})\mathcal{F}^*_{l'm'k'}(\vec{r})d^3r=\delta_{ll'}\delta_{mm'}\frac{(2\pi)^3}{k^2}\delta(k-k'), \tag{1.6}$$

$$\sum_{l,m,k}\mathcal{F}_{lmk}(\vec{r})\mathcal{F}^*_{lmk}(\vec{r}\,')=(2\pi)^3\delta(\vec{r}-\vec{r}\,'),$$

$$\sum_k=\int_0^\infty k^2dk, \tag{1.7}$$

$$\mathcal{F}_{lmk}(-\vec{r})=(-1)^l\mathcal{F}_{lmk}(\vec{r}). \tag{1.8}$$

The basis vector functions $\vec{\mathcal{F}}^{(\lambda)}_{lmk}(\vec{r})$ $(\lambda=-,0,+)$ are solutions of the vector Helmholtz equation,

$$(\Delta+k^2)\vec{\mathcal{F}}_{lmk}(\vec{r})=0, \tag{1.4$'$}$$

$$\vec{\mathcal{F}}^{(0)}_{lmk}(\vec{r})=\frac{i}{\sqrt{l(l+1)}}\vec{\nabla}\times\{\vec{r}\mathcal{F}_{lmk}(\vec{r})\}=j_l(kr)\vec{Y}_{llm}(\vec{n}), \tag{1.9}$$

$$\begin{aligned}\vec{\mathcal{F}}^{(+)}_{lmk}(\vec{r})&=\frac{-1}{\sqrt{l(l+1)}}\frac{i}{k}\vec{\nabla}\times\vec{\nabla}\times\{\vec{r}\mathcal{F}_{lmk}(\vec{r})\}\\&=\frac{1}{\sqrt{2l+1}}[\sqrt{l}j_{l+1}(kr)\vec{Y}_{ll+1m}(\vec{n})\\&\quad+\sqrt{l+1}j_{l-1}(kr)\vec{Y}_{ll-1m}(\vec{n})],\end{aligned} \tag{1.10}$$

$$\begin{aligned}\vec{\mathcal{F}}^{(-)}_{lmk}(\vec{r})&=-\frac{i}{k}\vec{\nabla}\mathcal{F}_{lmk}(\vec{r})=\frac{1}{\sqrt{2l+1}}[\sqrt{l}j_{l-1}(kr)\vec{Y}_{ll-1m}(\vec{n})\\&\quad-\sqrt{l+1}j_{l+1}(kr)\vec{Y}_{ll+1m}(\vec{n})].\end{aligned} \tag{1.11}$$

The spherical vectors are

$$[\vec{Y}_{ll'm}(\vec{n})]_\mu = \sum_{m'} C^{l'}_{m'}\ {}^{1}_{\mu}\ {}^{l}_{m} Y_{l'm'}(\vec{n}), \quad \mu = -1,0,1. \tag{1.12}$$

$\vec{\mathcal{F}}^{(\lambda)}_{lmk}(\vec{r})$, $(\lambda = 0,\pm)$ satisfy the normalization and completeness conditions,

$$\int \vec{\mathcal{F}}^{(\lambda)*}_{lmk}(\vec{r}) \cdot \vec{\mathcal{F}}^{(\lambda')}_{l'm'k'}(\vec{r}) d^3\vec{r} = \delta_{ll'}\delta_{mm'}\delta_{\lambda\lambda'}\frac{(2\pi)^3}{k^2}\delta(k-k'), \tag{1.13}$$

$$\sum_{l,m,k,\lambda} [\vec{\mathcal{F}}^{(\lambda)}_{lmk}(\vec{r})]^*_i [\vec{\mathcal{F}}^{(\lambda)}_{lmk}(\vec{r}\,')]_j = (2\pi)^3 \delta_{ij}\delta(\vec{r}-\vec{r}\,'), \tag{1.14}$$

$$\vec{\mathcal{F}}^{(\lambda)}_{lmk}(-\vec{r}) = (-1)^{l+\lambda}\vec{\mathcal{F}}^{(\lambda)}_{lmk}(\vec{r}), \quad \lambda = 0,\pm. \tag{1.15}$$

One has also

$$\vec{\mathcal{F}}^{(+)}_{lmk}(\vec{r}) = -\frac{1}{k\sqrt{l(l+1)}}\vec{\nabla}\times\vec{L}\mathcal{F}_{lmk}(\vec{r}) = -\frac{1}{k}\vec{\nabla}\times\vec{\mathcal{F}}^{(0)}_{lmk}(\vec{r}), \tag{1.16}$$

$$\vec{\mathcal{F}}^{(0)}_{lmk}(\vec{r}) = \frac{1}{\sqrt{l(l+1)}}\vec{L}\mathcal{F}_{lmk}(\vec{r}) = -\frac{1}{k}\vec{\nabla}\times\vec{\mathcal{F}}^{(+)}_{lmk}(\vec{r}), \tag{1.17}$$

$$\vec{L} = -i\vec{r}\times\vec{\nabla}. \tag{1.18}$$

$\vec{\mathcal{F}}^{(-)}_{lmk}$ is longitudinal,

$$\vec{\nabla}\times\vec{\mathcal{F}}^{(-)}_{lmk}(\vec{r}) = \vec{0}, \tag{1.19}$$

$$\vec{\nabla}\cdot\vec{\mathcal{F}}^{(-)}_{lmk}(\vec{r}) = ik\mathcal{F}_{lmk}(\vec{r}), \tag{1.20}$$

while $\vec{\mathcal{F}}^{(0)}_{lmk}$, $\vec{\mathcal{F}}^{(+)}_{lmk}$ are transversal,

$$\vec{\nabla}\cdot\vec{\mathcal{F}}^{(+)}_{lmk}(\vec{r}) = \vec{\nabla}\cdot\vec{\mathcal{F}}^{(0)}_{lmk}(\vec{r}) = 0. \tag{1.21}$$

So, instead of $\rho(\vec{r},t)$, $\vec{j}(\vec{r},t)$, one has an equivalent description of the system of charges and currents in terms of the electric, magnetic, and toroid formfactors $Q_{lm}(-k^2,t)$, $M_{lm}(-k^2,t)$, $T_{lm}(-k^2,t)$, which, by inversion of Eqs. (1.2), (1.3), are

$$Q_{lm}(-k^2,t) = \frac{(2l+1)!!}{(-ik)^l\sqrt{4\pi(2l+1)}}\int \rho(\vec{r},t) j^*_l(kr) Y^*_{lm}(\vec{n}) d^3r, \tag{1.22}$$

$$Q^*_{lm}(-k^2,t) = (-1)^m Q_{l,-m}(-k^2,t), \tag{1.22$'$}$$

$$M_{lm}(-k^2,t) = \frac{-i(2l+1)!!}{c(-ik)^l\sqrt{4\pi(2l+1)(l+1)/l}} \times \int j^*_l(kr)\vec{Y}^*_{llm}(\vec{n})\cdot\vec{j}(\vec{r},t)d^3r, \tag{1.23}$$

$$M^*_{lm}(-k^2,t) = (-1)^m M_{l,-m}(-k^2,t), \tag{1.23$'$}$$

$$T_{lm}(-k^2,t) = \frac{-(2l-1)!!\sqrt{l}}{c(-ik)^{l+1}\sqrt{4\pi(l+I)}} \times \int \Big\{ \sqrt{l} j^*_{l+1}(kr)\vec{Y}^*_{ll+1m}(\vec{n}) + \sqrt{l+1}\Big[j^*_{l-1}(kr) - \frac{4\pi(-ikr)^{l-1}}{(2l-1)!!}\Big] \times \vec{Y}^*_{ll-1m}(\vec{n})\Big\}\vec{j}(\vec{r},t)d^3r, \tag{1.24}$$

$$T^*_{lm}(-k^2,t) = (-1)^m T_{l,-m}(-k^2,t). \tag{1.24$'$}$$

Using

$$\mathcal{F}_{lmk}(\vec{r}) \underset{r\to 0}{\sim} \frac{4\pi(ikr)^l}{(2l+1)!!}Y_{lm}(\vec{n}), \tag{1.25}$$

the electric formfactors $Q_{lm}(-k^2,t)$ of Eq. (1.22) give in the $k^2\to 0$ limit the usual electric multipole moments,

$$Q_{lm}(t) = Q_{lm}(0,t) = \frac{\sqrt{4\pi}}{\sqrt{2l+1}}\int r^l Y^*_{lm}(\vec{n})\rho(\vec{r},t)d^3r. \tag{1.26}$$

Developing the formfactors $Q_{lm}(-k^2,t)$ in Taylor series [with respect to the first argument, $(-k^2)$], one gets the mean $2n$-order radii of the 2^l-pole charge distribution,

$$Q_{lm}(-k^2,t) = \sum_{n=0}^{\infty}\frac{(-k^2)^n}{n!}Q^{[n]}_{lm}(t), \tag{1.27}$$

$$Q^{[n]}_{lm}(t) = \frac{d^n}{d(-k^2)^n}Q_{lm}(-k^2,t)\big|_{k^2=0},$$

$$\overline{r^{2n}_{lm}}(t) = \frac{2^n(2l+2n+1)!!}{(2l+1)!!}Q^{[n]}_{lm}(t) = \frac{\sqrt{4\pi}}{\sqrt{2l+1}}\int r^{l+2n}Y^*_{lm}(\vec{n})\rho(\vec{r},t)d^3r. \tag{1.28}$$

To find the correct factors relating the higher-order derivatives of the formfactors to the multipole mean radii of various orders, one uses relations of the type

$$\frac{d^n}{d(-k^2)^n}\left[\frac{j^*_l(kr)}{k^l}\right] = \frac{(ir)^n}{2^n}\frac{j^*_{l+n}(kr)}{k^{l+n}}. \tag{1.29}$$

Radii of zero order ($n=0$), i.e., the first term in Eq. (1.27) are just the multipole moments themselves,

$$\overline{r_{lm}^{0}}(t)=Q_{lm}(k^2=0,t)=Q_{lm}^{[0]}(t)=Q_{lm}(t). \qquad (1.30)$$

The same situation holds for the magnetic and toroid multipole form factors,

$$M_{lm}(-k^2,t)=\sum_{n=0}^{\infty}\frac{(-k^2)^n}{n!}M_{lm}^{[n]}(t), \qquad (1.31)$$

$$M_{lm}^{[n]}(t)=\frac{d^n}{d(-k^2)^n}M_{lm}(-k^2,t)|_{k^2=0}, \qquad (1.31')$$

$$T_{lm}(-k^2,t)=\sum_{n=0}^{\infty}\frac{(-k^2)^n}{n!}T_{lm}^{[n]}(t), \qquad (1.32)$$

$$T_{lm}^{[n]}(t)=\frac{d^n}{d(-k^2)^n}T_{lm}(-k^2,t)|_{k^2=0}. \qquad (1.32')$$

The multipole magnetic moments are

$$\begin{aligned}M_{lm}(t)&=M_{lm}^{[0]}(t)=M_{lm}(0,t)=\frac{1}{l+1}\frac{\sqrt{4\pi}}{c\sqrt{2l+1}}\\&\times\int r^l[\vec{r}\times\vec{j}(\vec{r},t)]\cdot\vec{\nabla}Y_{lm}^{*}(\vec{n})d^3r\\&=-\frac{i}{c}\frac{\sqrt{4\pi l}}{\sqrt{(2l+1)(l+1)}}\int d^3r r^l\vec{Y}_{llm}^{*}(\vec{n})\cdot\vec{j}(\vec{r},t)\end{aligned} \qquad (1.33)$$

and the toroid ones,

$$\begin{aligned}T_{lm}(t)&=T_{lm}^{[0]}(t)=T_{lm}(0,t)\\&=-\frac{\sqrt{\pi l}}{c(2l+1)}\int r^{l+1}\Bigg\{\vec{Y}_{ll-1m}^{*}(\vec{n})\\&\quad+\frac{2\sqrt{l/(l+1)}}{2l+3}\vec{Y}_{ll+1m}^{*}(\vec{n})\Bigg\}\cdot\vec{j}(\vec{r},t)d^3r,\end{aligned} \qquad (1.34)$$

the radii of various $2n$ order are connected to the derivatives of the corresponding formfactors by

$$\begin{aligned}\overline{\rho_{lm}^{2n}}(t)&=\frac{2^n(2l+2n+1)!!}{(2l+1)!!}M_{lm}^{[n]}(t)\\&=-\frac{i}{c}\frac{\sqrt{4\pi l}}{\sqrt{(l+1)(2l+1)}}\int r^{2n+l}\vec{Y}_{llm}^{*}(\vec{n})\cdot\vec{j}(\vec{r},t)d^3r,\end{aligned} \qquad (1.35)$$

$$\overline{\rho_{lm}^{0}}=M_{lm}(k^2=0,t)=M_{lm}^{[0]}(t)=M_{lm}(t), \qquad (1.36)$$

in the magnetic case and by

$$\begin{aligned}\overline{R_{lm}^{2n}}(t)&=\frac{2^n(2l+2n+1)!!}{(2l+1)!!}T_{lm}^{[n]}(0,t)\\&=\frac{-1}{c(2l+1)}\sqrt{\frac{4\pi l}{l+1}}\int d^3r r^{l+2n+1}\Bigg[\frac{\sqrt{l}}{(2l+2n+3)}\\&\quad\times\vec{Y}_{ll+1m}^{*}(\vec{n})+\frac{\sqrt{l+1}}{2(n+1)}\vec{Y}_{ll-1m}^{*}(\vec{n})\Bigg]\cdot\vec{j}(\vec{r},t),\end{aligned} \qquad (1.37)$$

$$\overline{R_{lm}^{0}}(t)=T_{lm}(k^2=0,t)=T_{lm}^{[0]}(t)=T_{lm}(t), \qquad (1.38)$$

in the toroid case.

Instead of the electric, magnetic, and toroid formfactors $Q_{lm}(-k^2,t)$, $M_{lm}(-k^2,t)$, $T_{lm}(-k^2,t)$ [as given by Eqs. (1.22), (1.23), and (1.24)], which are functions of two variables (the momentum transfer variable $-k^2$ and time t) it is useful to express equivalently this full information about the electromagnetic structure of the system in terms of the mean-square radii of various type $\overline{r_{lm}^{2n}(t)}$, $\overline{\rho_{lm}^{2n}(t)}$, $\overline{R_{lm}^{2n}(t)}$ (electric, magnetic, toroid), multipolarity (l), and orders ($2n$) (a mean-square radius of order $n=0$ is just the corresponding multipole moment) given by Eqs. (1.28), (1.35), and (1.37). All these radii are still functions of time and their derivatives of various orders with respect to time will occur in the calculations. Since up to numerical factors all these radii are themselves derivatives of the formfactors $Q_{lm}(-k^2,t)$, $M_{lm}(-k^2,t)$, $T_{lm}(-k^2,t)$ of various orders (n) with respect to ($-k^2$) at $k^2=0$ [see Eqs. (1.27′) (1.31′), and (1.32′)], we shall work throughout this paper with the double-superscript quantities, $Q_{lm}^{(n)(\nu)}(0,t)$, $M_{lm}^{(n)(\nu)}(0,t)$, $T_{lm}^{(n)(\nu)}(0,t)$,

$$Q_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\frac{d^n}{d(-k^2)^n}Q_{lm}(-k^2,t)\bigg|_{k^2=0}\right], \qquad (1.39)$$

$$M_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\frac{d^n}{d(-k^2)^n}M_{lm}(-k^2,t)\bigg|_{k^2=0}\right], \qquad (1.40)$$

$$T_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\frac{d^n}{d(-k^2)^n}T_{lm}(-k^2,t)\bigg|_{k^2=0}\right], \qquad (1.41)$$

in which the first superscript indicates the order of derivation with respect to ($-k^2$) at $k^2=0$, while the second, the order of derivation with respect to time t of the corresponding formfactor. The relationships between these double-superscript quantities and the derivatives (of order ν) with respect to time of the (order n) mean-square radii (which include, for $n=0$, multipole moments) of electric, magnetic, and toroid type, are obviously [Eqs. (1.28), (1.35), and (1.37)]

$$Q_{lm}^{(n)(\nu)}(0,t)=\frac{(2l+1)!!}{2^n(2l+2n+1)!!}\frac{d^\nu}{dt^\nu}\overline{r_{lm}^{2n}}(t), \qquad (1.42)$$

$$M_{lm}^{(n)(\nu)}(0,t)=\frac{(2l+1)!!}{2^n(2l+2n+1)!!}\frac{d^\nu}{dt^\nu}\overline{\rho_{lm}^{2n}}(t), \qquad (1.43)$$

$$T_{lm}^{(n)(\nu)}(0,t)=\frac{(2l+1)!!}{2^n(2l+2n+1)!!}\frac{d^\nu}{dt^\nu}\overline{R_{lm}^{2n}}(t). \qquad (1.44)$$

For clarity purposes we stress again that the double-superscript quantities $Q_{lm}^{(n)(\nu)}(0,t)$, $M_{lm}^{(n)(\nu)}(0,t)$, $T_{lm}^{(n)(\nu)}(0,t)$ are nothing else but derivatives of order ν with respect to time t at any t of the derivatives of order n with respect to $(-k^2)$ taken at $k^2=0$ of the corresponding electric, magnetic, and toroid formfactors $Q_{lm}(-k^2,t)$, $M_{lm}(-k^2,t)$, $T_{lm}(-k^2,t)$ introduced previously. To avoid confusions, when a single superscript will sometimes occur, we shall put it inside square brackets when it means derivation at $(-k^2)$ [as in Eqs. (1.27′), (1.31′), and (1.32′)] and inside simple parantheses when it means derivation at t.

The quantities above $Q_{lm}^{(n)(\nu)}(0,t)$, $M_{lm}^{(n)(\nu)}(0,t)$, $T_{lm}^{(n)(\nu)}(0,t)$ give us the full information about the multipole content of the source and it will be in terms of them that we shall express our results for the radiation intensity, angular momentum loss by the system, and the recoil force. Although cumbersome, the formulas that we shall list represent exact results in the multipole analysis of the most general configuration of charges and currents. They allow for immediate particularization to the first multipole contributions to the physical quantities mentioned above, usually treated in textbooks, and in this way one succeeds in obtaining correct expressions by completing the results given there with terms (as a rule belonging to the toroid class of multipoles) that could be of the same order of magnitude as the usual ones. For the radiation intensity, angular momentum loss, and recoil force we shall give, at the end of Secs. IV, V, and VI, respectively, all these contributions [up to the $(1/c^5)$ order inclusively in the development over $(1/c)$ powers] that, to our knowledge, have not yet been reported.

II. POTENTIALS AND FIELDS OF A GENERAL SOURCE

To find the fields created by a most general distribution of charges and currents described by the charge density $\rho(\vec{r},t)$ and the current density $\vec{j}(\vec{r},t)$ that satisfy the continuity relation Eq. (1.1),

$$\frac{\partial\rho(\vec{r},t)}{\partial t}+\vec{\nabla}\cdot\vec{j}(\vec{r},t)=0,$$

we have to start as usual from the retarded scalar $\varphi(\vec{r},t)$ and vector $\vec{A}(\vec{r},t)$ potentials

$$\varphi(\vec{r},t)=\int d^3\vec{r}\,'\frac{\rho\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}, \qquad (2.1)$$

$$\vec{A}(\vec{r},t)=\frac{1}{c}\int d^3\vec{r}\,'\frac{\vec{j}\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}, \qquad (2.2)$$

and calculate the fields according to the well known formulas

$$\vec{E}(\vec{r},t)=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi(\vec{r},t), \qquad (2.3)$$

$$\vec{B}(\vec{r},t)=\vec{\nabla}\times\vec{A}(\vec{r},t). \qquad (2.4)$$

Since we want to have the fields expressed in terms of the Dubovik-Tscheshkov electric, magnetic, and toroid formfactors of the source, first we have to introduce Fourier transforms with respect to time for $\rho(\vec{r},t)$, $\vec{j}(\vec{r},t)$ and their corresponding multipole formfactors $Q_{lm}(-k^2,t)$, $M_{lm}(-k^2,t)$, $T_{lm}(-k^2,t)$ entering Eqs. (1.2) and (1.3). In order to fix notations in the course of this paper, we have to give below the corresponding formulas.

So, one has for the charge density,

$$\rho(\vec{r},\omega)=\int_0^\infty dt\,\rho(\vec{r},t)\sin(\omega t),$$

$$\rho(\vec{r},t)=\frac{2}{\pi}\int_0^\infty \rho(\vec{r},\omega)\sin(\omega t)d\omega, \qquad (2.5)$$

$$\rho(\vec{r},\omega)=\frac{1}{(2\pi)^3}\sum_{l,m,k}(-ik)^l\frac{\sqrt{4\pi(2l+1)}}{(2l+1)!!}\times Q_{lm}(-k^2,\omega)\mathcal{F}_{lmk}(\vec{r}), \qquad (2.6)$$

with

$$Q_{lm}(-k^2,\omega)=\int_0^\infty dt\,\sin(\omega t)Q_{lm}(-k^2,t),$$

$$Q_{lm}(-k^2,t)=\frac{2}{\pi}\int_0^\infty d\omega\,Q_{lm}(-k^2,\omega)\sin(\omega t), \qquad (2.7)$$

and for the current density

$$\vec{j}(\vec{r},\omega)=\int_0^\infty dt\,\vec{j}(\vec{r},t)\cos(\omega t),$$

$$\vec{j}(\vec{r},t)=\frac{2}{\pi}\int_0^\infty d\omega\,\vec{j}(\vec{r},\omega)\cos(\omega t), \qquad (2.8)$$

$$\vec{j}(\vec{r},\omega)=\frac{1}{(2\pi)^3}\sum_{l,m,k}(-ik)^{l-1}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\times\Big\{kcM_{lm}(-k^2,\omega)\vec{\mathcal{F}}_{lmk}^{(0)}(\vec{r})+[\dot{Q}_{lm}(0,\omega)+k^2cT_{lm}(-k^2,\omega)]\vec{\mathcal{F}}_{lmk}^{(+)}(\vec{r})+\sqrt{\frac{l}{l+1}}\dot{Q}_{lm}(-k^2,\omega)\vec{\mathcal{F}}_{lmk}^{(-)}(\vec{r})\Big\}, \qquad (2.9)$$

with

$$M_{lm}(-k^2,\omega)=\int_0^\infty dt\cos(\omega t)M_{lm}(-k^2,t),$$

$$M_{lm}(-k^2,t)=\frac{2}{\pi}\int_0^\infty d\omega\, M_{lm}(-k^2,\omega)\cos(\omega t),\tag{2.10}$$

$$T_{lm}(-k^2,\omega)=\int_0^\infty dt\cos(\omega t)T_{lm}(-k^2,t),$$

$$T_{lm}(-k^2,t)=\frac{2}{\pi}\int_0^\infty d\omega\, T_{lm}(-k^2,\omega)\cos(\omega t).\tag{2.11}$$

$\dot{Q}_{lm}(0,\omega)$ and generally $\dot{Q}_{lm}(-k^2,\omega)$ are simply notations (when the time argument is lacking, the dot obviously does not mean time derivative) and stand for

$$\dot{Q}_{lm}(0,\omega)\equiv\int_0^\infty dt\cos(\omega t)\dot{Q}_{lm}(0,t),$$

$$\dot{Q}_{lm}(-k^2,\omega)\equiv\int_0^\infty dt\cos(\omega t)\dot{Q}_{lm}(-k^2,t),$$

$$\dot{Q}_{lm}(0,t)=\frac{2}{\pi}\int_0^\infty d\omega\,\dot{Q}_{lm}(0,\omega)\cos(\omega t),$$

$$\dot{Q}_{lm}(-k^2,t)=\frac{2}{\pi}\int_0^\infty d\omega\,\dot{Q}_{lm}(-k^2,\omega)\cos(\omega t),$$

$$\dot{Q}_{lm}(0,\omega)\equiv\omega Q_{lm}(0,\omega),$$

$$\dot{Q}_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\equiv\omega Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right).\tag{2.12}$$

The continuity relation Eq. (1.1) reads

$$\vec{\nabla}\cdot\vec{j}(\vec{r},\omega)+\omega\rho(\vec{r},\omega)=0.\tag{2.13}$$

With these Fourier transforms one has to evaluate further the scalar potential

$$\varphi(\vec{r},t)=\varphi^{(1)}(\vec{r},t)+\varphi^{(2)}(\vec{r},t),\tag{2.14}$$

$$\varphi^{(1)}(\vec{r},t)=\frac{2}{\pi}\int_0^\infty d\omega\sin(\omega t)\int d^3\vec{r}\,'\rho(\vec{r}\,',\omega)\frac{\cos\left(\frac{\omega}{c}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|},\tag{2.15}$$

$$\varphi^{(2)}(\vec{r},t)=-\frac{2}{\pi}\int_0^\infty d\omega\cos(\omega t)\int d^3\vec{r}\,'\rho(\vec{r}\,',\omega)\frac{\sin\left(\frac{\omega}{c}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|},\tag{2.16}$$

and the vector potential

$$\vec{A}(\vec{r},t)=\vec{A}^{(1)}(\vec{r},t)+\vec{A}^{(2)}(\vec{r},t),\tag{2.17}$$

$$\vec{A}^{(1)}(\vec{r},t)=\frac{2}{\pi c}\int_0^\infty d\omega\sin(\omega t)\int d^3\vec{r}\,'\vec{j}(\vec{r}\,',\omega)\frac{\sin\left(\frac{\omega}{c}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|},\tag{2.18}$$

$$\vec{A}^{(2)}(\vec{r},t)=\frac{2}{\pi c}\int_0^\infty d\omega\cos(\omega t)\int d^3\vec{r}\,'\vec{j}(\vec{r}\,',\omega)\frac{\cos\left(\frac{\omega}{c}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|}.\tag{2.19}$$

To these purposes one needs the Green's functions. For the scalar potential case, one has (see Refs. [19], [20], [4]),

$$\frac{e^{i(\omega/c)|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}=\frac{1}{4\pi}\frac{\omega}{c}\sum_{l,m}\mathcal{F}^*_{lm(\omega/c)}(\vec{r}\,')\mathcal{H}_{lm(\omega/c)}(\vec{r}),\quad r>r',\tag{2.20}$$

$$\frac{\cos\left(\frac{\omega}{c}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|}=-\frac{\omega}{4\pi c}\sum_{l,m}\mathcal{F}^*_{lm(\omega/c)}(\vec{r}\,')\mathcal{N}_{lm(\omega/c)}(\vec{r}),\quad r>r',\tag{2.21}$$

$$\frac{\sin\left(\frac{\omega}{c}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|}=\frac{\omega}{4\pi c}\sum_{l,m}\mathcal{F}^*_{lm(\omega/c)}(\vec{r}\,')\mathcal{F}_{lm(\omega/c)}(\vec{r}),\quad r>r',\tag{2.22}$$

where $\mathcal{F}_{lm(\omega/c)}$ is given by Eq. (1.5),

$$\mathcal{F}_{lm(\omega/c)}(\vec{r})=j_l\left(\frac{\omega}{c}r\right)Y_{lm}(\vec{n}),\quad \vec{n}=\frac{\vec{r}}{r},\tag{1.5}$$

while $\mathcal{H}_{lm(\omega/c)}(\vec{r})$ and $\mathcal{N}_{lm(\omega/c)}(\vec{r})$ are defined like $\mathcal{F}_{lm(\omega/c)}(\vec{r})$, but with the spherical Bessel function of the first species j_l replaced, respectively, with the spherical Hankel function $h_l^{(+)}$ and the spherical Bessel function of the second species n_l (see Appendix A for conventions),

$$\mathcal{H}_{lm(\omega/c)}(\vec{r})=h_l^{(+)}\left(\frac{\omega}{c}r\right)Y_{lm}(\vec{n}),\tag{2.23}$$

$$\mathcal{N}_{lm(\omega/c)}(\vec{r})=n_l\left(\frac{\omega}{c}r\right)Y_{lm}(\vec{n}).\tag{2.24}$$

$\mathcal{H}_{lmk}$ and $\mathcal{N}_{lmk}$ satisfy the same normalization, completeness, and parity conditions as those satisfied by $\mathcal{F}_{lmk}$, Eqs. (1.6), (1.7), and (1.8),

$$\int \mathcal{H}_{lmk}(\vec{r})\mathcal{H}^*_{l'm'k'}(\vec{r})d^3r=\int \mathcal{N}_{lmk}(\vec{r})\mathcal{N}^*_{l'm'k'}(\vec{r})d^3r$$
$$=\int \mathcal{F}_{lmk}(\vec{r})\mathcal{F}^*_{l'm'k'}(\vec{r})d^3r$$
$$=\delta_{ll'}\delta_{mm'}\frac{(2\pi)^3}{k^2}\delta(k-k'), \tag{2.25}$$

$$\sum_{lmk} \mathcal{H}_{lmk}(\vec{r})\mathcal{H}^*_{lmk}(\vec{r}\,')$$
$$=\sum_{lmk} \mathcal{N}_{lmk}(\vec{r})\mathcal{N}^*_{lmk}(\vec{r}\,')=\sum_{lmk} \mathcal{F}_{lmk}(\vec{r})\mathcal{F}^*_{lmk}(\vec{r}\,')$$
$$=(2\pi)^3\delta(\vec{r}-\vec{r}\,'),\quad \sum_k=\int_0^\infty k^2dk, \tag{2.26}$$

$$\mathcal{H}_{lmk}(-\vec{r})=(-1)^l\mathcal{H}_{lmk}(\vec{r}),\ \ \mathcal{N}_{lmk}(-\vec{r})=(-1)^l\mathcal{N}_{lmk}(\vec{r}). \tag{2.27}$$

Also one has, as for $\vec{\mathcal{F}}^{(\lambda)}_{lmk}$ the relations,

$$\vec{\nabla}\times\vec{\mathcal{N}}^{(-)}_{lmk}(\vec{r})=0,\quad \vec{\nabla}\times\vec{\mathcal{H}}^{(-)}_{lmk}(\vec{r})=\vec{0},$$

$$\vec{\nabla}\cdot\vec{\mathcal{N}}^{(-)}_{lmk}(\vec{r})=ik\mathcal{N}_{lmk}(\vec{r}),\ \ \vec{\nabla}\cdot\vec{\mathcal{H}}^{(-)}_{lmk}(\vec{r})=ik\mathcal{H}_{lmk}(\vec{r}), \tag{2.27$'$}$$

$$\vec{\nabla}\cdot\vec{\mathcal{N}}^{(+)}_{lmk}(\vec{r})=\vec{\nabla}\cdot\vec{\mathcal{N}}^{(0)}_{lmk}(\vec{r})=0,$$

$$\vec{\nabla}\cdot\vec{\mathcal{H}}^{(+)}_{lmk}(\vec{r})=\vec{\nabla}\cdot\vec{\mathcal{H}}^{(0)}_{lmk}(\vec{r})=0.$$

In the case of the vector potential it is helpful to have representations for the Green's functions in terms of the basis vector functions for the vector Helmholtz equation. So, we shall use (see Refs. [19], [20], [4])

$$\frac{e^{i(\omega/c)|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}=\frac{1}{4\pi}\frac{\omega}{c}\sum_{l,m}\vec{\mathcal{F}}^{(\lambda)*}_{lm(\omega/c)}(\vec{r}\,')\cdot\vec{\mathcal{H}}^{(\lambda)}_{lm(\omega/c)}(\vec{r}),\quad r>r', \tag{2.28}$$

which results from the slightly more general expression

$$\frac{1}{3}\delta_{ik}\frac{e^{i(\omega/c)|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}=\frac{1}{4\pi}\frac{\omega}{c}\sum_{l,m,\lambda}[\vec{\mathcal{F}}^{(\lambda)*}_{lm(\omega/c)}(\vec{r}\,')]_i$$
$$\cdot[\vec{\mathcal{H}}^{(\lambda)}_{lm(\omega/c)}(\vec{r})]_k,\quad r>r', \tag{2.28$'$}$$

and

$$\frac{\cos\left(\frac{\omega}{c}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|}=-\frac{\omega}{4\pi c}\sum_{l,m}[\vec{\mathcal{F}}^{(+)*}_{lm(\omega/c)}(\vec{r}\,')\cdot\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})$$
$$+\vec{\mathcal{F}}^{(-)*}_{lm(\omega/c)}(\vec{r}\,')\cdot\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})$$
$$+\vec{\mathcal{F}}^{(0)*}_{lm(\omega/c)}(\vec{r}\,')\cdot\vec{\mathcal{N}}^{(0)}_{lm(\omega/c)}(\vec{r})],\quad r>r', \tag{2.29}$$

$$\frac{\sin\left(\frac{\omega}{c}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|}=\frac{\omega}{4\pi c}\sum_{l,m}[\vec{\mathcal{F}}^{(+)*}_{lm(\omega/c)}(\vec{r}\,')\cdot\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})$$
$$+\vec{\mathcal{F}}^{(-)*}_{lm(\omega/c)}(\vec{r}\,')\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})$$
$$+\vec{\mathcal{F}}^{(0)*}_{lm(\omega/c)}(\vec{r}\,')\cdot\vec{\mathcal{F}}^{(0)}_{lm(\omega/c)}(\vec{r})],\quad r>r', \tag{2.30}$$

where $\vec{\mathcal{H}}^{\lambda}_{lmk}$ and $\vec{\mathcal{N}}^{\lambda}_{lmk}$ $(\lambda=0,\pm)$ are defined like $\vec{\mathcal{F}}^{(\lambda=0,\pm)}_{lmk}$ [Eqs. (1.9)–(1.11)], but with h^+_l and n_l instead of j_l and satisfy exactly those relations satisfied by $\vec{\mathcal{F}}^{(\lambda)}_{lmk}$ that are unaffected by the replacement $j_l\to h^{(+)}_l$, n_l. So one has

$$\begin{pmatrix}\vec{\mathcal{H}}^{(0)}_{lmk}(\vec{r})\\ \vec{\mathcal{N}}^{(0)}_{lmk}(\vec{r})\end{pmatrix}=\frac{i}{\sqrt{l(l+1)}}\vec{\nabla}\times\begin{pmatrix}\vec{r}\mathcal{H}_{lmk}(\vec{r})\\ \vec{r}\mathcal{N}_{lmk}(\vec{r})\end{pmatrix}$$
$$=\begin{pmatrix}h^{(+)}_l(kr)\\ n_l(kr)\end{pmatrix}\vec{Y}_{llm}(\vec{n}), \tag{2.31}$$

$$\begin{pmatrix}\vec{\mathcal{H}}^{(+)}_{lmk}(\vec{r})\\ \vec{\mathcal{N}}^{(+)}_{lmk}(\vec{r})\end{pmatrix}=-\frac{1}{\sqrt{l(l+1)}}\frac{i}{k}\vec{\nabla}\times\vec{\nabla}\times\begin{pmatrix}\vec{r}\mathcal{H}_{lmk}(\vec{r})\\ \vec{r}\mathcal{N}_{lmk}(\vec{r})\end{pmatrix}, \tag{2.32}$$

$$\begin{pmatrix}\vec{\mathcal{H}}^{(-)}_{lmk}(\vec{r})\\ \vec{\mathcal{N}}^{(-)}_{lmk}(\vec{r})\end{pmatrix}=-\frac{i}{k}\vec{\nabla}\begin{pmatrix}\mathcal{H}_{lmk}(\vec{r})\\ \mathcal{N}_{lmk}(\vec{r})\end{pmatrix}, \tag{2.33}$$

and

$$\int \vec{\mathcal{H}}^{(\lambda)*}_{lmk}(\vec{r})\cdot\vec{\mathcal{H}}^{(\lambda')}_{l'm'k'}(\vec{r})d^3\vec{r}$$
$$=\int \vec{\mathcal{N}}^{(\lambda)*}_{lmk}(\vec{r})\cdot\vec{\mathcal{N}}^{(\lambda')}_{l'm'k'}(\vec{r})d^3\vec{r}$$
$$=\int \vec{\mathcal{F}}^{(\lambda)*}_{lmk}(\vec{r})\cdot\vec{\mathcal{F}}^{(\lambda')}_{l'm'k'}(\vec{r})d^3\vec{r}$$
$$=\delta_{ll'}\delta_{mm'}\delta_{\lambda\lambda'}\frac{(2\pi)^3}{k^2}\delta(k-k'), \tag{2.34}$$

$$\sum_{l,m,k,\lambda}[\vec{\mathcal{H}}^{(\lambda)}_{lmk}(\vec r)]^*_i[\vec{\mathcal{H}}^{(\lambda)}_{lmk}(\vec r\,')]_j$$

$$=\sum_{l,m,k,\lambda}[\vec{\mathcal{N}}^{(\lambda)}_{lmk}(\vec r)]^*_i[\vec{\mathcal{N}}^{(\lambda)}_{lmk}(\vec r\,')]_j$$

$$=(2\pi)^3\delta_{ij}\delta(\vec r-\vec r\,'), \tag{2.35}$$

$$\vec{\mathcal{H}}^{(\lambda)}_{lmk}(-\vec r)=(-1)^{l+\lambda}\vec{\mathcal{H}}^{(\lambda)}_{lmk}(\vec r),$$

$$\vec{\mathcal{N}}^{(\lambda)}_{lmk}(-\vec r)=(-1)^{l+\lambda}\vec{\mathcal{N}}^{(\lambda)}_{lmk}(\vec r),\quad \lambda=0\pm. \tag{2.36}$$

Equations (2.29) and (2.30) can be obtained through straightforward calculation by separating the real and imaginary parts in Eq. (2.28).

Using the forms given above for the Green's functions, one obtains from Eqs. (2.14)–(2.16) the scalar potential in the form

$$\varphi(\vec r,t)=\varphi^{(1)}(\vec r,t)+\varphi^{(2)}(\vec r,t),$$

$$\varphi^{(1)}(\vec r,t)=-\frac{1}{\pi^{3/2}c}\int_0^\infty d\omega\,\omega\,\sin(\omega t)\times\sum_{lm}\left(\frac{-i\omega}{c}\right)^l\frac{\sqrt{2l+1}}{(2l+1)!!}\times Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\mathcal{N}_{lm(\omega/c)}(\vec r),$$

$$\varphi^{(2)}(\vec r,t)=-\frac{1}{\pi^{3/2}c}\int_0^\infty d\omega\,\omega\,\cos(\omega t)\times\sum_{lm}\left(\frac{-i\omega}{c}\right)^l\frac{\sqrt{2l+1}}{(2l+1)!!}\times Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\mathcal{F}_{lm(\omega/c)}(\vec r). \tag{2.37}$$

For further purposes we rewrite the expression given above for the scalar potential more compactly as

$$\varphi(\vec r,t)=-\frac{1}{\pi^{3/2}c}\int_0^\infty d\omega\,\omega\times\sum_{lm}\left(-\frac{i\omega}{c}\right)^l\frac{\sqrt{2l+1}}{(2l+1)!!}[\sin(\omega t)\mathcal{N}_{lm(\omega/c)}(\vec r)+\cos(\omega t)\mathcal{F}_{lm(\omega/c)}(\vec r)]Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right). \tag{2.37$'$}$$

Analogously, one gets for the vector potential from Eqs. (2.17)–(2.19),

$$\vec A(\vec r,t)=\vec A^{(1)}(\vec r,t)+\vec A^{(2)}(\vec r,t),$$

$$\vec A^{(1)}(\vec r,t)=-\frac{1}{2\pi^2c^2}\int_0^\infty d\omega\,\omega\,\cos(\omega t)\times\sum_{l,m}\left(\frac{-i\omega}{c}\right)^{l-1}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\times\Bigg\{\omega M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\vec{\mathcal{N}}^{(0)}_{lm(\omega/c)}(\vec r)+\left[\omega Q_{lm}(0,\omega)+\frac{\omega^2}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec r)+\frac{\sqrt{l}}{\sqrt{l+1}}\omega Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec r)\Bigg\},$$

$$\vec A^{(2)}(\vec r,t)=\frac{1}{2\pi^2c^2}\int_0^\infty d\omega\,\omega\,\sin(\omega t)\times\sum_{l,m}\left(-\frac{i\omega}{c}\right)^{l-1}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\times\Bigg\{\omega M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\vec{\mathcal{F}}^{(0)}_{lm(\omega/c)}(\vec r)+\left[\omega Q_{lm}(0,\omega)+\frac{\omega^2}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec r)+\frac{\sqrt{l}}{\sqrt{l+1}}\omega Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec r)\Bigg\}. \tag{2.38}$$

So, with Eqs. (2.3), (2.37), and (2.38) and using

$$\vec\nabla\mathcal{F}_{lm(\omega/c)}(\vec r)=i\frac{\omega}{c}\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec r),$$

$$\vec\nabla\mathcal{N}_{lm(\omega/c)}(\vec r)=i\frac{\omega}{c}\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec r),$$

one finds the exact expression for the electric field $\vec E(\vec r,t)$ emitted by the most general type of source described by the electric, magnetic, and toroid multipole formfactors $Q_{lm}(-k^2,t)$, $M_{lm}(-k^2,t)$, $T_{lm}(-k^2,t)$ in terms of their Fourier transforms in time $Q_{lm}(-k^2,\omega)$, $M_{lm}(-k^2,\omega)$, $T_{lm}(-k^2,\omega)$ at $k=\omega/c$,

$$\vec{E}(\vec{r},t)=\frac{1}{\pi^{3/2}}\int_0^\infty d\omega\sum_{l,m}(-i)^{l-1}\left(\frac{\omega}{c}\right)^{l+2}\frac{\sqrt{2l+1}}{(2l+1)!!}[\sin(\omega t)\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})+\cos(\omega t)\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})]Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)$$
$$-\frac{1}{2\pi^2 c}\int_0^\infty d\omega\sum_{lm}(-i)^{l-1}\left(\frac{\omega}{c}\right)^{l+1}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\Bigg\{[\sin(\omega t)\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}\vec{r}+\cos(\omega t)\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})]$$
$$\times\frac{\sqrt{l}}{\sqrt{l+1}}\omega Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)+\omega[\sin(\omega t)\vec{\mathcal{N}}^{(0)}_{lm(\omega/c)}(\vec{r})+\cos(\omega t)\vec{\mathcal{F}}^{(0)}_{lm(\omega/c)}(\vec{r})]M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)$$
$$+[\sin(\omega t)\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})+\cos(\omega t)\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})]\left[\omega Q_{lm}(0,\omega)+\frac{\omega^2}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]\Bigg\}. \quad (2.39)$$

The analogous expression for the magnetic field $\vec{B}(\vec{r},t)$ can be obtained from Eq. (2.38) by simply taking the curls of the functions $\vec{N}^{(\lambda)}_{lm(\omega/c)}(\vec{r})$, $\vec{F}^{(\lambda)}_{lm(\omega/c)}(\vec{r})$. Using

$$\vec{\nabla}\times\vec{\mathcal{F}}^{(-)}_{lmk}(\vec{r})=\vec{\nabla}\times\vec{\mathcal{N}}^{(-)}_{lmk}(\vec{r})=0,$$

$$\vec{\nabla}\times\vec{\mathcal{F}}^{(0)}_{lmk}(\vec{r})=-\frac{\omega}{c}\vec{\mathcal{F}}^{(+)}_{lmk}(\vec{r}),\quad \vec{\nabla}\times\vec{\mathcal{F}}^{(+)}_{lmk}(\vec{r})=-\frac{\omega}{c}\vec{\mathcal{F}}^{(0)}_{lmk}(\vec{r}),$$

$$\vec{\nabla}\times\vec{\mathcal{N}}^{(0)}_{lmk}(\vec{r})=-\frac{\omega}{c}\vec{\mathcal{N}}^{(+)}_{lmk}(\vec{r}),\quad \vec{\nabla}\times\vec{\mathcal{N}}^{(+)}_{lmk}(\vec{r})=-\frac{\omega}{c}\vec{\mathcal{N}}^{(0)}_{lmk}(\vec{r}),$$

one finds the following exact expression for the magnetic field $\vec{B}(\vec{r},t)$ emitted by the most general type of source in terms of the Fourier transforms of the electric, magnetic, and toroid formfactors:

$$\vec{B}(\vec{r},t)=\frac{1}{2\pi^2}\int_0^\infty d\omega\sum_{l,m}(-i)^{l-1}\left(\frac{\omega}{c}\right)^{l+2}$$
$$\times\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\Bigg\{[-\sin(\omega t)\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})$$
$$+\cos(\omega t)\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})]M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)$$
$$+[-\sin(\omega t)\vec{\mathcal{F}}^{(0)}_{lm(\omega/c)}(\vec{r})+\cos(\omega t)\vec{\mathcal{N}}^{(0)}_{lm(\omega/c)}(\vec{r})]$$
$$\times\left[Q_{lm}(0,\omega)+\frac{\omega}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]\Bigg\}. \quad (2.40)$$

The exact formulas for the electric field Eq. (2.39) and for the magnetic field Eq. (2.40) express without any ambiguity the way in which the multipole content of the source (specified by the electric, magnetic, and toroid multipole formfactors) reflects itself in the fields created.

The expressions of the fields $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ given in Eqs. (2.39) and (2.40) we shall work with have been derived by means of the well known procedure [Eqs. (2.3) and (2.4)] from the retarded scalar and vector potentials $\varphi(\vec{r},t)$, $\vec{A}(\vec{r},t)$ defined by Eqs. (2.1) and (2.2) and calculated in terms of the electric, magnetic, and toroid multipole formfactors as in Eq. (2.37) [or Eq. (2.37′) and Eqs. (2.38)]. Some comments on the last equations may still be in order in connection with gauge invariance questions. Our potentials $\varphi(\vec{r},t)$, $\vec{A}(\vec{r},t)$ as given by Eqs. (2.37) and (2.37′) and Eqs. (2.38) do satisfy the Lorenz condition

$$\vec{\nabla}\cdot\vec{A}(\vec{r},t)+\frac{1}{c}\frac{\partial\varphi(\vec{r},t)}{\partial t}=0. \quad (2.41)$$

This can be immediately checked again with the aid of the relations

$$\vec{\nabla}\cdot\vec{\mathcal{N}}^{(0)}_{lmk}(\vec{r})=0,\quad \vec{\nabla}\cdot\vec{\mathcal{N}}^{(+)}_{lmk}(\vec{r})=0,\quad \vec{\nabla}\cdot\vec{\mathcal{N}}^{(-)}_{lmk}(\vec{r})=ik\mathcal{N}_{lmk}(\vec{r}),$$

$$k=\frac{\omega}{c},$$

when one finds

$$\vec{\nabla}\cdot\vec{A}^{(1)}(\vec{r},t)=\frac{1}{2\pi^2c^2}\int_0^\infty d\omega\,\omega^2\cos(\omega t)$$
$$\times\sum_{l,m}\left(\frac{-i\omega}{c}\right)^l\frac{\sqrt{4\pi(2l+1)}}{(2l+1)!!}$$
$$\times Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\mathcal{N}_{lm(\omega/c)}(\vec{r}),$$

$$\vec{\nabla}\cdot\vec{A}^{(2)}(\vec{r},t)=-\frac{1}{2\pi^2c^2}\int_0^\infty d\omega\,\omega^2\sin(\omega t)$$
$$\times\sum_{l,m}\left(\frac{-i\omega}{c}\right)^l\frac{\sqrt{4\pi(2l+1)}}{(2l+1)!!}$$
$$\times Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\mathcal{F}_{lm(\omega/c)}(\vec{r}),$$

$$\frac{1}{c}\frac{\partial\varphi^{(1)}(\vec{r},t)}{\partial t}=-\frac{1}{\pi^{3/2}c^2}\int_0^\infty d\omega\omega^2\cos(\omega t)\times\sum_{l,m}\left(-\frac{i\omega}{c}\right)^l\frac{\sqrt{(2l+1)}}{(2l+1)!!}\times Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\mathcal{N}_{lm(\omega/c)}(\vec{r}),$$

$$\frac{1}{c}\frac{\partial\varphi^{(2)}(\vec{r},t)}{\partial t}=\frac{1}{\pi^{3/2}c^2}\int_0^\infty d\omega\omega^2\sin(\omega t)\times\sum_{l,m}\left(\frac{-i\omega}{c}\right)^l\frac{\sqrt{(2l+1)}}{(2l+1)!!}\times Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\mathcal{F}_{lm(\omega/c)}(\vec{r}).$$

Therefore, one has separately

$$\vec{\nabla}\cdot\vec{A}^{(1)}(\vec{r},t)+\frac{1}{c}\frac{\partial\varphi^{(1)}(\vec{r},t)}{\partial t}=0, \tag{2.42}$$

$$\vec{\nabla}\cdot\vec{A}^{(2)}(\vec{r},t)+\frac{1}{c}\frac{\partial\varphi^{(2)}(\vec{r},t)}{\partial t}=0, \tag{2.43}$$

and the Lorenz condition Eq. (2.41) for $\vec{A}(\vec{r},t)=\vec{A}^{(1)}(\vec{r},t)+\vec{A}^{(1)}(\vec{r},t)$ and $\varphi(\vec{r},t)=\varphi^{(1)}(\vec{r},t)+\varphi^{(2)}(\vec{r},t)$ is verified.

So the gauge in which our expressions for φ, $\vec{A}$ are written down is the Lorenz gauge established by the Lorenz condition Eq. (2.41). To evidentiate the gauge freedom still left after satisfying the Lorenz condition, we display below a more general form of the potentials than the one given in Eqs. (2.37) and (2.38), with which we could have perfectly worked as well as from which the same fields $\vec{E}$, $\vec{B}$ as in Eqs. (2.39) and (2.40) would have resulted, both forms for the potentials [i.e., Eqs. (2.37) and (2.38) on one side and Eqs. (2.44) and (2.45) below, on the other] being gauge equivalent,

$$\varphi'(\vec{r},t)=\varphi'^{(1)}(\vec{r},t)+\varphi'^{(2)}(\vec{r},t),$$

$$\varphi'^{(1)}(\vec{r},t)=-\frac{C_1}{\pi^{3/2}c}\int_0^\infty d\omega\omega\sin(\omega t)\sum_{l,m}\left(\frac{-i\omega}{c}\right)^l\frac{\sqrt{2l+1}}{(2l+1)!!}\times Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\mathcal{N}_{lm(\omega/c)}(\vec{r}),$$

$$\varphi'^{(2)}(\vec{r},t)=-\frac{C_2}{\pi^{3/2}c}\int_0^\infty d\omega\omega\cos(\omega t)\times\sum_{l,m}\left(\frac{-i\omega}{c}\right)^l\frac{\sqrt{2l+1}}{(2l+1)!!}Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\times\mathcal{F}_{lm(\omega/c)}(\vec{r}); \tag{2.44}$$

$$\vec{A}'(\vec{r},t)=\vec{A}'^{(1)}(\vec{r},t)+\vec{A}'^{(2)}(\vec{r},t),$$

$$\vec{A}'^{(1)}(\vec{r},t)=-\frac{1}{2\pi^2c^2}\int_0^\infty d\omega\omega\cos(\omega t)\times\sum_{l,m}\left(\frac{-i\omega}{c}\right)^{l-1}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\times\left\{\omega M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\vec{\mathcal{N}}^{(0)}_{lm(\omega/c)}(\vec{r})+\left[\omega Q_{lm}(0,\omega)+\frac{\omega^2}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})+\frac{C_1\sqrt{l}}{\sqrt{l+1}}\omega Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})\right\},$$

$$\vec{A}'^{(2)}(\vec{r},t)=\frac{1}{2\pi^2c^2}\int_0^\infty d\omega\omega\sin(\omega t)\times\sum_{l,m}\left(\frac{-i\omega}{c}\right)^{l-1}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\times\left\{\omega M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\vec{\mathcal{F}}^{(0)}_{lm(\omega/c)}(\vec{r})+\left[\omega Q_{lm}(0,\omega)+\frac{\omega^2}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})+\frac{C_2\sqrt{l}}{\sqrt{l+1}}\omega Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})\right\}. \tag{2.45}$$

Indeed, the new potentials $\varphi'(\vec{r},t)$, $\vec{A}'(\vec{r},t)$ given by Eqs. (2.44) and (2.45) [and moreover, their two components separately specified by the superscripts (1), (2), $\varphi'^{(1),(2)}(\vec{r},t)$, $\vec{A}'^{(1),(2)}(\vec{r},t)$] are related to the old ones $\varphi(\vec{r},t)$, $\vec{A}(\vec{r},t)$ [with their components $\varphi^{(1),(2)}(\vec{r},t)$, $\vec{A}^{(1),(2)}(\vec{r},t)$] given by Eqs. (2.37) and (2.38) through the following gauge transformation:

$$\varphi'^{(1)}(\vec{r},t)=\varphi^{(1)}(\vec{r},t)-\frac{1}{c}\frac{\partial\Lambda^{(1)}(\vec{r},t)}{\partial t},\quad \vec{A}'^{(1)}(\vec{r},t)=\vec{A}^{(1)}(\vec{r},t)+\vec{\nabla}\Lambda^{(1)}(\vec{r},t), \tag{2.46}$$

$$\varphi'^{(2)}(\vec{r},t)=\varphi^{(2)}(\vec{r},t)-\frac{1}{c}\frac{\partial\Lambda^{(2)}(\vec{r},t)}{\partial t},\quad \vec{A}'^{(2)}(\vec{r},t)=\vec{A}^{(2)}(\vec{r},t)+\vec{\nabla}\Lambda^{(2)}(\vec{r},t), \tag{2.47}$$

$$\varphi'(\vec{r},t)=\varphi(\vec{r},t)-\frac{1}{c}\frac{\partial\Lambda(\vec{r},t)}{\partial t},\quad \vec{A}'(\vec{r},t)=\vec{A}(\vec{r},t)+\vec{\nabla}\Lambda(\vec{r},t), \tag{2.48}$$

$$\Lambda(\vec{r},t)=\Lambda^{(1)}(\vec{r},t)+\Lambda^{(2)}(\vec{r},t), \tag{2.49}$$

with

$$\Lambda^{(1)}(\vec{r},t)\equiv\frac{1}{2\pi^2c^2}\int_0^\infty d\omega\omega\cos(\omega t)\times\sum_{l,m}\left(\frac{-i\omega}{c}\right)^{l-1}\frac{\sqrt{4\pi(2l+1)}}{(2l+1)!!}\times\omega Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\frac{ic}{\omega}(C_1-1)\mathcal{N}_{lm(\omega/c)}(\vec{r}),\tag{2.50}$$

$$\Lambda^{(2)}(\vec{r},t)\equiv\frac{1}{2\pi^2c^2}\int_0^\infty d\omega\omega\cos(\omega t)\times\sum_{l,m}\left(-\frac{i\omega}{c}\right)^{l-1}\frac{\sqrt{4\pi(2l+1)}}{(2l+1)!!}\times\omega Q_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\frac{ic}{\omega}(C_2-1)\mathcal{F}_{lm(\omega/c)}(\vec{r}).\tag{2.51}$$

Equations (2.46)–(2.49) can be checked again immediately by noting that

$$\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})=-\frac{ic}{\omega}\vec{\nabla}\mathcal{N}_{lm(\omega/c)}(\vec{r}),$$

$$\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})=-\frac{ic}{\omega}\vec{\nabla}\mathcal{F}_{lm(\omega/c)}(\vec{r}).$$

Since

$$(\Delta+k^2)\mathcal{N}_{lm(\omega/c)}(\vec{r})=0,\quad(\Delta+k^2)\vec{\mathcal{F}}_{lm(\omega/c)}(\vec{r})=0,$$

$$\left(k=\frac{\omega}{c}\right),$$

the gauge functions $\Lambda^{(1)}$, $\Lambda^{(2)}$, $\Lambda=\Lambda^{(1)}+\Lambda^{(2)}$, satisfy the wave equation

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Lambda^{(1)}(\vec{r},t)=0,\quad\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Lambda^{(2)}(\vec{r},t)=0,\tag{2.52}$$

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Lambda(\vec{r},t)=0.\tag{2.53}$$

The new potentials $\varphi'(\vec{r},t)$, $\vec{A}'(\vec{r},t)$ [as well as their two components $\varphi'^{(1),(2)}(\vec{r},t)$, $\vec{A}'^{(1),(2)}(\vec{r},t)$ separately] satisfy again the Lorenz condition

$$\vec{\nabla}\cdot\vec{A}'^{(1)}(\vec{r},t)+\frac{1}{c}\frac{\partial\varphi'^{(1)}(\vec{r},t)}{\partial t}=0,\tag{2.54}$$

$$\vec{\nabla}\cdot\vec{A}'^{(2)}(\vec{r},t)+\frac{1}{c}\frac{\partial\varphi'^{(2)}(\vec{r},t)}{\partial t}=0,\tag{2.55}$$

$$\vec{\nabla}\cdot\vec{A}'(\vec{r},t)+\frac{1}{c}\frac{\partial\varphi'(\vec{r},t)}{\partial t}=0.\tag{2.56}$$

Therefore, the new potentials $\varphi'(\vec{r},t)$, $\vec{A}(\vec{r},t)$ from Eqs. (2.44) and (2.45) are more general than the old ones from Eqs. (2.37) and (2.38) and are gauge equivalent to them; both forms of the potentials satisfy the Lorenz condition but in the new ones φ', $\vec{A}'$ the remaining gauge freedom after the Lorenz condition that has been satisfied is explicitly displayed through the two remaining real arbitrary constants C_1, C_2. The old potentials φ, $\vec{A}$ of Eqs. (2.37) and (2.38) are obtained as particular cases from the new ones φ', $\vec{A}'$ of Eqs. (2.44) and (2.45) for

$$C_1=C_2=C=1.\tag{2.57}$$

Equation (2.57) fixes the gauge in which the potentials φ, $\vec{A}$ are given in Eqs. (2.37) and (2.38). Another convenient gauge would have been the Coulomb one, corresponding to the choice

$$C_1=C_2=C=0,\tag{2.58}$$

when the potentials φ'', $\vec{A}''$ satisfy the conditions

$$\varphi''(\vec{r},t)\equiv0,\;\;\nabla\vec{A}''(\vec{r},t)=0.\tag{2.59}$$

Due to their gauge equivalence discussed before, all these forms of the potentials lead obviously to the same fields $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ and since we have worked here only with these fields, all the results obtained in this paper are gauge invariant, as they must.

III. FIELDS AT LARGE DISTANCES

In order to calculate the radiation intensity, angular momentum loss, and recoil force we need to obtain from the exact expressions of the electric and magnetic fields Eqs. (2.39) and (2.40) formulas expressing the behavior of the fields at large distances. Next we shall calculate the fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ to order $O(1/r)$. To that purpose, we use the following asymptotical behavior of the spherical Bessel functions $j_l(\omega r/c)$, $n_l(\omega r/c)$ (see Appendix A for conventions and definitions) for $r\to\infty$, to $O(1/r)$:

$$j_l\left(\frac{\omega}{c}r\right)\sim4\pi i^l\,\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},$$

$$n_l\left(\frac{\omega}{c}r\right)\sim-4\pi i^l\,\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},$$

$$j_{l+1}\left(\frac{\omega}{c}r\right)\sim-4\pi i^{l+1}\,\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},$$

$$n_{l+1}\left(\frac{\omega}{c}r\right)\sim-4\pi i^{l+1}\,\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},$$

$$j_{l-1}\left(\frac{\omega}{c}r\right)\sim 4\pi i^{l-1}\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},$$

$$n_{l-1}\left(\frac{\omega}{c}r\right)\sim 4\pi i^{l-1}\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}, \qquad (3.1)$$

in order to find the $O(1/r)$ expressions of the veector functions $\vec{\mathcal{F}}^{(\lambda)}_{im(\omega/c)}(\vec{r})$, $\vec{\mathcal{N}}^{(\lambda)}_{im(\omega/c)}(\vec{r})$ $(\lambda=0,\pm1)$ entering the fields:

$$\vec{\mathcal{F}}^{(0)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}\frac{4\pi i^{l}}{\sqrt{2l+1}}\,\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}\,\vec{Y}_{llm}(\vec{n}),$$

$$\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}\frac{4\pi i^{l-1}}{\sqrt{2l+1}}\,\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}$$
$$\times[\sqrt{l}\vec{Y}_{ll+1m}(\vec{n})+\sqrt{l+1}\vec{Y}_{ll-1m}(\vec{n})],$$

$$\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}\frac{4\pi i^{l-1}}{\sqrt{2l+1}}\,\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}$$
$$\times[\sqrt{l}\vec{Y}_{ll-1m}(\vec{n})-\sqrt{l+1}\vec{Y}_{ll+1m}(\vec{n})]; \qquad (3.2)$$

$$\vec{\mathcal{N}}^{(0)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}-\frac{4\pi i^{l}}{\sqrt{2l+1}}\,\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}\,\vec{Y}_{llm}(\vec{n}),$$

$$\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}\frac{4\pi i^{l-1}}{\sqrt{2l+1}}\,\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}$$
$$\times[\sqrt{l}\vec{Y}_{ll+1m}(\vec{n})+\sqrt{l+1}\vec{Y}_{ll-1m}(\overline{n})];$$

$$\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}\frac{4\pi i^{l-1}}{\sqrt{2l+1}}\,\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}$$
$$\times[\sqrt{l}\overline{Y}_{ll-1m}(\vec{n})-\sqrt{l+1}\overline{Y}_{ll+1m,}(\vec{n})]; \qquad (3.3)$$

$$\vec{n}=\frac{\vec{r}}{r}.$$

So, on the basis of Eqs. (2.39) and (2.40), the electric and magnetic fields $\vec{E}(\vec{r},t),\vec{B}(\vec{r},t)$ evaluated at large distances in order $O(1/r)$, in terms of the multipole formfactors $M_{lm}(-\omega^2/c^2,\omega)$, $T_{lm}(-\omega^2/c^2,\omega)$ and the electric multipole moments $Q_{lm}(0,\omega)$, are

$$\vec{E}(\vec{r},t)\overset{O(1/r)}{\sim}\left(-\frac{2}{\pi c}\right)\frac{1}{r}\int_0^\infty d\omega\sum_{l,m}\frac{\omega^{l+1}}{c^{1}}$$
$$\times\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\cdot\left\{-iM_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right.$$
$$\times\sin\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)\vec{Y}_{llm}(\vec{n})$$
$$+\left[Q_{lm}(0,\omega)+\frac{\omega}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]$$
$$\times\cos\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)$$
$$\left.\times\left[\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n})+\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})\right]\right\}, \qquad (3.4)$$

$$\vec{B}(\vec{r},t)\overset{O(1/r)}{\sim}\left(-\frac{2}{\pi}\right)\frac{1}{r}\int_0^\infty d\omega$$
$$\times\sum_{l,m}\frac{\omega^{l+1}}{c^{l+1}}\,\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}$$
$$\times\left\{M_{lm}\left(-\frac{\omega^2}{c^2}\omega\right)\sin\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)\right.$$
$$\times\left[\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n})+\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})\right]$$

$$+i\left[Q_{lm}(0,\omega)+\frac{\omega}{c}T_{lm}\left(-\frac{\omega^2}{c^2,\omega}\right)\right]$$

$$\times\cos\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)\vec{Y}_{llm}(\vec{n})\Bigg\}. \tag{3.5}$$

$\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ as given above verify the transversality condition in the wave zone,

$$\frac{\vec{r}}{r}\times\vec{E}(\vec{r},t)=\vec{B}(\vec{r},t). \tag{3.6}$$

At this point, our aim is to get rid of the integral over ω and we can do that by introducing the double-superscript quantities from Eqs. (1.39)–(1.44), which are, up to numbers, time derivatives (the second superscript) of the multipole mean-square radii of any type (electric, magnetic, toroid) and order (marked by the first superscript). We consider first the magnetic field $\vec{B}(\vec{r},t)$ as given by Eq. (3.5) and treat separately contributions of the terms with $l=$even and $l=$odd to the sum over l.

In the $l=$even case, $l=2k$ ($k=1/2,k=$integer), with the Fourier transformations Eqs. (2.7), (2.10), and (2.11) one has

$$\omega^{l+1}M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)=(-1)^{l/2+1}\int_0^\infty dt'\,M_{lm}^{(l+1)}\left(-\frac{\omega^2}{c^2},t'\right)\sin(\omega t'),$$

$$\omega^{l+1}Q_{lm}(0,\omega)=(-1)^{l/2}\int_0^\infty dt'\,Q_{lm}^{(l+1)}(0,t')\cos(\omega t'), \tag{3.7}$$

$$\omega^{l+2}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)=(-1)^{l/2+1}\int_0^\infty dt'\,T_{lm}^{(l+2)}\left(-\frac{\omega^2}{c^2},t'\right)\cos(\omega t'),$$

where the (single) superscript denotes the order of derivation with respect to the second (time) argument of the formfactors. Using

$$\sin\left(\omega t-\frac{\omega r}{c}+l\frac{\pi}{2}\right)=(-1)^{l/2}\sin\left(\omega t-\frac{\omega r}{c}\right),$$

$$\cos\left(\omega t-\frac{\omega r}{c}+l\frac{\pi}{2}\right)=(-1)^{l/2}\cos\left(\omega t-\frac{\omega r}{c}\right), \tag{3.8}$$

in Eq. (3.5), and then Eqs. (3.7), one gets contributions to $\vec{B}(\vec{r},t)$ from terms containing, e.g.,

$$\int_0^\infty d\omega\,\omega^{l+1}M_{lm}\left(-\frac{\omega^2}{c^2},\omega^2\right)\sin\left(\omega t-\frac{\omega r}{c}\right)=\frac{(-1)^{l/2+1}}{2}\int_0^\infty d\omega\int_0^\infty dt'\,M_{lm}^{(l+1)}\left(-\frac{\omega^2}{c^2},t'\right)\cdot\cos\left(\omega t'-\omega t+\frac{\omega r}{c}\right). \tag{3.9}$$

Now developing the $(l+1)$ derivative (with respect to time) of the magnetic formfactor under the integrals in terms of the (essentially) magnetic radii, i.e., developing as in Eq. (1.31),

$$M_{lm}^{(l+1)}\left(-\frac{\omega^2}{c^2},t'\right)=\sum_{n=0}^{\infty}\frac{\left(-\frac{\omega^2}{c^2}\right)^n}{n!}M_{lm}^{(n)(l+1)}(0,t'),$$

one introduces the double-superscript quantities $M_{lm}^{(n)(l+1)}(0,t')$ defined in Eq. (1.40) and succeeds so to take both integrals (over t' and over ω) in Eq. (3.9) and express the result in terms of a Taylor series,

$$\int_0^\infty d\omega\,\omega^{l+1}M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]=(-1)^{l/2+1}\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{n!\,c^{2n}}M_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right). \tag{3.10}$$

To get Eq. (3.10), use has been made of the relation

$$\int_0^\infty \omega^{2n}\cos\left[\omega\left(t'-t+\frac{r}{c}\right)\right]d\omega=(-1)^n\pi\delta^{(2n)}\left(t'-t+\frac{r}{c}\right). \tag{3.11}$$

Analogously, one finds

$$\int_0^\infty d\omega\,\omega^{l+1}Q_{lm}(0,\omega)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]=(-1)^{l/2}\frac{\pi}{2}Q_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right), \tag{3.12}$$

$$\int_0^\infty d\omega\,\omega^{l+2}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]=(-1)^{l/2+1}\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{n!\,c^{2n}}T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right), \tag{3.13}$$

which completes the calculation of the $l=$even part of $\vec{B}(\vec{r},t)$ in $O(1/r)$.

In the $l=$odd case $[l=2k+1,k=(l-1)/2=$integer$]$ the analysis goes on the same lines, with only minor modifications, which compensate themselves, so that the final result for $\vec{B}(\vec{r},t)$ at large distances $[O(1/r)]$ remains the same, irrespective of whether l is even or odd in the sum over l. In the wave zone, the electric field $\vec{E}(\vec{r},t)$ can be computed from $\vec{B}(\vec{r},t)$ as

$$\vec{E}(\vec{r},t)=-\frac{\vec{r}}{r}\times\vec{B}(\vec{r},t). \tag{3.14}$$

So, we obtain the following expressions for the electric and magnetic fields at large distances in order $O(1/r)$ expressed in terms of the double derivatives of the formfactors, where the first superscript indicates the order of derivation with respect to the first argument of the formfactor at zero value of this argument, while the second superscript indicates the order of derivation with respect to the second argument. [i.e., in terms of the double-superscript quantities introduced in Eqs. (1.39)–(1.44), which are, up to numbers, time derivatives (the second superscript) of the multipole mean-square radii (electric, magnetic, and toroid) of any order (the first superscript)],

$$\begin{aligned}\vec{E}(\vec{r},t)\overset{O(1/r)}{\sim}&\frac{1}{r}\sum_{lm}\frac{1}{c^{l+1}}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\\&\times\Bigg\{-Q_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right)\\&\times\left[\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n})+\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})\right]\\&-i\sum_{n=0}^{\infty}\frac{1}{n!c^{2n}}M_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\vec{Y}_{llm}(\vec{n})\\&+\frac{1}{c}\sum_{n=0}^{\infty}\frac{1}{n!c^{2n}}T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right)\\&\times\left[\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n})+\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})\right]\Bigg\},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\vec{B}(\vec{r},t)\overset{O(1/r)}{\sim}&\frac{1}{r}\sum_{lm}\frac{1}{c^{l+1}}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\\&\times\Bigg\{-iQ_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right)\vec{Y}_{llm}(\vec{n})\\&+\sum_{n=0}^{\infty}\frac{1}{n!c^{2n}}M_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\\&\times\left[\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n})+\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})\right]\\&+\frac{i}{c}\sum_{n=0}^{\infty}\frac{1}{n!c^{2n}}T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right)\vec{Y}_{llm}(\vec{n})\Bigg\},\end{aligned} \tag{3.16}$$

$$\frac{\vec{r}}{r}\times\vec{E}(\vec{r},t)\overset{O(1/r)}{=}\vec{B}(\vec{r},t).$$

We stress again that the double-superscript quantities $Q_{lm}^{(n)(\nu)}(0,t)$, $M_{lm}^{(n)(\nu)}(0,t)$, $T_{lm}^{(n)(\nu)}(0,t)$, have been defined through Eqs. (1.39)–(1.44) [in the equations above they appear, of course, at the retarded time $(t-r/c)$], which, for clarity purposes, we give again here:

$$Q_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\frac{d^n}{d(-k^2)^n}Q_{lm}(-k^2,t)\bigg|_{k^2=0}\right], \tag{1.39}$$

$$M_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\frac{d^n}{d(-k^2)^n}M_{lm}(-k^2,t)\bigg|_{k^2=0}\right], \tag{1.40}$$

$$T_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\frac{d^n}{d(-k^2)^n}T_{lm}(-k^2,t)\bigg|_{k^2=0}\right]. \tag{1.41}$$

For the calculation of the angular momentum loss by the system in Sec. V one needs $\vec{n}\vec{E}$, $\vec{n}\vec{B}$, $\vec{n}\times\vec{E}$, $\vec{n}\times\vec{B}$, ($\vec{n}=\vec{r}/r$) at large distances. We shall first see to order $O(1/r)$ what the results are and since $\vec{n}\vec{E},\vec{n}\vec{B}$ turn out to be vanishing in this order, we shall evaluate further $\vec{n}\vec{E}$ and $\vec{n}\vec{B}$ to the next $O(1/r^2)$ order to get the relevant first nonvanishing contributions.

For the time being, let us confine ourselves to the first order $O(1/r)$. Regardless of approximations one has

$$\vec{n}\vec{\mathcal{F}}_{lm(\omega/c)}^{(0)}(\vec{r})=\vec{n}\vec{\mathcal{N}}_{lm(\omega/c)}^{(0)}(\vec{r})=0,(\vec{n}=\vec{r}/r), \tag{3.17}$$

since $(\vec{Y}_{llm})_r=0$, while for the $(+)$, $(-)$ superscripts, using

$$[Y_{ll+lm}(\vec{n})]_r=-\frac{\sqrt{l+1}}{\sqrt{2l+1}}Y_{lm}(\vec{n}),$$

$$[Y_{ll-lm}(\vec{n})]_r=\frac{\sqrt{l}}{\sqrt{2l+1}}Y_{lm}(\vec{n}),$$

one has

$$\vec{n}\vec{\mathcal{F}}_{lm(\omega/c)}^{(+)}(\vec{r})=\frac{\sqrt{l(l+1)}}{(2l+1)}\left[-j_{l+1}\left(\frac{\omega}{c}r\right)+j_{l-1}\left(\frac{\omega}{c}r\right)\right]Y_{lm}(\vec{n}), \tag{3.18}$$

$$\vec{n}\vec{\mathcal{F}}_{lm(\omega/c)}^{(-)}(\vec{r})=\frac{1}{(2l+1)}\left[lj_{l-1}\left(\frac{\omega}{c}r\right)+(l+1)j_{l+1}\left(\frac{\omega}{c}r\right)\right]Y_{lm}(\vec{n}), \tag{3.19}$$

$$\vec{n}\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})=\frac{\sqrt{l(l+1)}}{(2l+1)}\left[-n_{l+1}\left(\frac{\omega}{c}r\right)+n_{l-1}\left(\frac{\omega}{c}r\right)\right]Y_{lm}(\vec{n}), \tag{3.20}$$

$$\vec{n}\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})=\frac{1}{(2l+1)}\left[l n_{l-1}\left(\frac{\omega}{c}r\right)+(l+1)n_{l+1}\left(\frac{\omega}{c}r\right)\right]Y_{lm}(\vec{n}). \tag{3.21}$$

Therefore, using the asymptotical behavior to $O(1/r)$ of j_l and n_l (see Appendix A), one gets

$$\vec{n}\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})\underset{r\to\infty}{\overset{O(1/r)}{\sim}}0, \tag{3.22}$$

$$\vec{n}\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})\underset{r\to\infty}{\overset{O(1/r)}{\sim}}-4\pi i^{l+1}\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}Y_{lm}(\vec{n}), \tag{3.23}$$

$$\vec{n}\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})\underset{r\to\infty}{\overset{O(1/r)}{\sim}}0, \tag{3.24}$$

$$\vec{n}\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})\underset{r\to\infty}{\overset{O(1/r)}{\sim}}-4\pi i^{l+1}\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}Y_{lm}(\vec{n}). \tag{3.25}$$

With Eqs. (2.39), (2.40) one sees that indeed to order $O(1/r)$ one has

$$\frac{\vec{r}}{r}\vec{E}(\vec{r},t)\underset{r\to\infty}{\overset{O(1/r)}{\sim}}0, \tag{3.26}$$

$$\frac{\vec{r}}{r}\vec{B}(\vec{r},t)\underset{r\to\infty}{\overset{O(1/r)}{\sim}}0. \tag{3.27}$$

Now, by looking to the expressions of the fields $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ to $O(1/r)$ [Eqs. (3.4) and (3.5)], one sees that to find $\vec{n}\times\vec{E}$, $\vec{n}\times\vec{B}$ to the same $1/r$ order, one needs the vector products between $\vec{n}=\vec{r}/r$ and the appearing vector spherical harmonics. With the unit vectors: $\vec{e}_r=\vec{n}=\vec{r}/r$, $\vec{e}_\theta$, $\vec{e}_\varphi$, one has

$$\begin{aligned}\vec{e}_r\times\vec{Y}_{llm}(\vec{n})&=\vec{e}_\varphi(\vec{Y}_{llm})_\theta-\vec{e}_\theta(\vec{Y}_{llm})_\varphi\\&=i\frac{\sqrt{2l+1}}{\sqrt{l+1}}\left[\vec{Y}_{ll-1m}-\vec{e}_r\frac{\sqrt{l}}{\sqrt{2l+1}}Y_{lm}\right]\\&=i\frac{\sqrt{2l+1}}{\sqrt{l}}\left[\vec{Y}_{ll+1m}+\vec{e}_r\frac{\sqrt{l+1}}{\sqrt{2l+1}}Y_{lm}\right],\end{aligned} \tag{3.28}$$

since

$$(\vec{Y}_{llm})_\theta=i\frac{\sqrt{2l+1}}{\sqrt{l+1}}(\vec{Y}_{ll-1m})_\varphi=i\frac{\sqrt{2l+1}}{\sqrt{l}}(\vec{Y}_{ll+1m})_\varphi,$$

$$(\vec{Y}_{llm})_\varphi=-i\frac{\sqrt{2l+1}}{\sqrt{l+1}}(\vec{Y}_{ll-1m})_\theta=-i\frac{\sqrt{2l+1}}{\sqrt{l}}(\vec{Y}_{ll+1m})_\theta.$$

Eliminating the terms with $\vec{e}_rY_{lm}$ in Eqs. (3.28), one gets

$$\vec{n}\times\vec{Y}_{llm}(\vec{n})=i\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})+i\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n}). \tag{3.29}$$

Also, one has

$$\begin{aligned}\vec{e}_r\times\vec{Y}_{ll+1m}(\vec{n})&=\vec{e}_\varphi(\vec{Y}_{ll+1m})_\theta-\vec{e}_\theta(\vec{Y}_{ll+1m})_\varphi\\&=i\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{llm}(\vec{n}),\end{aligned} \tag{3.30}$$

since

$$(\vec{Y}_{ll+1m})_\theta=i\frac{\sqrt{l}}{\sqrt{2l+1}}(\vec{Y}_{llm})_\varphi,$$

$$(\vec{Y}_{ll+1m})_\varphi=-i\frac{\sqrt{l}}{\sqrt{2l+1}}(\vec{Y}_{llm})_\theta,$$

and, analogously,

$$\vec{e}_r\times\vec{Y}_{ll-1m}(\vec{n})=i\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{llm}(\vec{n}). \tag{3.31}$$

Using the vector products of Eqs. (3.29)–(3.31), one finds the following general relations expressing $\vec{n}\times\vec{\mathcal{F}}^{(\lambda)}_{lm(\omega/c)}(\vec{r})$, $\vec{n}\times\vec{\mathcal{N}}^{(\lambda)}_{lm(\omega/c)}(\vec{r})$ ($\vec{n}=\vec{e}_r=\vec{r}/r,\lambda=0,\pm$):

$$\vec{n}\times\vec{\mathcal{F}}^{(0)}_{lm(\omega/c)}(\vec{r})=j_l\left(\frac{\omega}{c}r\right)\left[i\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})+i\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n})\right], \tag{3.32}$$

$$\vec{n}\times\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})=\frac{i}{(2l+1)}\left[lj_{l+1}\left(\frac{\omega}{c}r\right)+(l+1)j_{l-1}\left(\frac{\omega}{c}r\right)\right]\vec{Y}_{llm}(\vec{n}),\tag{3.33}$$

$$\vec{n}\times\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})=\frac{i\sqrt{l(l+1)}}{(2l+1)}\left[j_{l-1}\left(\frac{\omega}{c}r\right)-j_{l+1}\left(\frac{\omega}{c}r\right)\right]\vec{Y}_{llm}(\vec{n}),\tag{3.34}$$

$$\vec{n}\times\vec{\mathcal{N}}^{(0)}_{lm(\omega/c)}(\vec{r})=n_l\left(\frac{\omega}{c}r\right)\left[i\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})+i\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n})\right],\tag{3.35}$$

$$\vec{n}\times\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})=\frac{i}{(2l+1)}\left[ln_{l+1}\left(\frac{\omega}{c}r\right)+(l+1)n_{l-1}\left(\frac{\omega}{c}r\right)\right]\vec{Y}_{llm}(\vec{n}),\tag{3.36}$$

$$\vec{n}\times\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})=\frac{i\sqrt{l(l+1)}}{(2l+1)}\left[n_{l-1}\left(\frac{\omega}{c}r\right)-n_{l+1}\left(\frac{\omega}{c}r\right)\right]\vec{Y}_{llm}(\vec{n}).\tag{3.37}$$

Now, using the large r behavior of the spherical Bessel functions to order $1/r$, one finds from Eqs. (3.32)–(3.37) above the desired asymptotical behavior [to $O(1/r)$] of the vector products $\vec{n}\times\vec{\mathcal{F}}^{(\lambda)}_{lm(\omega/c)}(\vec{r})$, $\vec{n}\times\vec{\mathcal{N}}^{(\lambda)}_{lm(\omega/c)}(\vec{r})$ ($\vec{n}=\vec{r}/r$):

$$\vec{n}\times\vec{\mathcal{F}}^{(0)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}\frac{4\pi i^{l+1}}{\sqrt{2l+1}}[\sqrt{l+1}\vec{Y}_{ll-1m}(\vec{n})+\sqrt{l}\vec{Y}_{ll+1m}(\vec{n})]\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},\tag{3.32$'$}$$

$$\vec{n}\times\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}4\pi i^l\vec{Y}_{llm}(\vec{n})\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},\tag{3.33$'$}$$

$$\vec{n}\times\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}0;\tag{3.34$'$}$$

$$\vec{n}\times\vec{\mathcal{N}}^{(0)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}-\frac{4\pi i^{l+1}}{\sqrt{2l+1}}[\sqrt{l+1}\vec{Y}_{ll-1m}(\vec{n})+\sqrt{l}\vec{Y}_{ll+1m}(\vec{n})]\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},\tag{3.35$'$}$$

$$\vec{n}\times\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}4\pi i^l\vec{Y}_{llm}(\vec{n})\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r},\tag{3.36$'$}$$

$$\vec{n}\times\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r})\overset{O(1/r)}{\sim}0.\tag{3.37$'$}$$

With the aid of the Eqs. (3.32′)–(3.37′) given above, one finally obtains the desired asymptotical behavior at large distances, in order $O(1/r)$, of the vector product $(\vec{r}/r)\times\vec{E}(\vec{r},t)$ of the electric field, in terms of the electric, magnetic, and toroid formfactors

$$\begin{aligned}\frac{\vec{r}}{r}\times\vec{E}(\vec{r},t)\overset{O(1/r)}{\sim}&-\frac{4}{\sqrt{\pi}}\frac{1}{r}\int_0^\infty d\omega\sum_{i,m}\frac{\omega^{l+1}}{c^{l+1}}\\&\times\frac{\sqrt{l+1}}{(2l+1)!!\sqrt{l}}\left\{\sqrt{l+1}M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right.\\&\times\sin\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)\vec{Y}_{ll-1m}(\vec{n})\\&+\sqrt{l}M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\\&\times\sin\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)\vec{Y}_{ll+1m}(\vec{n})\\&+i\sqrt{2l+1}\left[Q_{lm}(0,\omega)+\frac{\omega}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]\\&\left.\times\cos\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)\vec{Y}_{llm}(\vec{n})\right\}.\end{aligned}\tag{3.38}$$

Similarly, one finds for the vector product of $\vec{n}$ with the magnetic field $\vec{B}$ at large r in order $O(1/r)$ the expression:

$$\frac{\vec{r}}{r}\times\vec{B}(\vec{r},t)\overset{O(1/r)}{\sim}-\frac{4}{\sqrt{\pi}}\frac{1}{r}\int_0^\infty d\omega\sum_{i,m}\frac{\omega^{l+1}}{c^{l+1}}\frac{\sqrt{l+1}}{(2l+1)!!\sqrt{l}}\times\Bigg\{i\sqrt{2l+1}M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\times\sin\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)\vec{Y}_{llm}(\vec{n})-\left[Q_{lm}(0,\omega)+\frac{\omega}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right]\cos\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)\times[\sqrt{l+1}\vec{Y}_{ll-1m}(\vec{n})+\sqrt{l}\vec{Y}_{ll+1m}(\vec{n})]\Bigg\}. \quad (3.39)$$

Now we shall use Eqs. (3.38) and (3.39) to find the corresponding expressions for $\vec{n}\times\vec{E}$, $\vec{n}\times\vec{B}$ to order $O(1/r)$ in terms of the double-superscript quantities $Q_{lm}^{(n)(\nu)}(0,t)$, $M_{lm}^{(n)(\nu)}(0,t)$, $T_{lm}^{(n)(\nu)}(0,t)$ [Eqs. (1.39)–(1.41)], just as we did in going from Eqs. (3.4) and (3.5) to Eqs. (3.15) and (3.16) in the case of $\vec{E}$, $\vec{B}$ [order $O(1/r)$]. One finally finds

$$\frac{\vec{r}}{r}\times\vec{E}(\vec{r},t)\overset{O(1/r)}{\sim}\frac{2i\sqrt{\pi}}{r}\sum_{l,m}\frac{1}{c^{l+1}}\frac{\sqrt{(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\times\left[-Q_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right)+\sum_{n=0}^{\infty}\frac{1}{n!c^{2n+1}}\times T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right)\right]\vec{Y}_{llm}(\vec{n})+\frac{2\sqrt{\pi}}{r}\times\sum_{lm}\sum_{n=0}^{\infty}\frac{1}{n!c^{l+2n+1}}\frac{\sqrt{l+1}}{\sqrt{l}(2l+1)!!}\times M_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)[\sqrt{l+1}\vec{Y}_{ll-1m}(\vec{n})+\sqrt{l}\vec{Y}_{ll+1m}(\vec{n})], \quad (3.40)$$

$$\frac{\vec{r}}{r}\times\vec{B}(\vec{r},t)\overset{O(1/r)}{\sim}\frac{2\sqrt{\pi}}{r}\sum_{l,m}\frac{\sqrt{l+1}}{\sqrt{l}(2l+1)!!}\times\left[\frac{1}{c^{l+1}}Q_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right)-\sum_{n=0}^{\infty}\frac{1}{n!c^{l+2n+2}}T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right)\right]\times[\sqrt{l+1}\vec{Y}_{ll-1m}(\vec{n})+\sqrt{l}\vec{Y}_{ll+1m}(\vec{n})]+\frac{2i\sqrt{\pi}}{r}\sum_{lm}\sum_{n=0}^{\infty}\frac{\sqrt{(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\frac{1}{n!c^{l+2n+1}}\times M_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\vec{Y}_{llm}(\vec{n}). \quad (3.41)$$

With these formulas we end our considerations restricted to the order $O(1/r)$ and evaluate in the next, $1/r^2$ order, $\vec{n}\vec{E}$, $\vec{n}\vec{B}$. So, we shall find the first nonvanishing contributions to these quantities, since in $O(1/r)$ they are zero [Eqs. (3.26) and (3.27)]. To this purpose, one needs the next terms in the asymptotical behavior of the spherical Bessel functions as compared with Eqs. (3.1). We shall use

$$j_l(x)\underset{x\to\infty}{\overset{O(1/x^2)}{\sim}}4\pi i^l\frac{\sin\left(x-l\frac{\pi}{2}\right)}{x}+2\pi i^l l(l+1)\frac{\cos\left(x-l\frac{\pi}{2}\right)}{x^2},$$

$$n_l(x)\underset{x\to\infty}{\overset{O(1/x^2)}{\sim}}-4\pi i^l\frac{\cos\left(x-l\frac{\pi}{2}\right)}{x}+2\pi i^l l(l+1)\frac{\sin\left(x-l\frac{\pi}{2}\right)}{x^2}, \quad (3.42)$$

which follow from the development

$$J_{l+1/2}(x)=\sqrt{\frac{2}{\pi x}}\left\{\sin\left(x-l\frac{\pi}{2}\right)\sum_{k=0}^{[l/2]}\frac{(-1)^k(l+2k)!}{(2k)!(l-2k)!(2x)^{2k}}+\cos\left(x-l\frac{\pi}{2}\right)\times\sum_{k=0}^{[(l-1)/2]}\frac{(-1)^k(l+2k+1)!}{(2k+1)!(l+2k-1)!(2x)^{2k+1}}\right\},$$

and the relations

$$j_l(x)=(2\pi)^{3/2}\frac{i^l}{\sqrt{x}}J_{l+1/2}(x),n_l(x)=(2\pi)^{3/2}\frac{i^l}{\sqrt{x}}N_{l+1/2}(x),$$

$$N_{l+1/2}(x)=(-1)^{l-1}J_{-l-1/2}(x).$$

With Eqs. (3.42) one has to $O(1/r^2)$ ($\vec{n}=\vec{n}/r$),

$$\vec{n}\cdot\vec{\mathcal{F}}^{(+)}_{lm(\omega/c)}(\vec{r})\sim-4\pi i^{l+1}\sqrt{l(l+1)}\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega^2}{c^2}r^2}Y_{lm}(\vec{n}),$$

$$\vec{n}\cdot\vec{\mathcal{F}}^{(-)}_{lm(\omega/c)}(\vec{r})\sim\left[-4\pi i^{l+1}\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r}+2\pi i^{l+1}(l^2+l+2)\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega^2}{c^2}r^2}\right]Y_{lm}(\vec{n}),$$

$$\vec{n}\cdot\vec{\mathcal{N}}^{(+)}_{lm(\omega/c)}(\vec{r})\sim 4\pi i^{l+1}\sqrt{l(l+1)}\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega^2}{c^2}r^2}Y_{lm}(\vec{n}),$$

$$\vec{n}\cdot\vec{\mathcal{N}}^{(-)}_{lm(\omega/c)}(\vec{r}) \sim\left[-4\pi i^{l+1}\frac{\sin\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega}{c}r} -2\pi i^{l+1}(l^2+l+2)\frac{\cos\left(\frac{\omega}{c}r-l\frac{\pi}{2}\right)}{\frac{\omega^2}{c^2}r^2}\right]Y_{lm}(\vec{n}). \tag{3.43}$$

So, from Eqs. (2.39) and (2.40) and Eqs. (3.43) above, one finds the desired leading contributions to $\vec{n}\cdot\vec{E}$ and $\vec{n}\cdot\vec{B}$ in $O(1/r^2)$,

$$\frac{\vec{r}}{r}\cdot\vec{E}(\vec{r},t)\overset{O(1/r)}{\sim}\frac{4}{r^2\pi^{1/2}}\int_0^\infty d\omega \times\sum_{l,m}\frac{(l+1)\sqrt{2l+1}}{(2l+1)!!}\left(\frac{\omega}{c}\right)^l \times\left[Q_{lm}(0,\omega)+\frac{\omega}{c}T_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\right] \times\sin\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right)Y_{lm}(\vec{n}), \tag{3.44}$$

$$\frac{\vec{r}}{r}\cdot\vec{B}(\vec{r},t)\overset{O(1/r)}{\sim}-\frac{4}{r^2\pi^{1/2}}\int_0^\infty d\omega\sum_{l,m}\frac{(l+1)\sqrt{2l+1}}{(2l+1)!!}\left(\frac{\omega}{c}\right)^l \times M_{lm}\left(-\frac{\omega^2}{c^2},\omega\right)\cos\left(\omega t-\frac{\omega}{c}r+l\frac{\pi}{2}\right) \times Y_{lm}(\vec{n}). \tag{3.45}$$

To have the corresponding expressions of $\vec{n}\cdot\vec{E}$ and $\vec{n}\cdot\vec{B}$ to $O(1/r^2)$ in terms of time derivatives of radii [i.e., in terms of the double-superscript quantities $Q^{(n)(\nu)}_{lm}(0,t)$, $M^{(n)(\nu)}_{lm}(0,t)$, $T^{(n)(\nu)}_{lm}(0,t)$ defined by Eqs. (1.39)–(1.41)], one has again to use Eqs. (3.10)–(3.13) and consider separately the $l=$even and $l=$odd contributions to the sum over l. One finally finds

$$\frac{\vec{r}}{r}\cdot\vec{E}(\vec{r},t)\overset{O(1/r)}{\sim}\frac{2\pi^{1/2}}{r^2}\sum_{l,m}\frac{(l+1)\sqrt{2l+1}}{(2l+1)!!c^l}\left[Q^{(0)(l)}_{lm}\left(0,t-\frac{r}{c}\right) -\sum_{n=0}^\infty\frac{1}{n!c^{2n+1}}T^{(n)(l+2n+1)}_{lm}\left(0,t-\frac{r}{c}\right)\right] \times Y_{lm}(\vec{n}), \tag{3.46}$$

$$\frac{\vec{r}}{r}\cdot\vec{B}(\vec{r},t)\overset{O(1/r^2)}{\sim}-\frac{2\pi^{1/2}}{r^2}\sum_{l,m}\frac{(l+1)\sqrt{2l+1}}{(2l+1)!!c^l} \times\left[\sum_{n=0}^\infty\frac{1}{n!c^{2n}}M^{(n)(l+2n)}_{lm}\left(0,t-\frac{r}{c}\right)\right]Y_{lm}(\vec{n}). \tag{3.47}$$

At this point we end the analysis of the large distance behavior of the fields; we have at our disposal all the quantities necessary for the further calculation of the angular momentum loss, recoil force, and the radiation intensity in terms of the multipole content of the most general type of source.

IV. RADIATION INTENSITY

Here we calculate the radiation intensity (for a general distribution of charges and currents) in the case of the most general time variation of the sources, in terms of quantities of the type $\mathcal{M}^{(n)(l+q)}_{lm}(0,t-r/c)$ where $\mathcal{M}_{lm}$ stands for any multipole form factor $Q_{lm}(k^2,t)$, $M_{lm}(k^2,t)$, $T_{lm}(k^2,t)$ and the first superscript denotes the (n) derivative with respect to $(-k^2)$ at $k^2=0$, while the second superscript denotes the $(l+q)$ derivative with respect to t at $t-r/c$.

We have first to evaluate the Poynting vector

$$\vec{S}=\frac{c}{4\pi}(\vec{E}\times\vec{B}), \tag{4.1}$$

and express the radiation intensity as the surface integral of the radial component of $\vec{S}$, S_r, over a sphere of large radius r [21],

$$I=r^2\int d\Omega S_r=\frac{cr^2}{4\pi}\int d\Omega(\vec{E}\times\vec{B})_r. \tag{4.2}$$

The result should be essentially r independent, of course. The electric and magnetic fields. $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ need to be evaluated to order $O(1/r)$ and the corresponding formulas are those of Eqs. (3.15) and (3.16).

One needs to evaluate the surface integral over the r component of the vector,

$$\vec{E}\times\vec{B}=\frac{4\pi}{r^2}\sum_{l,m,l',m'}\frac{1}{c^{l+l'+2}}\frac{\sqrt{(2l+1)(l+1)(2l'+1)(l'+1)}}{\sqrt{l}\sqrt{l'}(2l+1)!!(2l'+1)!!}\Bigg\{-i\sum_{n=0}^{\infty}\frac{1}{n!c^{2n}}M_{lm}^{(n)(l+2n+1)}\vec{Y}_{llm}-Q_{lm}^{(0)(l+1)}$$
$$\cdot\left[\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}+\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}\right]+\frac{1}{c}\sum_{n=0}^{\infty}\frac{1}{n!c^{2n}}T_{lm}^{(n)(l+2n+2)}\left[\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}+\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}\right]\Bigg\}$$
$$\times\Bigg\{\sum_{n'=0}^{\infty}\frac{1}{n'!c^{2n'}}M_{l'm'}^{(n')(l'+2n'+1)}\left[\frac{\sqrt{l'}}{\sqrt{2l'+1}}\vec{Y}_{l'l'+1m'}+\frac{\sqrt{l'+1}}{\sqrt{2l'+1}}\vec{Y}_{l'l'-1m'}\right]$$
$$-iQ_{l'm'}^{(0)(l'+1)}\vec{Y}_{l'l'm'}+\frac{i}{c}\sum_{n'=0}^{\infty}\frac{1}{n'!c^{2n'}}T_{l'm'}^{(n')(l'+2n'+2)}\vec{Y}_{l'l'm'}\Bigg\}. \tag{4.3}$$

The argument of the vector spherical harmonics $\vec{Y}$ is $\vec{n}=\vec{r}/r$. The surface integration in Eq. (4.2) over the vector products of spherical vector harmonics $\vec{Y}_{ll'm}$ have been performed by reducing them to scalar products of $\vec{Y}_{ll'm}$, when the integral can be taken immediately. So, using formulas of the type

$$(\vec{Y}_{l'l'm'}\times\vec{Y}_{ll+1m})_r=-i\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{l'l'm'}\cdot\vec{Y}_{llm}, \tag{4.4}$$

$$(\vec{Y}_{l'l'+1m'}\times\vec{Y}_{ll-1m})_r=i\frac{\sqrt{l'}}{\sqrt{2l'+1}}\vec{Y}_{l'l'm'}\cdot\vec{Y}_{ll-1m}, \tag{4.5}$$

and

$$\int d\Omega(\vec{Y}_{ll+1m}\times\vec{Y}_{l'l'm'})_r=-i(-1)^m\frac{\sqrt{l}}{\sqrt{l+1}}\delta_{ll'}\delta_{m,-m'}, \tag{4.6}$$

$$\int d\Omega(\vec{Y}_{ll-1m}\times\vec{Y}_{l'l'm'})_r=-i(-1)^m\frac{\sqrt{l+1}}{\sqrt{2l+1}}\delta_{ll'}\delta_{m,-m'}, \tag{4.7}$$

one gets for the intensity of radiation the following general formula in terms of the derivatives of the electric, magnetic, and toroidal form factors (with respect to both arguments $-k^2$ and t), in which the $(1/c)$ powers in front of various contributions are explicitly evidentiated,

$$I=\sum_{l,m}\frac{1}{c^{2l+1}}\frac{(l+1)}{l(2l-1)!!(2l+1)!!}\Bigg\{\left|Q_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right)\right|^2$$
$$-\sum_n\frac{1}{n!c^{2n+1}}\left[Q_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right)T_{lm}^{(n)(l+2n+2)*}\left(0,t-\frac{r}{c}\right)\right.$$
$$\left.+Q_{lm}^{(0)(l+1)*}\left(0,t-\frac{r}{c}\right)T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right)\right]$$
$$+\sum_{n,n'}\frac{1}{n!n'!c^{2n+2n'}}\left[M_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right.$$
$$\times M_{lm}^{(n')(l+2n'+1)*}\left(0,t-\frac{r}{c}\right)+\frac{1}{c^2}T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right)$$
$$\left.\left.\times T_{lm}^{(n')(l+2n'+2)*}\left(0,t-\frac{r}{c}\right)\right]\right\}. \tag{4.8}$$

Equation (4.8) above expresses the radiation intensity for a most general type of source, characterized by the electric, magnetic, and toroid multipole formfactors $Q_{lm}(-k^2,t)$, $M_{lm}(-k^2,t)$, $T_{lm}(-k^2,t)$, for an arbitrary time dependence of the latter, in terms of their derivatives at both arguments [see Eqs. (1.39)–(1.41)], which are nothing else but time derivatives (up to numerical known factors) of the mean-square radii of the multipole distributions of various types and order [see Eqs. (1.42)–(1.44)]. Equation (4.8) exploits exactly and fully the multipole content of the source, for an arbitrary time dependence, in the calculation of the intensity of radiation emmited by that source.

We recall that the first superscript indicates the order of derivation with respect to the first argument, $-k^2$, of the formfactor, while the second, the order of derivation with respect to the second argument, t. All terms appearing in Eq. (4.8) are real, being equal to their complex conjugates. Equation (4.8) of the radiation intensity I is obviously positive definite also, as better seen from the following equivalent but more compact form in which it can be written:

$$I=\sum_{l,m}\frac{1}{c^{2l+1}}\frac{(l+1)}{l(2l-1)!!(2l+1)!!}\Bigg[\Bigg|Q_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right)$$
$$-\frac{1}{c}\sum_n\frac{1}{n!c^{2n}}T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right)\Bigg|^2$$
$$+\left|\sum_n\frac{1}{n!c^{2n}}M_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right|^2\Bigg]. \tag{4.8$'$}$$

From the above formula, one recovers easily in the dipole case $(l=1,n=n'=0)$, the known expressions ([4,6]), for the radiation intensities in the case of the electric dipole,

$$I=\frac{2}{3c^3}\ddot{\vec{d}}^2, \tag{4.9}$$

the magnetic dipole:

$$I=\frac{2}{3c^3}\ddot{\vec{m}}^2, \tag{4.10}$$

and the toroid dipole:

$$I=\frac{2}{3c^5}\dddot{\vec{t}}^2. \tag{4.11}$$

But to the order $(1/c^5)$ there are other contributions too, and from the general expression Eq. (4.8) we can straightforwardly derive the correct formula for the radiation intensity valid to the order $(1/c^5)$ inclusively in terms of the lower electric, magnetic, and toroid multipole quantities. It is

$$I=\frac{2}{3c^3}\ddot{\vec{d}}^2+\frac{2}{3c^3}\ddot{\vec{m}}^2-\frac{4}{3c^4}\ddot{\vec{d}}\dddot{\vec{t}}+\frac{2}{3c^5}\dddot{\vec{t}}^2+\frac{2}{15c^5}\ddot{\vec{m}}\cdot\dddot{\vec{\rho}^2}$$
$$+\frac{1}{5c^5}\dddot{Q}^2_{\alpha\beta}+\frac{1}{20c^5}\ddot{m}^2_{\alpha,\beta}. \tag{4.12}$$

The lower multipoles appearing above are those of Appendix C. For convenience, we recall that $\vec{d}$, $\vec{m}$, $\vec{t}$ are the electric (charge), magnetic, and toroid dipoles, $\vec{\rho}^2$ the first mean-square radius of the magnetic dipole distribution, $Q_{\alpha,\beta}$, $m_{\alpha,\beta}$ the electric and magnetic quadrupole moments. We note also that (summation over repeated indicies is understood)

$$\dddot{Q}^2_{\alpha\beta}\equiv\dddot{Q}_{\alpha\beta}\dddot{Q}_{\alpha\beta}=\dddot{Q}^2_{xx}+\dddot{Q}^2_{yy}+\dddot{Q}^2_{zz}+2\dddot{Q}^2_{xy}+2\dddot{Q}^2_{xz}+2\dddot{Q}^2_{yz}$$
$$=2(\dddot{Q}^2_{xx}+\dddot{Q}^2_{yy}+\dddot{Q}^2_{xy}+\dddot{Q}^2_{xz}+\dddot{Q}^2_{yz}+\dddot{Q}_{xx}\dddot{Q}_{yy}),$$
$$\ddot{m}^2_{\alpha\beta}\equiv\ddot{m}_{\alpha\beta}\ddot{m}_{\alpha\beta}=\ddot{m}^2_{xx}+\ddot{m}^2_{yy}+\ddot{m}^2_{zz}+2\ddot{m}^2_{xy}+2\ddot{m}^2_{xz}+2\ddot{m}^2_{yz}$$
$$=2(\ddot{m}^2_{xx}+\ddot{m}^2_{yy}+\ddot{m}^2_{xy}+\ddot{m}^2_{xz}+\ddot{m}^2_{yz}+\ddot{m}_{xx}\ddot{m}_{yy}),$$

where the relations

$$Q_{xx}+Q_{yy}+Q_{zz}=0,\ \ Q_{ij}=Q_{ji},$$
$$m_{xx}+m_{yy}+m_{zz}=0,\ \ m_{ij}=m_{ji},$$

have been used. Our $Q_{\alpha\beta}$ is $(1/6)D_{\alpha\beta}$ of Ref. [21].

We note that by working with the fields only to the $(1/c^3)$ order, in Ref. [21] are lost not only terms given by the toroid dipole [the third and fourth in Eq. (4.12)], but also the fifth, amounting to radiation on account of the interference between the magnetic dipole moment and the mean-square radius of the magnetic dipole distribution.

V. ANGULAR MOMENTUM LOSS

We consider an arbitrary system of charges and currents described by the charge density $\rho(\vec{r},t)$ and the current density $\vec{j}(\vec{r},t)$ that satisfy the continuity relation

$$\frac{\partial\rho(\vec{r},t)}{\partial t}+\mathrm{div}\,\vec{j}(\vec{r},t)=0$$

for all $\vec{r}$ and t.

We have to evaluate, using the electric and magnetic fields of the source $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ at large distances, the angular momentum loss by the source per unit time [21]

$$\frac{d\vec{M}}{dt}=\lim_{r\to\infty}\frac{r^3}{4\pi}\int d\Omega[(\vec{n}\cdot\vec{E})(\vec{n}\times\vec{E})+(\vec{n}\cdot\vec{B})(\vec{n}\times\vec{B})]. \tag{5.1}$$

The integration has to be done over a spherical surface of large radius r and the limit $r\to\infty$ must subsequently be taken. As it is well known from textbooks (see, e.g., Ref. [21]), Eq. (5.1) is derived by taking into account that the total angular momentum lost by the system per unit time is just the flux of angular momentum of the radiation field through a spherical surface of large radius r:

$$\frac{dM_i}{dt}=\int \varepsilon_{ijk}x_j\sigma_{kl}n_l dS,\ \ dS=r^2d\Omega,\ \ \vec{n}=\frac{\vec{r}}{r}, \tag{5.2}$$

where σ_{ij} is the three-dimensional Maxwell stress tensor,

$$\sigma_{ij}=\frac{1}{4\pi}\left[E_iE_j+B_iB_j-\frac{1}{2}\delta_{ij}(E^2+B^2)\right]. \tag{5.3}$$

Applying Eq. (5.1) to the radiation fields at large distances one cannot, however, take the fields only to the $1/r$ order, since in this order $\vec{n}\cdot\vec{E}=\vec{n}\cdot\vec{B}=0$ and the integrand vanishes. One can use the fields $\vec{E}$ and $\vec{B}$ to $O(1/r)$ only to get the factors $\vec{n}\times\vec{E}$ and $\vec{n}\times\vec{B}$ in Eq. (5.1). The longitudinal components $\vec{n}\vec{E}$ and $\vec{n}\vec{B}$ appear on account of contributions of the next order $O(1/r^2)$ and it is so that the integral in Eq. (5.1) is of order $O(1/r^3)$ and the distance r essentially disappears from the result, as it must. The expressions of $\vec{n}\times\vec{E}$, $\vec{n}\times\vec{B}$ to order $O(1/r)$ and those for $\vec{n}\vec{E}$, $\vec{n}\vec{B}$ to order $O(1/r^2)$ in terms of the multipole content of the source, i.e., in terms of time derivatives of all the electric, magnetic, and toroid mean-square radii of the corresponding distributions, have been obtained in Sec. III, Eqs. (3.40) and (3.41) and Eqs. (3.46) and (3.47), respectively.

The contributions of the electric and magnetic fields to the angular momentum loss as given by Eq. (5.1) will be considered separately,

$$\frac{d\vec{M}}{dt}=\frac{d\vec{M}^{(el)}}{dt}+\frac{d\vec{M}^{(mag)}}{dt}, \tag{5.4}$$

with

$$\frac{d\vec{M}^{(el)}}{dt}=\lim_{r\to\infty}\frac{r^3}{4\pi}\int[(\vec{n}\cdot\vec{E})(\vec{n}\times\vec{E})], \tag{5.5}$$

$$\frac{d\vec{M}^{(mag)}}{dt}=\lim_{r\to\infty}\frac{r^3}{4\pi}\int[(\vec{n}\cdot\vec{B})(\vec{n}\times\vec{B})]. \tag{5.6}$$

The contribution of the electric field will be dealt with first. With Eqs. (3.40) and (3.46), we have then to calculate

$$\frac{d\vec{M}^{(el)}}{dt}=\lim_{r\to\infty}\frac{r^3}{4\pi}(\vec{J}_-+\vec{J}_++\vec{J}_0), \tag{5.7}$$

where

$$\vec{J}_-=\frac{2\sqrt{\pi}}{r^2}\left\{\sum_{l,m}\frac{(l+1)\sqrt{2l+1}}{(2l+1)!!c^l}\left[Q_{lm}^{(0)(l)}\left(0,t-\frac{r}{c}\right)-\sum_n\frac{1}{n!c^{2n+1}}T_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right]\right\}\times\frac{2\sqrt{\pi}}{r}\left\{\sum_{l'm'n'}\frac{1}{n'!c^{l'+2n'+1}}\frac{l'+1}{\sqrt{l'}(2l'+1)!!}\times M_{l'm'}^{(n')(l'+2n'+1)}\left(0,t-\frac{r}{c}\right)\right\}\times\int d\Omega Y_{lm}(\vec{n})\vec{Y}_{l'l'-1m'}(\vec{n}), \tag{5.8}$$

$$\vec{J}_+=\frac{2\sqrt{\pi}}{r^2}\left\{\sum_{l,m}\frac{(l+1)\sqrt{2l+1}}{(2l+1)!!c^l}\left[Q_{lm}^{(0)(l)}\left(0,t-\frac{r}{c}\right)-\sum_n\frac{1}{n!c^{2n+1}}T_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right]\right\}\times\frac{2\sqrt{\pi}}{r}\left\{\sum_{l'm'n'}\frac{1}{n'!c^{l'+2n'+1}}\frac{l'+1}{\sqrt{l'}(2l'+1)!!}\times M_{l'm'}^{(n')(l'+2n'+1)}\left(0,t-\frac{r}{c}\right)\right\}\times\int d\Omega Y_{lm}(\vec{n})\vec{Y}_{l'l'+1m'}(\vec{n}), \tag{5.9}$$

$$\vec{J}_0=\frac{2\sqrt{\pi}}{r^2}\left\{\sum_{l,m}\frac{(l+1)\sqrt{2l+1}}{(2l+1)!!c^l}\left[Q_{lm}^{(0)(l)}\left(0,t-\frac{r}{c}\right)-\sum_n\frac{1}{n!c^{2n+1}}T_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right]\right\}\frac{2i\sqrt{\pi}}{r}\times\left\{\sum_{l'm'}\frac{1}{c^{l'+1}}\frac{\sqrt{(2l'+1)(l'+1)}}{\sqrt{l'}(2l'+1)!!}\times\left[-Q_{l'm'}^{(0)(l'+1)}\left(0,t-\frac{r}{c}\right)+\sum_{n'}\frac{1}{n'!c^{2n'+1}}T_{l'm'}^{(n')(l'+2n'+2)}\left(0,t-\frac{r}{c}\right)\right]\right\}\times\int d\Omega Y_{lm}(\vec{n})\vec{Y}_{l'l'm'}(\vec{n}). \tag{5.10}$$

As it is seen from the Eqs. (5.8)–(5.10) above, the distance r drops out from the expression Eq. (5.7) of the electric part of the angular momentum loss per unit time, as it must.

The integrals over the solid angle still remaining in Eqs. (5.8)–(5.10) can be simply evaluated with the aid of the Cartesian components of the spherical vectors $\vec{Y}_{ll'm}$ ($l'=l,l'=l\pm1$) (expressed in terms of Y_{lm}) given in Appendix D and the usual normalization relations

$$\int d\Omega Y_{lm}(\vec{n})Y^*_{l'm'}(\vec{n})=\delta_{ll'}\delta_{mm'},$$

$$\int d\Omega Y_{lm}(\vec{n})Y_{l'm'}(\vec{n})=(-1)^m\delta_{ll'}\delta_{m,-m'}.$$

The needed integrals then are

$$\int d\Omega Y_{lm}(\vec{n})\vec{Y}_{l'l'm'}(\vec{n})=\vec{e}_x\left[\frac{(-1)^m}{2}\sqrt{\frac{(l+m)(l-m+1)}{l(l+1)}}\delta_{ll'}\delta_{m,-m'+1}+\frac{(-1)^m}{2}\sqrt{\frac{(l-m)(l+m+1)}{l(l+1)}}\delta_{ll'}\delta_{m,-m'-1}\right]+\vec{e}_y\left[\frac{i(-1)^m}{2}\sqrt{\frac{(l+m)(l-m+1)}{l(l+1)}}\delta_{ll'}\delta_{m,-m'+1}-\frac{i(-1)^m}{2}\sqrt{\frac{(l-m)(l+m+1)}{l(l+1)}}\delta_{ll'}\delta_{m,-m'-1}\right]+\vec{e}_z\left[(-1)^{m+1}\frac{m}{\sqrt{l(l+1)}}\delta_{ll'}\delta_{m,-m'}\right], \tag{5.11}$$

$$\int d\Omega Y_{lm}(\vec{n})\vec{Y}_{l'l'-1m'}(\vec{n})=\vec{e}_x\left[-\frac{(-1)^m}{2}\sqrt{\frac{(l-m+1)(l-m+2)}{(l+1)(2l+1)}}\times\delta_{l,l'-1}\delta_{m,-m'+1}+\frac{(-1)^m}{2}\sqrt{\frac{(l+m+1)(l+m+2)}{(l+1)(2l+1)}}\delta_{l,l'-1}\delta_{m,-m'-1}\right]+\vec{e}_y\left[-\frac{i(-1)^m}{2}\sqrt{\frac{(l-m+1)(l-m+2)}{(l+1)(2l+1)}}\times\delta_{l,l'-1}\delta_{m,-m'+1}-\frac{i(-1)^m}{2}\sqrt{\frac{(l+m+1)(l+m+2)}{(l+1)(2l+1)}}\delta_{l,l'-1}\delta_{m,-m'-1}\right]+\vec{e}_z\left[(-1)^m\sqrt{\frac{(l+m+1)(l-m+1)}{(l+1)(2l+1)}}\delta_{l,l'-1}\delta_{m,-m'}\right], \tag{5.12}$$

$$\int d\Omega Y_{lm}(\vec{n})\vec{Y}_{l'l'+1m'}(\vec{n})$$
$$=\vec{e}_x\left[-\frac{(-1)^m}{2}\sqrt{\frac{(l+m-1)(l+m)}{l(2l+1)}}\delta_{l,l'+1}\delta_{m,-m'+1}+\frac{(-1)^m}{2}\sqrt{\frac{(l-m-1)(l-m)}{l(2l+1)}}\delta_{l,l'+1}\delta_{m,-m'-1}\right]$$
$$+\vec{e}_y\left[-\frac{i(-1)^m}{2}\sqrt{\frac{(l+m-1)(l+m)}{l(2l+1)}}\times\delta_{l,l'+1}\delta_{m,-m'+1}-\frac{i(-1)^m}{2}\sqrt{\frac{(l-m-1)(l-m)}{l(2l+1)}}\delta_{l,l'+1}\delta_{m,-m'-1}\right]$$
$$+\vec{e}_z\left[-(-1)^m\sqrt{\frac{(l-m)(l+m)}{l(2l+1)}}\delta_{l,l'+1}\delta_{m,-m'}\right]. \tag{5.13}$$

In this way one finds the desired result for the electric field contribution to the rate of the angular momentum lost by the system through radiation of electromagnetic waves. Writing separately the contributions of $\vec{J}_-$, $\vec{J}_+$, $\vec{J}_0$ to $d\vec{M}^{el}/dt$ according to Eq. (5.7) as

$$\frac{d\vec{M}^{(el)}}{dt}=\left(\frac{d\vec{M}^{(el)}}{dt}\right)^{(\vec{J}_-)}+\left(\frac{d\vec{M}^{(el)}}{dt}\right)^{(\vec{J}_+)}+\left(\frac{d\vec{M}^{(el)}}{dt}\right)^{(\vec{J}_0)},$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)^{(\vec{J}_a)}=\lim_{r\to\infty}\frac{r^3}{4\pi}\vec{J}_a,(a)=-,+,0, \tag{5.7'}$$

one has

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)^{(\vec{J}_-)}_x=\frac{1}{2}\sum_{l,m,n'}\frac{(-1)^{m+1}(l+2)}{(2l+1)!!(2l+3)!!}\times\frac{1}{n'!c^{2l+2n'+2}}\left[Q^{(0)(l)}_{lm}\left(0,t-\frac{r}{c}\right)-\sum_n\frac{1}{n!c^{2n+1}}T^{(n)(l+2n+1)}_{lm}\left(0,t-\frac{r}{c}\right)\right]$$
$$\times\left[\sqrt{(l-m+1)(l-m+2)}\times M^{(n')(l+2n'+2)}_{l+1,1-m}\left(0,t-\frac{r}{c}\right)-\sqrt{(l+m+1)(l+m+2)}\times M^{(n')(l+2n'+2)}_{l+1,-1-m}\left(0,t-\frac{r}{c}\right)\right], \tag{5.14}$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)^{(\vec{J}_-)}_y=\frac{1}{2}\sum_{l,m,n'}\frac{i(-1)^{m+1}(l+2)}{(2l+1)!!(2l+3)!!}\times\frac{1}{n'!c^{2l+2n'+2}}\left[Q^{(0)(l)}_{lm}\left(0,t-\frac{r}{c}\right)-\sum_n\frac{1}{n!c^{2n+l}}T^{(n)(l+2n+1)}_{lm}\left(0,t-\frac{r}{c}\right)\right]$$
$$\times\left[\sqrt{(l-m+1)(l-m+2)}\times M^{(n')(l+2n'+2)}_{l+1,1-m}\left(0,t-\frac{r}{c}\right)+\sqrt{(l+m+1)(l+m+2)}\times M^{(n')(l+2n'+2)}_{l+1,-1-m}\left(0,t-\frac{r}{c}\right)\right], \tag{5.15}$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)^{(\vec{J}_-)}_z=\sum_{l,m,n'}\frac{(-1)^m(l+2)}{(2l+1)!!(2l+3)!!}\frac{\sqrt{(l+m+1)(l-m+1)}}{n'!c^{2l+2n'+2}}\times\left[Q^{(0)(l)}_{lm}\left(0,t-\frac{r}{c}\right)-\sum_n\frac{1}{n!c^{2n+1}}T^{(n)(l+2n+1)}_{lm}\left(0,t-\frac{r}{c}\right)\right]$$
$$\times M^{(n')(l+2n'+2)}_{l+1,-m}\left(0,t-\frac{r}{c}\right); \tag{5.16}$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)^{(\vec{J}_+)}_x=\frac{1}{2}\sum_{l,m,n'}\frac{(-1)^{m+1}(l+1)}{(2l-1)!!(2l+1)!!}\frac{1}{n'!c^{2l+2n'}}\times\left[Q^{(0)(l)}_{lm}\left(0,t-\frac{r}{c}\right)-\sum_n\frac{1}{n!c^{2n+1}}T^{(n)(l+2n+1)}_{lm}\left(0,t-\frac{r}{c}\right)\right]$$
$$\times\left[\sqrt{(l+m-1)(l+m)}M^{(n')(l+2n')}_{l-1,1-m}\left(0,t-\frac{r}{c}\right)-\sqrt{(l-m-1)(l-m)}M^{(n')(l+2n')}_{l-1,-1-m}\left(0,t-\frac{r}{c}\right)\right], \tag{5.17}$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)_y^{(\vec{J}_+)} = \frac{1}{2}\sum_{l,m,n'} \frac{i(-1)^{m+1}(l+1)}{(2l-1)!!(2l+1)!!}\,\frac{1}{n'!c^{2l+2n'}} \times\left[Q_{lm}^{(0)(l)}\left(0,t-\frac{r}{c}\right) - \sum_n \frac{1}{n!c^{2n+1}}\,T_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right] \times\left[\sqrt{(l+m-1)(l+m)}\,M_{l-1,1-m}^{(n')(l+2n')}\left(0,t-\frac{r}{c}\right) + \sqrt{(l-m-1)(l-m)}\,M_{l-1,-1-m}^{(n')(l+2n')}\left(0,t-\frac{r}{c}\right)\right], \tag{5.18}$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)_z^{(\vec{J}_+)} = \sum_{l,m,n'} \frac{(-1)^{m+1}(l+1)}{(2l-1)!!(2l+1)!!}\,\frac{\sqrt{(l-m)(l+m)}}{n'!c^{2l+2n'}} \times\left[Q_{lm}^{(0)(l)}\left(0,t-\frac{r}{c}\right) - \sum_n \frac{1}{n!c^{2n+1}}\,T_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right] \times M_{l-1,-m}^{(n')(l+2n')}\left(0,t-\frac{r}{c}\right); \tag{5.19}$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)_x^{(\vec{J}_0)} = \frac{i}{2}\sum_{l,m} \frac{(-1)^{m}(l+1)}{l(2l-1)!!(2l+1)!!}\,\frac{1}{c^{2l+1}}\left[Q_{lm}^{(0)(l)}\left(0,t-\frac{r}{c}\right) - \sum_n \frac{1}{n!c^{2n+1}}\,T_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right] \times\Bigg\{\sqrt{(l+m)(l-m+1)}\left[-Q_{l,1-m}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right) + \sum_{n'} \frac{1}{n'!c^{2n'+l}}\,T_{l,1-m}^{(n')(l+2n'+2)}\left(0,t-\frac{r}{c}\right)\right] + \sqrt{(l-m)(l+m+1)}\left[-Q_{l,-1-m}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right) + \sum_{n'} \frac{1}{n'!c^{2n'+1}}\,T_{l,-1-m}^{(n')(l+2n'+2)}\left(0,t-\frac{r}{c}\right)\right]\Bigg\}, \tag{5.20}$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)_y^{(\vec{J}_0)} = \frac{1}{2}\sum_{l,m} \frac{(-1)^{m}(l+1)}{l(2l-1)!!(2l+1)!!}\,\frac{1}{c^{2l+1}}\left[Q_{lm}^{(0)(l)}\left(0,t-\frac{r}{c}\right) - \sum_n \frac{1}{n!c^{2n+l}}\,T_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right] \times\Bigg\{\sqrt{(l+m)(l-m+1)}\left[-Q_{l,1-m}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right) + \sum_{n'} \frac{1}{n'!c^{2n'+l}}\,T_{l,1-m}^{(n')(l+2n'+2)}\left(0,t-\frac{r}{c}\right)\right] - \sqrt{(l-m)(l+m+1)}\left[-Q_{l-1-m}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right) + \sum_{n'} \frac{1}{n'!c^{2n'+l}}\,T_{l,-m}^{(n')(l+2n'+2)}\left(0,t-\frac{r}{c}\right)\right]\Bigg\}, \tag{5.21}$$

$$\left(\frac{d\vec{M}^{(el)}}{dt}\right)_z^{(\vec{J}_0)} = \sum_{l,m} \frac{i(-1)^{m}m(l+1)}{l(2l-1)!!(2l+1)!!}\,\frac{1}{c^{2l+1}}\left[Q_{lm}^{(0)(l)}\left(0,t-\frac{r}{c}\right) - \sum_n \frac{1}{n!c^{2n+1}}\,T_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\right] \times\left[-Q_{l,-m}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right) + \sum_{n'} \frac{1}{n'!c^{2n'+1}}\,T_{l,-m}^{(n')(l+2n'+2)}\left(0,t-\frac{r}{c}\right)\right]. \tag{5.22}$$

It is worth noting that the magnetic-type multipole moments and radii of the system contribute to $d\vec{M}^{(el)}/dt$ (through interferences with the electric and toroid corespondents) only through the $\vec{J}_\pm$-pieces [Eqs. (5.14)–(5.19)], while they are absent from the $\vec{J}_0$ piece [Eqs. (5.20)–(5.22)], in which only interferences of the type (electric $+$toroid)$\times$(electric$+$toroid) are there.

The evaluation of the magnetic-field contribution to the angular momentum loss, given by Eq. (5.6), goes entirely on the same line as in the electric field case, so we skip over details and give directly the lengthy result, but splitting, however, the long expression in a slightly different manner as before, by evidentiating perhaps more suggestively the various types of interferences. So one has

$$\frac{d\vec{M}^{(mag)}}{dt}=\left(\frac{d\vec{M}^{(mag)}}{dt}\right)_{M,M}+\left(\frac{d\vec{M}^{(mag)}}{dt}\right)_{M,Q+T} \tag{5.23}$$

where the subscript M, M indicates interferences between the magnetic type multipole radii themselves, while the subscript M, $Q+T$ indicates interferences between magnetic and electric+toroid type radii,

$$\left(\frac{dM_x^{(mag)}}{dt}\right)_{M,M}=-\frac{i}{2}\sum_{l,m,n,n'}\frac{(-1)^m(l+1)}{l(2l-1)!!(2l+1)!!}\frac{\sqrt{(l+m)(l-m+1)}}{n!n'!c^{2l+2n+2n'+1}}\times M_{lm}^{(n)(l+2n)}\left(0,t-\frac{r}{c}\right)M_{l,1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right)$$
$$-\frac{i}{2}\sum_{l,m,n,n'}\frac{(-1)^m(l+1)}{l(2l-1)!!(2l+1)!!}\frac{\sqrt{(l-m)(l+m+1)}}{n!n'!c^{2l+2n+2n'+1}}\times M_{lm}^{(n)(l+2n)}\left(0,t-\frac{r}{c}\right)M_{l,-1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right), \tag{5.24}$$

$$\left(\frac{dM_y^{(mag)}}{dt}\right)_{M,M}=\frac{1}{2}\sum_{l,m,n,n'}\frac{(-1)^m(l+1)}{l(2l-1)!!(2l+1)!!}\frac{\sqrt{(l+m)(l-m+1)}}{n!n'!c^{2l+2n+2n'+1}}\times M_{lm}^{(n)(l+2n)}\left(0,t-\frac{r}{c}\right)M_{l,1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right)$$
$$-\frac{1}{2}\sum_{l,m,n,n'}\frac{(-1)^m(l+1)}{l(2l-1)!!(2l+1)!!}\frac{\sqrt{(l-m)(l+m+1)}}{n!n'!c^{2l+2n+2n'+1}}\times M_{lm}^{(n)(l+2n)}\left(0,t-\frac{r}{c}\right)M_{l,-1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right), \tag{5.25}$$

$$\left(\frac{dM_z^{(mag)}}{dt}\right)_{M,M}=i\sum_{l,m,n,n'}\frac{(-1)^m m(l+1)}{l(2l-1)!!(2l+1)!!}\frac{1}{n!n'!c^{2l+2n+2n'+1}}\times M_{lm}^{(n)(l+2n)}\left(0,t-\frac{r}{c}\right)M_{l,-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right); \tag{5.26}$$

$$\left(\frac{dM_x^{(mag)}}{dt}\right)_{M,Q+T}=\frac{1}{2}\sum_{l,m,n}\frac{(-1)^m}{(2l-1)!!(2l+1)!!}\frac{1}{n!c^{l+2n}}M_{lm}^{(n)(l+2n)}\left(0,t-\frac{r}{c}\right)$$
$$\times\Bigg\{\frac{(l+2)\sqrt{(l-m+1)(l-m+2)}}{(2l+1)(2l+3)}\left[\frac{Q_{l+1,1-m}^{(0)(l+2)}\left(0,t-\frac{r}{c}\right)}{c^{l+2}}-\sum_{n'=0}^{\infty}\frac{T_{l+1,1-m}^{(n')(l+2n'+3)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+3}}\right]$$
$$-\frac{(l+2)\sqrt{(l+m+1)(l+m+2)}}{(2l+1)(2l+3)}\left[\frac{Q_{l+1,-1-m}^{(0)(l+2)}\left(0,t-\frac{r}{c}\right)}{c^{l+2}}-\sum_{n'=0}^{\infty}\frac{T_{l+1,-1-m}^{(n')(l+2n'+3)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+3}}\right]$$
$$+(l+1)\sqrt{(l+m-1)(l+m)}\left[\frac{Q_{l-1,-1-m}^{(0)(l)}\left(0,t-\frac{r}{c}\right)}{c^{l}}-\sum_{n'=0}^{\infty}\frac{T_{l-1,1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+1}}\right]$$
$$-(l+1)\sqrt{(l-m-1)(l-m)}\left[\frac{Q_{l-1,-1-m}^{(0)(l)}\left(0,t-\frac{r}{c}\right)}{c^{l}}-\sum_{n'=0}^{\infty}\frac{T_{l-1,-1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+1}}\right]\Bigg\}, \tag{5.27}$$

$$\left(\frac{dM_y^{(mag)}}{dt}\right)_{M,Q+T}=\frac{i}{2}\sum_{l,m,n}\frac{(-1)^m}{(2l-1)!!(2l+1)!!}\frac{1}{n!c^{l+2n}}M_{lm}^{(n)(l+2n)}\left(0,t-\frac{r}{c}\right)$$
$$\times\left\{\frac{(l+2)\sqrt{(l-m+1)(l-m+2)}}{(2l+1)(2l+3)}\left[\frac{Q_{l+1,1-m}^{(0)(l+2)}\left(0,t-\frac{r}{c}\right)}{c^{l+2}}-\sum_{n'=0}^{\infty}\frac{T_{l+1,1-m}^{(n')(l+2n'+3)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+3}}\right]\right.$$
$$+\frac{(l+2)\sqrt{(l+m+1)(l+m+2)}}{(2l+1)(2l+3)}\left[\frac{Q_{l+1,-1-m}^{(0)(l+2)}\left(0,t-\frac{r}{c}\right)}{c^{l+2}}-\sum_{n'=0}^{\infty}\frac{T_{l+1,-1-m}^{(n')(l+2n'+3)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+3}}\right]$$
$$+(l+1)\sqrt{(l+m-1)(l+m)}\left[\frac{Q_{l-1,1-m}^{(0)(l)}\left(0,t-\frac{r}{c}\right)}{c^{l}}-\sum_{n'=0}^{\infty}\frac{T_{l-1,1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+1}}\right]$$
$$\left.+(l+1)\sqrt{(l-m-1)(l-m)}\left[\frac{Q_{l-1,-1-m}^{(0)(l)}\left(0,t-\frac{r}{c}\right)}{c^{l}}-\sum_{n'=0}^{\infty}\frac{T_{l-1,-1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+1}}\right]\right\}, \tag{5.28}$$

$$\left(\frac{dM_z^{(mag)}}{dt}\right)_{M,Q+T}=\sum_{l,m,n}\frac{(-1)^m}{(2l-1)!!(2l+1)!!}\frac{1}{n!c^{l+2n}}M_{lm}^{(n)(l+2n)}\left(0,t-\frac{r}{c}\right)$$
$$\times\left\{-\frac{(l+2)\sqrt{(l+m+1)(l-m+1)}}{(2l+1)(2l+3)}\left[\frac{Q_{l+1,-m}^{(0)(l+2)}\left(0,t-\frac{r}{c}\right)}{c^{l+2}}-\sum_{n'=0}^{\infty}\frac{T_{+1,-m}^{(n')(l+2n'+3)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+3}}\right]\right.$$
$$\left.+(l+1)\sqrt{(l-m)(l+m)}\left[\frac{Q_{l-1,-m}^{(0)(l)}\left(0,t-\frac{r}{c}\right)}{c^{l}}-\sum_{n'=0}^{\infty}\frac{T_{l-1-m}^{(n')(l+2n'+1)}\left(0,t-\frac{r}{c}\right)}{n'!c^{l+2n'+1}}\right]\right\}. \tag{5.29}$$

So we have completed the calculation of the angular momentum loss per unit time. The electric field contribution in Eq. (5.4) is given by Eqs. (5.7′) and Eqs. (5.14)–(5.22), while the contribution of the magnetic field in Eq. (5.4) is given by Eqs. (5.23)–(5.29). The results have been expressed by giving formulas for the Cartesian components of $d\vec{M}/dt$. Ellaborating still a little bit these results and writing them for the spherical components of the vector $d\vec{M}/dt$, they can be put in a more compact form in terms of known Clebsch-Gordan coefficients.

So, in spherical components ($\mu=-1,0,+1$), one finally obtains for the angular momentum loss per unit time the following general expression:

$$\frac{dM_\mu}{dt}=\sum_{l,m}\frac{(-1)^{m+1}}{(2l-1)!!(2l+1)!!}\frac{1}{c^{2l}}$$
$$\times\left[\frac{l+2}{2l+1}\frac{\sqrt{l+1}}{\sqrt{2l+3}}C_{m+\mu,-\mu,m}^{\,l+1,1,l}A\right.$$
$$+(l+1)\sqrt{l(2l-1)}\,C_{m+\mu,-\mu,m}^{\,l-1,1,l}B$$
$$\left.+\frac{i}{c}(l+1)\frac{\sqrt{l+1}}{\sqrt{l}}C_{m+\mu,-\mu,m}^{\,l,1,l}C\right], \tag{5.30}$$

where A, B, C stand for

$$A=\sum_{n'}\frac{1}{c^{2n'+2}n'!}[Q_{lm}^{(0)(l)}M_{l+1,-\mu-m}^{(n')(l+2n'+2)}-M_{lm}^{(n')(l+2n')}Q_{l+1,-m-\mu}^{(0)(l+2)}]+\sum_{n,n'}\frac{1}{c^{2n+2n'+3}n!n'!}[M_{lm}^{(n')(l+2n')}T_{l+1,-m-\mu}^{(n)(l+2n+3)}-T_{lm}^{(n)(l+2n+1)}M_{l+1,-\mu-m}^{(n')(l+2n'+2)}], \tag{5.31}$$

$$B=\sum_{n'}\frac{1}{c^{2n'}n'!}[Q_{lm}^{(0)(l)}M_{l-1,-\mu-m}^{(n')(l+2n')}-M_{lm}^{(n')(l+2n')}Q_{l-1,-m-\mu}^{(0)(l)}]+\sum_{n,n'}\frac{1}{c^{2n+2n'+1}n!n'!}[M_{lm}^{(n')(l+2n')}T_{l-1,-m-\mu}^{(n)(l+2n+1)}-T_{lm}^{(n)(l+2n+1)}M_{l-1,-\mu-m}^{(n')(l+2n')}], \tag{5.32}$$

$$C=-Q_{lm}^{(0)(l)}Q_{l,-\mu-m}^{(0)(l+1)}+\sum_{n'}\frac{1}{c^{2n+1}n!}[Q_{lm}^{(0)(l)}T_{l,-\mu-m}^{(n)(l+2n+2)}+T_{lm}^{(n)(l+2n+1)}Q_{l,-\mu-m}^{(0)(l+1)}]-\sum_{n,n'}\frac{1}{c^{2n+2n'}n!n'!}\left[M_{lm}^{(n)(l+2n')}M_{l,-\mu-m}^{(n')(l+2n'+1)}+\frac{1}{c^2}T_{lm}^{(n)(l+2n+1)}T_{l,-\mu-m}^{(n')(l+2n'+2)}\right]. \tag{5.33}$$

The connection with the Cartesian components is as usually,

$$\frac{dM_+}{dt}=-\frac{1}{\sqrt{2}}\left(\frac{dM_x}{dt}+i\frac{dM_y}{dt}\right),$$

$$\frac{dM_{(-)}}{dt}=\frac{1}{\sqrt{2}}\left(\frac{dM_x}{dt}-i\frac{dM_y}{dt}\right),$$

$$\frac{dM_{(0)}}{dt}=\frac{dM_z}{dt}. \tag{5.34}$$

The argument of all the double-superscript quantities $Q_{lm}^{(n)(\nu)}$, $M_{lm}^{(n)(\nu)}$, $T_{lm}^{(n)(\nu)}$ [which up to numerical factors are just time derivatives of the multipole electric, magnetic, and toroid radii of the system, see Eqs. (1.42)–(1.44)] is $t-r/c$. We recall for convenience the definitions of the double-superscript quantities in terms of the corresponding multipole electric, magnetic, and toroid form factors [Eqs. (1.39)–(1.41)]:

$$Q_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\frac{d^n}{d(-k^2)^n}Q_{lm}(-k^2,t)\bigg|_{k^2=0}\right],$$

$$M_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\left|\frac{d^n}{d(-k^2)^n}M_{lm}(-k^2,t)\right|_{k^2=0}\right],$$

$$T_{lm}^{(n)(\nu)}(0,t)=\frac{d^\nu}{dt^\nu}\left[\frac{d^n}{d(-k^2)^n}T_{lm}(-k^2,t)\bigg|_{k^2=0}\right],$$

and their relationship with the multipole charge (electric), magnetic, and toroid mean-square radii [respectively, $\overline{r_{lm}^{2n}}(t)$, $\overline{\rho_{lm}^{2n}}(t)$, $\overline{R_{lm}^{2n}}(t)$] of various orders (zero order means just the corresponding multipole moment itself) [Eqs. (1.42)–(1.44)]:

$$Q_{lm}^{(n)(\nu)}(0,t)=\frac{(2l+1)!!}{2^n(2l+2n+1)!!}\frac{d^\nu}{dt^\nu}\overline{r_{lm}^{2n}}(t),$$

$$M_{lm}^{(n)(\nu)}(0,t)=\frac{(2l+1)!!}{2^n(2l+2n+1)!!}\frac{d^\nu}{dt^\nu}\overline{\rho_{lm}^{2n}}(t),$$

$$T_{lm}^{(n)(\nu)}(0,t)=\frac{(2l+1)!!}{2^n(2l+2n+1)!!}\frac{d^\nu}{dt^\nu}\overline{R_{lm}^{2n}}(t).$$

All these radii express exactly the multipole content of the most general type of source. They characterize completely the system; in this paper, their time dependence is left arbitrary.

At this point, a comment on the time dependence in the formulas obtained for the angular momentum lost per unit time by the system through radiation may be in order (as a matter of fact, this comment holds also for the calculation of the recoil force, radiation intensity, etc.). The question is that apparently (i.e., rigourously, from the mathematical point of view) the evaluation of the limits $r\to\infty$ like the one in Eq. (5.1) might rise some delicate problems, since the fields $\vec{E},\vec{B}$ under the integral are by definition solutions of the Maxwell equations that are zero at infinity. But in principle one may evaluate the angular momentum lost per unit time by the system through a sphere of arbitrary radius r,

$$\frac{d\vec{M}(\vec{r},t)}{dt}=\frac{r^3}{4\pi}\int d\Omega[(\vec{n}\cdot\vec{E})(\vec{n}\times\vec{E})+(\vec{n}\cdot\vec{B})(\vec{n}\times\vec{B})]. \tag{5.1$'$}$$

It is practically hard to do that in the vicinity of the source, so one does it for large r but not just in the limit $r\to\infty$; one must still have non-negligible terms of order $1/r^3$ from the integral to compensate the r^3 in front of it. That is why, in general, one may speak of $d\vec{M}(r,t)/dt$ as a function of r. One should remember that the argument of the radii in the final result for $d\vec{M}/dt$ is $t-r/c$. This means that if an observer at a distance r_1 from the source measures the angular momentum that flows through the surface of radius r_1 during the time interval T and other observer situated at the distance $r_2>r_1$ does the same thing and starts measuring at the retarded time (when the waves reached him), after the same time interval T he will find the same answer as the first observer. That is why $d\vec{M}/dt$, the recoil force, etc., still depend (unessentially) on r through the retarded time $t-r/c$ in the argument of the radii from the final result.

We shall exploit the above general formulas Eqs. (5.30)–(5.33) by writing down the expression of the angular momentum loss correct up to terms of $1/c^5$ in the development over $1/c$ powers. For contact with relations given in the literature, we have come back to Cartesian components and use the corresponding connections with the spherical components for the first multipole moments and radii given in Appendix C. We find

$$\frac{dM_\alpha}{dt}=C(QQ)+C(MM)+C(QM)+C(QT)+C(MT)+C(TT), \tag{5.35}$$

where the various diagonal and interference (electric, magnetic, and toroid) contributions are

$$C(QQ)=-\frac{2}{3c^3}\varepsilon_{\alpha ij}\dot{d}_i\ddot{d}_j-\frac{2}{5c^5}\varepsilon_{\alpha ij}\ddot{Q}_{\beta i}\dddot{Q}_{\beta j}, \tag{5.36}$$

$$C(MM)=-\frac{2}{3c^3}\varepsilon_{\alpha ij}\dot{m}_i\ddot{m}_j-\frac{1}{5c^5}\left(-\frac{1}{3}\varepsilon_{\alpha ij}\ddot{m}_i\dddot{\overline{\rho_j^2}}+\frac{1}{3}\varepsilon_{\alpha ij}\dot{m}_i\ddddot{\overline{\rho_j^2}}+\frac{1}{2}\varepsilon_{\alpha ij}\ddot{m}_{\beta i}\dddot{m}_{\beta j}\right), \tag{5.37}$$

$$C(QM)=-\frac{1}{5c^4}(\dddot{m}_{\alpha i}\dot{d}_i+\ddot{m}_{\alpha i}\ddot{d}_i-2\dddot{Q}_{\alpha i}\dot{m}_i-2\ddot{Q}_{\alpha i}\ddot{m}_i), \tag{5.38}$$

$$C(QT)=\frac{2}{3c^4}\varepsilon_{\alpha ij}\dot{d}_i\dddot{t}_j+\frac{2}{3c^4}\varepsilon_{\alpha ij}\ddot{t}_i\ddot{d}_j, \tag{5.39}$$

$$C(MT)=-\frac{1}{15c^5}(2\ddddot{t}_{\alpha i}\dot{m}_i+2\dddot{t}_{\alpha i}\ddot{m}_i-3\dddot{m}_{\alpha i}\ddot{t}_i-3\ddot{m}_{\alpha i}\dddot{t}_i), \tag{5.40}$$

$$C(TT)=-\frac{2}{3c^5}\varepsilon_{\alpha ij}\ddot{t}_i\dddot{t}_j. \tag{5.41}$$

In the expressions above, dots mean time derivatives, the argument of the multipole quantities involved is $t-r/c$, and d_i is the electric (charge) dipole moment, Q_{ij} is the electric quadrupole moment, m_i the magnetic dipole moment, $\overline{\rho_i^2}$ the first mean-square radius of the magnetic dipole distribution, m_{ij} the magnetic quadrupole moment, t_i the toroid dipole moment, t_{ij} the toroid quadrupole moment. All these first usual multipole quantities are listed for convenience in Appendix C together with their corresponding spherical-basis analogs. Reordering the above formulas in powers of $1/c$ one has

$$\frac{dM_\alpha}{dt}=-\frac{2}{3c^3}\varepsilon_{\alpha\beta\gamma}(\dot{d}_\beta\ddot{d}_\gamma+\dot{m}_\beta\ddot{m}_\gamma)+\frac{1}{c^4}\left[\frac{1}{5}(-\dddot{m}_{\alpha\beta}\dot{d}_\beta-\ddot{m}_{\alpha\beta}\ddot{d}_\beta+2\dddot{Q}_{\alpha\beta}\dot{m}_\beta+2\ddot{Q}_{\alpha\beta}\ddot{m}_\beta)+\frac{2}{3}\varepsilon_{\alpha\beta\gamma}(\dot{d}_\beta\dddot{t}_\gamma+\ddot{t}_\beta\ddot{d}_\gamma)\right]-\frac{1}{5c^5}\left[\frac{1}{3}(2\ddddot{t}_{\alpha\beta}\dot{m}_\beta+2\dddot{t}_{\alpha\beta}\ddot{m}_\beta-3\dddot{m}_{\alpha\beta}\ddot{t}_\beta-3\ddot{m}_{\alpha\beta}\dddot{t}_\beta)+\varepsilon_{\alpha\beta\gamma}\left(2\ddot{Q}_{\delta\beta}\dddot{Q}_{\delta\gamma}+\frac{10}{3}\ddot{t}_\beta\dddot{t}_\gamma+\frac{1}{3}\dot{m}_\beta\ddddot{\overline{\rho_\gamma^2}}-\frac{1}{3}\ddot{m}_\beta\dddot{\overline{\rho_\gamma^2}}+\frac{1}{2}\ddot{m}_{\delta\beta}\dddot{m}_{\delta\gamma}\right)\right]+\left(\text{terms of higher order than }\frac{1}{c^5}\right). \tag{5.42}$$

All the time derivatives (dots) of the multipole quantities appearing in the Eqs. (5.42) above, as said before, have as argument $t-r/c$. We mention that in the expression of the rate of the angular momentum loss $d\vec{M}/dt$ calculated in Ref. [21] (the Problem 2 at the end of Paragraph 72, the 1988 Russian edition) only the first term of our Eq. (5.42) is given, while Eq. (5.42) completes the result given in Ref. [21] with the remaining contributions of order $1/c^3$ (of magnetic type), of order $1/c^4$ (which include a toroid dipole piece) and of order $1/c^5$ (which also include toroid pieces, this time appearing not only the toroid dipole moment, but the toroid quadrupole moment as well).

VI. RECOIL FORCE

The recoil force is the momentum lost by the system per unit time, i.e., the flux of momentum taken off from the system by radiation of electromagnetic waves [21],

$$F_i=\int\sigma_{ij}n_j dS,\quad dS=r^2d\Omega,\quad \vec{n}=\frac{\vec{r}}{r}. \tag{6.1}$$

σ_{ij} is the three-dimensional Maxwell stress tensor given by Eq. (5.3) and the integration has to be done over a sphere of large radius r.

A system that emits electromagnetic radiation suffers then the recoil force

$$\vec{F}=\lim_{r\to\infty}\left\{-\frac{r^2}{4\pi}\int d\Omega\left[\vec{E}(\vec{E}\cdot\vec{n})+\vec{B}(\vec{B}\cdot\vec{n})-\vec{n}\frac{(\vec{E}^2+\vec{B}^2)}{2}\right]\right\},\quad \vec{n}=\frac{\vec{r}}{r}. \tag{6.2}$$

We need the electric and magnetic fields $\vec{E}$ and $\vec{B}$ to order $O(1/r)$. To this order the fields are transversal,

$$\vec{E}\cdot\vec{n}=0,\quad \vec{B}\cdot\vec{n}=0,$$

and we have to calculate

$$\vec{F}=-\frac{r}{8\pi}\int d\Omega\,\vec{r}(\vec{E}^2+\vec{B}^2), \tag{6.3}$$

i.e.,

$$F_x=-\frac{r}{8\pi}\int d\Omega\, x(\vec{E}^2+\vec{B}^2), \tag{6.4}$$

$$F_y=-\frac{r}{8\pi}\int d\Omega\, y(\vec{E}^2+\vec{B}^2), \tag{6.5}$$

$$F_z = -\frac{r}{8\pi}\int d\omega z(\vec{E}^2+\vec{B}^2), \tag{6.6}$$

for $r\to\infty$, with $\vec{E}$ and $\vec{B}\sim O(1/r)$, so that r in front of the integrals will finally drop from the result, as it must.

Our aim is to evaluate the recoil force in terms of the multipole content of the source in the most general situation, i.e., for all types of source's multipoles (electric, magnetic, toroid), any multipolarity order and an arbitrary time dependence of the quantities involved. This means that we want to express the recoil force in terms of (essentially) the time derivatives of the system's various multipole radii, i.e., in terms of the double-superscript quantities defined in Eqs. (1.39)–(1.41) [which are related to the radii as in Eqs. (1.42)–(1.44)]. To this purpose, as it is seen from Eqs. (6.2)–(6.6) above, one needs only the $O(1/r)$ multipole content of the fields $\vec{E},\vec{B}$ as given by Eqs. (3.15), (3.16) of Sec. III. The calculation, however, is tedious and we shall give here only the main steps and final results, presenting nonetheless sometimes certain details about technicalities that involve some lengthy expressions and relegating to appendixes for some other formulas. The calculation goes as follows:

Step (1). Since the fields $\vec{E},\vec{B}$ in order $O(1/r)$, given by Eqs. (3.15) and (3.16), satisfy

$$\frac{\vec{r}}{r}\times\vec{E}(\vec{r},t) \overset{O(1/r)}{=} \vec{B}(\vec{r},t),$$

one has

$$\vec{E}^2(\vec{r},t)=\vec{B}^2(\vec{r},t),$$

and Eqs. (6.4)–(6.6) mean that one has to calculate (large r, $\vec{n}=\vec{r}/r$),

$$F_x = -r^2\frac{\sqrt{2}}{4\sqrt{3\pi}}\int d\Omega[Y_{1-1}(\vec{n})-Y_{11}(\vec{n})]\vec{B}^2(\vec{r},t), \tag{6.7}$$

$$F_y = -r^2 i\frac{\sqrt{2}}{4\sqrt{3\pi}}\int d\Omega[Y_{11}(\vec{n})+Y_{1-1}(\vec{n})]\vec{B}^2(\vec{r},t), \tag{6.8}$$

$$F_z = -r^2\frac{1}{2\sqrt{3\pi}}\int d\Omega Y_{10}(\vec{n})\vec{B}^2(\vec{r},t), \tag{6.9}$$

i.e., one has to find the surface integrals

$$I_1 = \int d\Omega Y_{11}(\vec{n})\vec{B}^2(\vec{r},t), \tag{6.10}$$

$$I_{-1} = \int d\Omega Y_{1-1}(\vec{n})\vec{B}^2(\vec{r},t), \tag{6.11}$$

$$I_0 = \int d\Omega Y_{10}(\vec{n})\vec{B}^2(\vec{r},t). \tag{6.12}$$

Then the recoil force will be, in the Cartesian basis

$$F_x = -r^2\frac{\sqrt{2}}{4\sqrt{3\pi}}(I_{-1}-I_1), \tag{6.13}$$

$$F_y = -r^2\frac{\sqrt{2}}{4\sqrt{3\pi}}(iI_{-1}+iI_1), \tag{6.14}$$

$$F_z = -r^2\frac{1}{2\sqrt{3\pi}}I_0, \tag{6.15}$$

and in spherical basis

$$F_{+1} = -\frac{1}{\sqrt{2}}(F_x+iF_y) = -\frac{r^2}{2\sqrt{3\pi}}I_1, \tag{6.16}$$

$$F_{-1} = \frac{1}{\sqrt{2}}(F_x-iF_y) = -\frac{r^2}{2\sqrt{3\pi}}I_{-1}, \tag{6.17}$$

$$F_0 = F_z = -\frac{r^2}{2\sqrt{3\pi}}I_0. \tag{6.18}$$

Step (2). We start considering the integral I_1 of Eq. (6.10). One has to evaluate it with $\vec{B}(\vec{r},t)$ under the integral as given by Eq. (3.16), i.e., one has to calculate the lengthy expression

$$\begin{aligned} I_1 = \frac{1}{r^2}\int d\Omega Y_{11}(\vec{n})&\Bigg\{\sum_{l,m}\frac{1}{c^{l+1}}\frac{\sqrt{4\pi(2l+1)(l+1)}}{\sqrt{l}(2l+1)!!}\Bigg[\sum_{n=0}^{\infty}\frac{1}{n!c^{2n}}M_{lm}^{(n)(l+2n+1)}\left(0,t-\frac{r}{c}\right)\\ &\times\left(\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}(\vec{n})+\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m}(\vec{n})\right)-iQ_{lm}^{(0)(l+1)}\left(0,t-\frac{r}{c}\right)\vec{Y}_{llm}(\vec{n})+\frac{i}{c}\sum_{n=0}^{\infty}\frac{1}{n!c^{2n}}T_{lm}^{(n)(l+2n+2)}\left(0,t-\frac{r}{c}\right)\\ &\times\vec{Y}_{llm}(\vec{n})\Bigg]\Bigg\}\cdot\Bigg\{\sum_{l'm'}\frac{1}{c^{l'+1}}\frac{\sqrt{4\pi(2l'+1)(l'+1)}}{\sqrt{l'}(2l'+1)!!}\Bigg[\frac{1}{n'!c^{2n'}}M_{l'm'}^{(n')(l'+2n'+1)}\left(0,t-\frac{r}{c}\right)\left(\frac{\sqrt{l'}}{\sqrt{2l'+1}}\vec{Y}_{l'l'+1m'}(\vec{n})\right.\\ &\left.+\frac{\sqrt{l'+1}}{\sqrt{2l'+1}}\vec{Y}_{l'l'-1m'}(\vec{n})\right)-iQ_{l'm'}^{(0)(l'+1)}\left(0,t-\frac{r}{c}\right)\vec{Y}_{l'l'm'}(\vec{n})+\frac{i}{c}\sum_{n'=0}^{\infty}\frac{1}{n'!c^{2n'}}T_{l'm'}^{(n')(l'+2n'+2)}\left(0,t-\frac{r}{c}\right)\vec{Y}_{l'l'm'}(\vec{n})\Bigg]\Bigg\}. \end{aligned} \tag{6.19}$$

So, to find I_1, one splits it into six types of different pieces and calculates them separately,

$$I_1 = I_1^{MM} + I_1^{QQ} + I_1^{TT} + I_1^{QM} + I_1^{TM} + I_1^{QT}. \tag{6.20}$$

We give here only the expression of I_1^{MM}, the other pieces having the obvious corresponding form

$$I_1^{MM} = \sum_{l,m,l',m'} \sum_{n,n'} \frac{1}{c^{l+l'+2n+2n'+2}} \frac{4\pi\sqrt{(l+1)(l'+1)}}{n!n'!\sqrt{ll'}(2l+1)!!(2l'+1)!!} M_{lm}^{(n)(l+2n+1)} M_{l'm'}^{(n')(l'+2n'+1)} \times\Bigg[\sqrt{ll'}\int d\Omega Y_{11}\vec{Y}_{ll+1m}\cdot\vec{Y}_{l'l'+1m'} + 2\sqrt{l(l'+1)}\int d\Omega Y_{11}\vec{Y}_{ll+1m}\cdot\vec{Y}_{l'l'-1m'} + \sqrt{(l+1)(l'+1)} \times\int d\Omega Y_{11}\vec{Y}_{ll-1m}\cdot\vec{Y}_{l'l'-1m'}\Bigg]. \tag{6.21}$$

From now on we shall drop the arguments $(0,t-r/c)$ and $\vec{n}$ in the spherical vector functions when unnecessary.

Step (3). It is seen that essentially one has to compute many surface integrals over products of three spherical harmonics, which can be performed in terms of $3j$ Wigner coefficients with the aid of the formula [22],

$$\int d\Omega Y_{l_1m_1}(\vec{n})Y_{l_2m_2}(\vec{n})Y_{l_3m_3}(\vec{n}) = \frac{\sqrt{(2l_1+1)(2l_2+1)(2l_3+1)}}{\sqrt{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix} \times\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}. \tag{6.22}$$

In this section, unlike our procedure in Secs. IV and V when we used in the calculations the usual Clebsch-Gordan coefficients, we preferred to work with the more symmetrical $3j$ Wigner symbols [22]. We recall the known relationship,

$$\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{l_1-l_2-m_3}\frac{1}{\sqrt{2l_3+1}} C^{l_1,l_2,l_3}_{m_1,m_2,-m_3}. \tag{6.22$'$}$$

The Cartesian components of $\vec{Y}_{ll'm}(\vec{n})$ that enter the expressions of the fields Eqs. (3.15) and (3.16), and hence the expression to be evaluated and which have to be expressed in terms of the usual spherical harmonics $Y_{lm}(\vec{n})$ through $3j$ symbols, are given for convenience in Appendix D. So, one finds

$$I_1^{MM} = \frac{1}{r^2}\sum_{l,n,n'} \frac{(-1)^l}{c^{2l+2n+2n'+1}} \times\frac{8l(l+1)\sqrt{3\pi l}}{n!n'!(2l+1)!!(2l-1)!!\sqrt{(2l-1)(2l+1)}} \times\sum_m M_{lm}^{(n)(l+2n+1)} M_{l-1,-1-m}^{(n')(l+2n')}\begin{pmatrix} l & l-1 & 1\\ m & -1-m & 1\end{pmatrix} + \frac{1}{r^2}\sum_{l,n,n'}\frac{(-1)^{l+1}}{c^{2l+2n+2n'+3}} \times\frac{4(l+2)(2l^2+4l+1)\sqrt{3\pi}}{n!n'!(2l+1)!!(2l+3)!!\sqrt{(l+1)(2l+1)(2l+3)}} \times\sum_m M_{lm}^{(n)(l+2n+1)} M_{l+1,-1-m}^{(n')(l+2n'+2)}\begin{pmatrix} l & l+1 & 1\\ m & -1-m & 1\end{pmatrix}, \tag{6.23}$$

and analogous expressions for all the other pieces in Eq. (6.20),

$$I_1^{QQ} = \frac{1}{r^2}\sum_{l,m}\frac{(-1)^l}{c^{2l+1}}\frac{2\sqrt{3\pi}(l+1)}{(2l-1)!!(2l-3)!!\sqrt{l(2l-1)(2l+1)}} \times\sum_m Q_{lm}^{(0)(l+1)} Q_{l-1,-1-m}^{(0)(l)}\begin{pmatrix} l & l-1 & 1\\ m & -1-m & 1\end{pmatrix} + \frac{1}{r^2}\sum_{l,m}\frac{(-1)^{l-1}}{c^{2l+3}} \times\frac{2\sqrt{3\pi}(l+2)}{(2l-1)!!(2l+1)!!\sqrt{(l+1)(2l+1)(2l+3)}} \times\sum_m Q_{lm}^{(0)(l+1)} Q_{l+1,-1-m}^{(0)(l+2)}\begin{pmatrix} l & l+1 & 1\\ m & -1-m & 1\end{pmatrix}, \tag{6.24}$$

$$I_1^{TT} = \frac{1}{r^2}\sum_{l,n,n'}\frac{(-1)^l}{n!n'!c^{2l+2n+2n'+2}} \times\frac{2\sqrt{3\pi}(l+1)}{(2l-1)!!(2l-3)!!\sqrt{l(2l-1)(2l+1)}} \times\sum_m T_{lm}^{(n)(l+2n+2)} T_{l-1,-1-m}^{(n')(l+2n'+1)}\begin{pmatrix} l & l-1 & 1\\ m & -1-m & 1\end{pmatrix} + \frac{1}{r^2}\sum_{l,n,n'}\frac{(-1)^{l-1}}{n!n'!c^{2l+2n+2n'+4}}$$

$$\times\frac{2\sqrt{3\pi}(l+2)}{(2l-1)!!(2l+1)!!\sqrt{(l+1)(2l+1)(2l+3)}}$$

$$\times\sum_{m} T_{lm}^{(n)(l+2n+2)}T_{l+1,-1-m}^{(n')(l+2n'+3)}\begin{pmatrix} l & l+1 & 1\\ m & -1-m & 1\end{pmatrix},\tag{6.25}$$

$$I_1^{QM}=\frac{4i}{r^2}\sum_{l,n}\frac{(-1)^{l-1}}{n!c^{2n+2l+2}}\frac{\sqrt{3\pi(l+1)(2l+1)}}{l\sqrt{l}(2l+1)!!(2l-1)!!}$$

$$\times\sum_{m} Q_{l,-1-m}^{(0)(l+1)}M_{lm}^{(n)(l+2n+1)}\begin{pmatrix} l & l & 1\\ -1-m & m & 1\end{pmatrix},\tag{6.26}$$

$$I_1^{TM}=\frac{4i}{r^2}\sum_{l,n,n'}\frac{(-1)^{l}}{n!n'!c^{2n+2n'+2l+3}}$$

$$\times\frac{1}{(2l+1)!!(2l-1)!!}\frac{\sqrt{3\pi(l+1)(2l+1)}}{l\sqrt{l}}$$

$$\times\sum_{m} T_{l,-1-m}^{(n')(l+2n'+2)}M_{lm}^{(n)(l+2n+1)}\begin{pmatrix} l & l & 1\\ -1-m & m & 1\end{pmatrix},\tag{6.27}$$

$$I_1^{QT}=\frac{4}{r^2}\sum_{l,n'}\frac{(-1)^{l+1}}{n'!c^{2n'+2l+2}}\frac{(l+1)}{(2l-1)!!(2l-3)!!}$$

$$\times\frac{\sqrt{3\pi}}{\sqrt{l(2l+1)(2l-1)}}\sum_{m} Q_{lm}^{(0)(l+1)}T_{l-1,-1-m}^{(n')(l+2n'+1)}$$

$$\times\begin{pmatrix} l & l-1 & 1\\ m & -1-m & 1\end{pmatrix}+\frac{4}{r^2}\sum_{l,n'}\frac{(-1)^{l}}{n'!c^{2n'+2l+4}}$$

$$\times\frac{(l+2)}{(2l-1)!!(2l+1)!!}\frac{\sqrt{3\pi}}{\sqrt{(l+1)(2l+1)(2l+3)}}$$

$$\times\sum_{m} Q_{lm}^{(0)(l+1)}T_{l+1,-1-m}^{(n')(l+2n'+3)}\begin{pmatrix} l & l+1 & 1\\ m & -1-m & 1\end{pmatrix}.\tag{6.28}$$

So, one completes the calculation of the integral I_1 from Eq. (6.10).

Step (4). The integral I_{-1}, Eq. (6.11), is calculated from the result obtained above for I_1, Eqs. (6.20), (6.23)–(6.28), using the obvious relation

$$I_{-1}=-I_1^{*},\tag{6.29}$$

and the properties under complex conjugation of the double-superscript quantities $Q_{lm}^{(n)(\nu)}$, $M_{lm}^{(n)(\nu)}$, which are of the type

$$M_{lm}^{(n)(\nu)*}=(-1)^m M_{l-m}^{(n)(\nu)},\quad \text{etc.},\tag{6.30}$$

following from the same properties of the whole electric, magnetic, and toroid formfactors $Q_{lm}(-k^2,t)$, $M_{lm}(-k^2,t), T_{lm}(-k^2,t)$ they are originating from [see Eqs. (1.22′)–(1.24′)]. One changes then $m\to -m$ (the summation over m remains again from $-l$ to l because of symmetry) and one uses the known relation [22]

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}=(-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3\\ -m_1 & -m_2 & -m_3\end{pmatrix},\tag{6.31}$$

for the appearing $3j$ symbols, to get finally the expression of I_{-1} we are looking for, which is entirely similar to the one for I_1 given above (with the expected due changes in the indices and $3j$ symbols).

Step (5). One computes in the same way the integral I_0 of Eq. (6.12). The result is

$$I_0=I_0^{MM}+I_0^{QQ}+I_0^{TT}+I_0^{QM}+I_0^{TM}+I_0^{QT},\tag{6.32}$$

where

$$I_0^{MM}=\frac{1}{r^2}\sum_{l,m,n,n'}\frac{(-1)^{l}}{n!n'!c^{2n+2n'+2l+1}}$$

$$\times\frac{8l(l+1)\sqrt{3\pi l}}{(2l-1)!!(2l+1)!!\sqrt{(2l-1)(2l+1)}}$$

$$\times M_{lm}^{(n)(l+2n+1)}M_{l-1,-m}^{(n')(l+2n')}\begin{pmatrix} l & l-1 & 1\\ m & -m & 0\end{pmatrix}$$

$$+\frac{1}{r^2}\sum_{l,m,n,n'}\frac{(-1)^{l+1}}{n!n'!c^{2n+2n'+2l+3}}$$

$$\times\frac{4(2l^2+4l+1)(l+2)\sqrt{3\pi}}{(2l+1)!!(2l+3)!!\sqrt{(l+1)(2l+1)(2l+3)}}$$

$$\times M_{lm}^{(n)(l+2n+1)}M_{l+1,-m}^{(n')(l+2n'+2)}\begin{pmatrix} l & l+1 & 1\\ m & -m & 0\end{pmatrix},\tag{6.33}$$

$$I_0^{QQ}=\frac{1}{r^2}\sum_{l,m}\frac{(-1)^{l}}{c^{2l+1}}\frac{2\sqrt{3\pi}(l+1)}{(2l-1)!!(2l-3)!!\sqrt{l(2l-1)(2l+1)}}$$

$$\times Q_{lm}^{(0)(l+1)}Q_{l-1,-m}^{(0)(l)}\begin{pmatrix} l & l-1 & 1\\ m & -m & 0\end{pmatrix}+\frac{1}{r^2}\sum_{l,m}\frac{(-1)^{l-1}}{c^{2l+3}}$$

$$\times\frac{2\sqrt{3\pi}(l+2)}{(2l-1)!!(2l+1)!!\sqrt{(l+1)(2l+1)(2l+3)}}$$

$$\times Q_{lm}^{(0)(l+1)}Q_{l+1,-m}^{(0)(l+2)}\begin{pmatrix} l & l+1 & 1\\ m & -m & 0\end{pmatrix},\tag{6.34}$$

$$I_0^{TT}=\frac{1}{r^2}\sum_{l,m,n,n'}\frac{(-1)^l}{n!n'!c^{2l+2n+2n'+2}}\times\frac{2\sqrt{3\pi}(l+1)}{(2l-1)!!(2l-3)!!\sqrt{l(2l-1)(2l+1)}}\times T_{lm}^{(n)(l+2n+2)}T_{l-1,-m}^{(n')(l+2n'+1)}\begin{pmatrix} l & l-1 & 1\\ m & -m & 0\end{pmatrix}+\frac{1}{r^2}\sum_{l,m,n,n'}\frac{(-1)^{l-1}}{n!n'!c^{2l+2n+2n'+4}}\times\frac{2\sqrt{3\pi}(l+2)}{(2l-1)!!(2l+1)!!\sqrt{(l+1)(2l+1)(2l+3)}}\times T_{lm}^{(n)(l+2n+2)}T_{l+1,-m}^{(n')(l+2n'+3)}\begin{pmatrix} l & l+1 & 1\\ m & -m & 0\end{pmatrix}, \tag{6.35}$$

$$I_0^{QM}=\frac{4i}{r^2}\sum_{l,m,n}\frac{(-1)^{l+1}}{n!c^{2n+2l+2}}\frac{\sqrt{3\pi(l+1)(2l+1)}}{l\sqrt{l}}\times\frac{1}{(2l-1)!!(2l+1)!!}Q_{l,-m}^{(0)(l+1)}M_{lm}^{(n)(l+2n+1)}\times\begin{pmatrix} l & l & 1\\ -m & m & 0\end{pmatrix}, \tag{6.36}$$

$$I_0^{TM}=\frac{4i}{r^2}\sum_{l,m,n,n'}\frac{(-1)^l}{n!n'!c^{2l+2n+2n'+3}}\frac{\sqrt{3\pi(l+1)(2l+1)}}{l\sqrt{l}}\frac{1}{(2l-1)!!(2l+1)!!}T_{l,-m}^{(n')(l+2n'+2)}M_{lm}^{(n)(l+2n+1)}\begin{pmatrix} l & l & 1\\ -m & m & 0\end{pmatrix}, \tag{6.37}$$

$$I_0^{QT}=\frac{1}{r^2}\sum_{l,m,n'}\frac{(-1)^{l+1}}{n'!c^{2l+2n'+2}}\frac{4\sqrt{3\pi}(l+1)}{\sqrt{l(2l-1)(2l+1)}}\frac{(l+1)}{(2l-1)!!(2l-3)!!}Q_{lm}^{(0)(l+1)}T_{l-1,-m}^{(n')(l+2n'+1)}\begin{pmatrix} l & l-1 & 1\\ m & -m & 0\end{pmatrix}+\frac{1}{r^2}\sum_{l,m,n'}\frac{(-1)^l}{n'!c^{2l+2n'+4}}\frac{4\sqrt{3\pi}(l+2)}{\sqrt{(l+1)(2l+1)(2l+3)}}\frac{(l+2)}{(2l-1)!!(2l+1)!!}Q_{lm}^{(0)(l+1)}T_{l+1,-m}^{(n')(l+2n'+3)}\begin{pmatrix} l & l+1 & 1\\ m & -m & 0\end{pmatrix}. \tag{6.38}$$

Step (6). With I_1, I_{-1}, I_0 of Eqs. (6.10)–(6.12) calculated we have ended this part of the work and have at our disposal the recoil force according to Eqs. (6.13)–(6.15) or Eqs. (6.16)–(6.19).

Now we list the final expression of the recoil force for the most general configuration of charges and currents. The result is expressed in the spherical basis ($\mu=-1,0,+1$) with the aid of $3j$-symbols, in terms of derivatives of the system's electric, magnetic, and toroid formfactors. The order of derivation with respect to the momentum transfer ($-k^2$) is specified by the first superscript, while the order of time derivatives is given by the second superscript.

$$F_\mu=[Q,Q]+[M,M]+[T,T]+[Q,M]+[Q,T]+[M,T], \tag{6.39}$$

$$[Q,Q]=\sum_l\frac{(-1)^{l+1}}{c^{2l+1}}\frac{(l+1)}{(2l-1)!!(2l-3)!!\sqrt{l(2l-1)(2l+1)}}\sum_m\begin{pmatrix} l & l-1 & 1\\ m & -\mu-m & \mu\end{pmatrix}Q_{lm}^{(0)(l+1)}Q_{l-1,-\mu-m}^{(0)(l)}+\sum_l\frac{(-1)^l}{c^{2l+3}}\frac{(l+2)}{(2l-1)!!(2l+1)!!\sqrt{(l+1)(2l+1)(2l+3)}}\sum_m\begin{pmatrix} l & l+1 & 1\\ m & -\mu-m & \mu\end{pmatrix}Q_{lm}^{(0)(l+1)}Q_{l+1,-\mu-m}^{(0)(l+2)}, \tag{6.40}$$

$$[M,M]=\sum_{l,n,n'}\frac{(-1)^{l+1}}{n!n'!c^{2l+2n+2n'+1}}\times\frac{4l(l+1)\sqrt{l}}{(2l-1)!!(2l+1)!!\sqrt{(2l-1)(2l+1)}}\times\sum_m\begin{pmatrix} l & l-1 & 1\\ m & -\mu-m & \mu\end{pmatrix}\times M_{lm}^{(n)(l+2n+1)}M_{l-1,-\mu-m}^{(n')(l+2n')}+\sum_{l,n,n'}\frac{(-1)^l}{n!n'!c^{2l+2n+2n'+3}}\times\frac{2(l+2)(2l^2+4l+1)}{(2l+1)!!(2l+3)!!\sqrt{(l+1)(2l+1)(2l+3)}}\times\sum_m\begin{pmatrix} l & l+1 & 1\\ m & -\mu-m & \mu\end{pmatrix}\times M_{lm}^{(n)(l+2n+1)}M_{l+1,-\mu-m}^{(n')(l+2n'+2)}, \tag{6.41}$$

$$[T,T]=\sum_{l,n,n'}\frac{(-1)^{l+1}}{n!n'!c^{2l+2n+2n'+2}}\times\frac{(l+1)}{(2l-1)!!(2l-3)!!\sqrt{l(2l-1)(2l+1)}}\times\sum_{m}\begin{pmatrix} l & l-1 & 1\\ m & -\mu-m & \mu\end{pmatrix}T_{lm}^{(n)(l+2n+2)}T_{l-1,-\mu-m}^{(n')(l+2n'+1)}+\sum_{l,n,n'}\frac{(-1)^{l}}{n!n'!c^{2l+2n+2n'+4}}\times\frac{(l+2)}{(2l-1)!!(2l+1)!!\sqrt{(l+1)(2l+1)(2l+3)}}\times\sum_{m}\begin{pmatrix} l & l+1 & 1\\ m & -\mu-m & \mu\end{pmatrix}T_{lm}^{(n)(l+2n+2)}T_{l+1,-\mu-m}^{(n')(l+2n'+3)}, \tag{6.42}$$

$$[Q,M]=-2i\sum_{l,n}\frac{(-1)^{l}}{n!c^{2l+2n+2}}\frac{\sqrt{(l+1)(2l+1)}}{(2l-1)!!(2l+1)!!l\sqrt{l}}\times\sum_{m}\begin{pmatrix} l & l & 1\\ -\mu-m & m & \mu\end{pmatrix}Q_{l,-\mu-m}^{(0)(l+1)}M_{lm}^{(n)(l+2n+1)}, \tag{6.43}$$

$$[Q,T]=2\sum_{l,n'}\frac{(-1)^{l}}{n'!c^{2l+2n'+2}}\times\frac{(l+1)}{(2l-1)!!(2l-3)!!l\sqrt{l(2l-1)(2l+1)}}\times\sum_{m}\begin{pmatrix} l & l-1 & 1\\ m & -\mu-m & \mu\end{pmatrix}Q_{im}^{(0)(l+1)}T_{l-1,-\mu-m}^{(n')(l+2n'+1)}+2\sum_{l,n'}\frac{(-1)^{l+1}}{n'!c^{2l+2n'+4}}\times\frac{(l+2)}{(2l-1)!!(2l+1)!!\sqrt{(l+1)(2l+1)(2l+3)}}\times\sum_{m}\begin{pmatrix} l & l+1 & 1\\ m & -\mu-m & \mu\end{pmatrix}Q_{lm}^{(0)(l+1)}T_{l+1,-\mu-m}^{(n')(l+2n'+3)}, \tag{6.44}$$

$$[M,T]=-2i\sum_{l,n,n'}\frac{(-1)^{l+1}}{n!n'!c^{2l+2n+2n'+3}}\times\frac{\sqrt{(l+1)(2l+1)}}{(2l-1)!!(2l+1)!!l\sqrt{l}}\sum_{m}\begin{pmatrix} l & l & 1\\ -\mu-m & m & \mu\end{pmatrix}\times T_{l,-\mu-m}^{(n')(l+2n'+2)}M_{lm}^{(n)(l+2n+1)}. \tag{6.45}$$

The summations are to be taken for $m=-l,\ldots,+l$; $l=1,2,\ldots$; n, $n'=0,1,2,\ldots$. The appearing $3j$ symbols are those from Ref. [22], related to the usual Clebsch-Gordan coefficients as follows:

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}=(-1)^{j_1-j_2-m_3}(2j_3+1)^{-1/2}C_{m_1,m_2,-m_3}^{j_1,j_2,j_2}.$$

The Cartesian components of the recoil force F_x, F_y, F_z are related to F_μ ($\mu=-1,0,1$) by Eqs. (6.16)–(6.18),

$$F_{+1}=-\frac{1}{\sqrt{2}}(F_x+iF_y), \tag{6.16$'$}$$

$$F_{-1}=\frac{1}{\sqrt{2}}(F_x-iF_y), \tag{6.17$'$}$$

$$F_0=F_z. \tag{6.18$'$}$$

From the general expression of F_μ ($\mu=-1,0,1$) one finds easily the expression of the recoil force exact up to the $1/c^5$ order inclusively,

$$F_\alpha=-\frac{2}{3c^4}\varepsilon_{\alpha\beta\gamma}\ddot{d}_\beta\ddot{m}_\gamma-\frac{1}{5c^5}\dddot{m}_{\alpha\beta}\ddot{m}_\beta-\frac{2}{5c^5}\dddot{Q}_{\alpha\beta}\ddot{d}_\beta-\frac{2}{3c^5}\varepsilon_{\alpha\beta\gamma}\ddot{m}_\beta\dddot{t}_\gamma, \tag{6.46}$$

In the formula above, α, β, $\gamma\in(x,y,z)$ and summation over repeated indices is understood, as always in this paper. The electric, magnetic, and toroid dipole moments $\vec{d}$, $\vec{m}$, $\vec{t}$, and the electric and magnetic quadrupole moments $Q_{\alpha\beta}$, $m_{\alpha\beta}$ are those from Appendix C. Note that our $Q_{\alpha\beta}$ is $Q_{\alpha\beta}=1/6D_{\alpha\beta}$ from Ref. [21]. Our Eq. (6.46) completes the result given in Ref. [21] (in the Problem 2 at the end of the Paragraph 71, the 1988 Russian edition); indeed, by working with the fields correct only up to the $(1/c^3)$ order, in this references one misses two terms of order $(1/c^5)$ [the second and the fourth terms in Eq. (6.46)] and only the first (of order $1/c^4$) and the third (of order $1/c^5$) are appearing. It seems interesting to us that besides contributions coming from the usual electric and magnetic multipoles, the recoil force, starting already to the $1/c^5$ order, receives a contribution coming from the less usual toroid dipole moment [the last, fourth term in Eq. (6.46)],

$$\vec{F}^{\text{magnetic toroid}}=-\frac{2}{3c^5}\ddot{\vec{m}}\times\dddot{\vec{t}}.$$

Looking at Eq. (6.46) one sees that this magnetic-toroid interference is to be considered on the same footing with the third $(1/c^5)$ term (computed in Ref. [21]) and the second $(1/c^5)$ term (missed in Ref. [21]).

VII. CONCLUSIONS AND DISCUSSION

The main conclusion of this paper is that for the most general configuration of charges and currents (including toroidal current structures, which in other approaches are either unproperly treated or partly lost through approximations) described by the charge density $\rho(\vec{r},t)$ and the current density $\vec{J}(\vec{r},t)$ satisfying only the continuity relation, the radiation intensity as well as the rates of momentum and angular momentum loss through radiation, within classical electrodynamics, can be calculated exactely for any time dependence of the sources.

We did that calculation with the aid of the multipole decomposition of Ref. [4] in which, by the introduction of a third family of multipoles, the toroid ones, one achieves a complete parametrization of $\rho(\vec{r},t)$, $\vec{J}(\vec{r},t)$ (without any approximation) in terms of quantities, which at least, in principle, do have a direct physical significance. The final results are expressed (through quite long formulas, we admit, but exact) in terms of time derivatives of the system's electric (charge), magnetic, and toroid mean square radii (of various orders) and are obtained for an arbitrary time dependence of the latter. The exact formula for the radiation intensity is given by Eq. (4.8) [or Eq. (4.8′)], the one for the angular momentum loss is expressed by Eqs. (5.30)–(5.33) and the one for the recoil force by Eqs. (6.39)–(6.45). Although quite long (which is understandable in view of their generality), all these equations represent exact results in the correct multipole analysis of configurations of charges and currents that include toroidal sources. One immediate bonus from these expressions comes by particularizing them to the contributions of the first multipoles. Then corrections to the previously known familiar formulas are found, mostly on account of the toroid moments and their interference with the usual electric and magnetic ones, to the best of our knowledge not yet reported. Such corrections [to the order $(1/c^5)$ inclusively] are given, e.g., in Eq. (5.42) for the angular momentum loss and in Eq. (6.46) for the recoil force.

Finding an exact multipole analysis was a longstanding problem in electrodynamics and confusions have arised in the past due to the different number of independent quantities required for a complete characterization of the sources on one side and of their radiation fields on the other side. Indeed, sources should be described by three classes of multipoles while the radiation they emit is described by only two classes, the usual El and Ml waves. This reduction occurs on account of the additional restrictions on the scalar and vector potentials φ, $\vec{A}$, which, apart from the Lorenz condition $\vec{\nabla}\cdot\vec{A}+\partial\varphi/\partial t=0$ (the analog of the continuity relation for ρ and $\vec{j}$, $\vec{\nabla}\cdot\vec{j}+\partial\rho/\partial t=0$) are still subject to gauge invariance constraints, while ρ and $\vec{j}$ are not. This problem has been fully solved in Ref. [4] and in this context the exact results obtained in our work bring yet another support to the clarification of this matter of principle. The toroid multipoles introduced in Ref. [4] turn out to be indispensable in getting the closed formulas for the angular momentum loss, recoil force, and radiation intensity for an arbitrary source derived and presented here.

We mention also that our Eqs. (5.30)–(5.33) include not only an exhaustive generalization of a previous result concerning the angular momentum loss by a radiating toroid dipole [18] to any 2^l-toroid multipole, but furnishes also exact formulas for contributions coming from interferences between various toroid multipoles and the usual electric and magnetic ones.

Our work goes far beyond the multipole analysis presented for pedagogical purposes in certain chapters or paragraphs of the known books on classical electrodynamics, including the marvellous books by Jackson [24] or Landau and Lifschitz [21], which served over decades to the formation of so many physicists and certainly will continue to do this.

Since for the present research aims some new elements are necessary, we find it useful to shortly give here some explanatory remarks concerning the relations between our work and the presentations related to it in such textbooks.

Our Eqs. (4.12), (5.42), and (6.46) being different from those by Landau-Lifschitz (Ref. [21]), next we shall show why and where the toroid moments are lost in this treatise. Since the radiation emitted by the first element of the toroid family of multipoles, the toroid dipole, comes along with the radiation emitted by, e.g., the electric quadrupole, we refer to the Paragraph 71 "Quadrupole and magnetic-dipole radiation" of Ref. [21] and use below the notations and numbering of the equations from there.

Developing the integrand in the previous Eq. (66.2) that expresses the retarded vector potential,

$$\vec{A}=\frac{1}{cR_0}\int \vec{j}_{t'+\cdot\vec{r}\cdot\vec{n}/c}dV \tag{66.2}$$

in powers over $\vec{r}\cdot\vec{n}/c$ $(t'=t-R_0/c,\vec{n}=\vec{R}_0/R_0)$ and keeping only the first two terms, the authors found

$$\vec{A}=\frac{1}{cR_0}\int \vec{j}_{t'}dV+\frac{1}{c^2R_0}\frac{\partial}{\partial t'}\int (\vec{r}\cdot\vec{n})\vec{j}_{t'}dV.$$

Setting $\vec{j}=\rho\vec{v}$ and going to point charges, they got

$$\vec{A}=\frac{1}{cR_0}\sum e\vec{v}+\frac{1}{c^2R_0}\frac{\partial}{\partial t}\sum e\vec{v}(\vec{r}\cdot\vec{n}), \tag{71.1}$$

where the summation is over the charges and the index t' is here and further omitted. Now, with the relation

$$\vec{v}(\vec{r}\cdot\vec{n})=\frac{1}{2}\frac{\partial}{\partial t}\vec{r}(\vec{n}\cdot\vec{r})+\frac{1}{2}\vec{v}(\vec{n}\cdot\vec{r})-\frac{1}{2}\vec{r}(\vec{n}\cdot\vec{v})$$
$$=\frac{1}{2}\frac{\partial}{\partial t}\vec{r}(\vec{n}\cdot\vec{r})+\frac{1}{2}[(\vec{r}\times\vec{v})\times\vec{n}],$$

they have got for $\vec{A}$ the expression

$$\vec{A}=\frac{\dot{\vec{d}}}{cR_0}+\frac{1}{2c^2R_0}\frac{\partial^2}{\partial t^2}\sum e\vec{r}(\vec{n}\cdot\vec{r})+\frac{1}{cR_0}(\dot{\vec{m}}\times\vec{n}), \tag{71.2}$$

where $\vec{d}$ is the electric dipole moment and $\vec{m}=1/2c\Sigma e(\vec{r}\times\vec{v})$ is the magnetic dipole moment of the system. Now comes the first questionable point. To work out the expression of $\vec{A}$ from their Eq. (71.2), the authors used their Eqs. (66.3) for the fields $\vec{H}$ and $\vec{E}$.

$$\vec{H}=\frac{1}{c}\dot{\vec{A}}\times\vec{n}, \quad \vec{E}=\frac{1}{c}[(\dot{\vec{A}}\times\vec{n})\times\vec{n}], \qquad (66.3)$$

which is valid only in the plane-wave approximation where $\vec{H}$ and $\vec{E}$ are *strictly* perpendicular and both strictly perpendicular on $\vec{n}$, by adding to the right-hand side rhs of Eq. (71.2) a term proportional to $\vec{n}$ on account of the fact that the fields $\vec{H}$, $\vec{E}$ would not be affected by this operation, and forming so the electric quadrupole moment [in their notation $D_{\alpha\beta}=\Sigma e(3x_\alpha x_\beta-\delta_{\alpha\beta}r^2)$]. So they find

$$\vec{A}=\frac{\dot{\vec{d}}}{cR_0}+\frac{1}{6c^2R_0}\frac{\partial^2}{\partial t^2}\sum e[3\vec{r}(\vec{n}\cdot\vec{r})-\vec{n}r^2]+\frac{1}{cR_0}(\dot{\vec{m}}\times\vec{n}),$$

wherefrom they got their final form for the vector potential

$$\vec{A}=\frac{\dot{\vec{d}}}{cR_0}+\frac{1}{6c^2R_0}\ddot{\vec{D}}+\frac{1}{cR_0}(\dot{\vec{m}}\times\vec{n}), \qquad (71.3)$$

with $\vec{D}$ being the vector of components $D_\alpha=D_{\alpha\beta}n_\beta$ (summation over repeated indices understood). It is with this form of $\vec{A}$ that the authors finally found [using again Eqs. (66.3) for the fields valid only in the plane-wave approximation] their form of the fields,

$$\vec{H}=\frac{1}{c^2R_0}\left\{(\ddot{\vec{d}}\times\vec{n})+\frac{1}{6c}(\dddot{\vec{D}}\times\vec{n})+[(\ddot{\vec{m}}\times\vec{n})\times\vec{n}]\right\}, \qquad (71.4)$$

$$\vec{E}=\frac{1}{c^2R_0}\left\{[(\ddot{\vec{d}}\times\vec{n})\times\vec{n}]+\frac{1}{6c}[(\dddot{\vec{D}}\times\vec{n})\times\vec{n}]+(\vec{n}\times\ddot{\vec{m}})\right\}.$$

Up to this point, everything would be all right if the considerations were limited to the fields: indeed, one has contributions to them coming, among others, from the electric quadrupole moment of the system. But with these fields, one starts calculating next physical quantities that are *quadratic* in fields, such as the radiation intensity,

$$I=\frac{2}{3c^3}\ddot{\vec{d}}^2+\frac{1}{180c^5}\dddot{\vec{D}}^2_{\alpha\beta}+\frac{2}{3c^2}\ddot{\vec{m}}^2, \qquad (71.5)$$

immediately next and the recoil force

$$F_\alpha=-\frac{1}{c^4}\left\{\frac{1}{15c}\dddot{\vec{D}}_{\alpha\beta}\ddot{\vec{d}}_\beta+\frac{2}{3}(\ddot{\vec{d}}\times\ddot{\vec{m}})_\alpha\right\},$$

in the second problem at the end of the same Paragraph 71. And that is not permissible, and it is here that the toroid dipole moment contribution, among others, get lost.

The rhs of the last two expressions, for I and F_α, contain pieces correctly computed, however, these pieces are not the whole result, but only part of it, in the same sense as from the first-order expansions of two functions $f(x)=a+bx+\cdots$, $g(x)=c+dx+\cdots$, one cannot get the expansion of the product $f(x)g(x)$ correct to the *second* order in x, but only to the first. To have the second order of $f(x)g(x)$ one has to have the expansions of $f(x)$, $g(x)$ valid to second order also; otherwise one loses terms. Next, we shall show simply how one must proceed in order to get the full contribution to the rhs of the relations for I and F in the last two equations, thus bringing the Landau-Lifschitz results for I and F in full agreement with ours. One has first to get what is missing to $\vec{A}$ as given by Eq. (71.3), then to give up the plane-wave approximations Eqs. (66.3) for the fields and to compute them correctly from the basic formulas,

$$\vec{E}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi, \quad \vec{H}=\vec{\nabla}\times\vec{A},$$

in terms of both the vector and the *scalar* potential φ. So we shall recover our expressions Eqs. (4.12), (5.42), and (6.46) obtained in our general approach, but this time within the procedure of Landau-Lifschitz [21], by completing it with what is missing. And what is missing is the consideration of the next term in the initial development of $\vec{A}$ in powers of $\vec{r}\cdot\vec{n}/c$, together with an analogous treatment of the scalar potential φ. We do it now in the same spirit as in the Landau-Lifschitz treatment, with only minor algebraical complications due to the new $(\vec{r}\cdot\vec{n}/c)^2$ term.

So, starting from Eq. (66.2) and the corresponding one for the scalar potential φ and keeping also the next term in $\vec{r}\cdot\vec{n}/c$, one gets

$$\vec{A}=\frac{1}{cR_0}\int\vec{j}_{t'}dV+\frac{1}{c^2R_0}\frac{\partial}{\partial t'}\int(\vec{r}\cdot\vec{n})\vec{j}_{t'}dV+\frac{1}{2c^3R_0}\frac{\partial^2}{\partial t'^2}\int(\vec{r}\cdot\vec{n})^2\vec{j}_{t'}dV,$$

$$\varphi=\frac{1}{R_0}\int\rho_{t'}dV+\frac{1}{cR_0}\frac{\partial}{\partial t'}\int(\vec{r}\cdot\vec{n})\rho_{t'}dV+\frac{1}{2c^2R_0}\frac{\partial^2}{\partial t'^2}\int(\vec{r}\cdot\vec{n})^2\rho_{t'}dV+\frac{1}{6c^3R_0}\frac{\partial^3}{\partial t'^3}\int(\vec{r}\cdot\vec{n})^3\rho_{t'}dV,$$

and so, for point charges, instead of Eq. (71.1) one must consider

$$\vec{A}=\frac{1}{cR_0}\sum e\vec{v}+\frac{1}{c^2R_0}\frac{\partial}{\partial t}\sum e\vec{v}(\vec{r}\cdot\vec{n})+\frac{1}{2c^3R_0}\frac{\partial^2}{\partial t^2}\sum e(\vec{r}\cdot\vec{n})^2\vec{v}, \qquad (71.1')$$

$$\varphi=\frac{1}{R_0}\sum e+\frac{1}{cR_0}\frac{\partial}{\partial t}\sum e(\vec{r}\cdot\vec{n})+\frac{1}{2c^2R_0}\frac{\partial^2}{\partial t^2}\sum e(\vec{r}\cdot\vec{n})^2+\frac{1}{6c^3R_0}\frac{\partial^3}{\partial t^3}\sum e(\vec{r}\cdot\vec{n})^3,$$

and instead of the equation preceding Eq. (71.3) one should use here

$$\vec{A}=\frac{\dot{\vec{d}}}{cR_0}+\frac{1}{6c^2R_0}\frac{\partial^2}{\partial t^2}\sum e[3\vec{r}(\vec{n}\cdot\vec{r})-\vec{n}r^2+\vec{n}r^2]$$

$$+\frac{1}{cR_0}\dot{\vec{m}}\times\vec{n}+\frac{1}{2c^3R_0}\frac{\partial^2}{\partial t^2}\sum e(\vec{n}\cdot\vec{r})^2\vec{v},$$

$$\varphi=\frac{1}{R_0}Q+\frac{1}{cR_0}(\vec{n}\cdot\dot{\vec{d}})+\frac{1}{2c^2R_0}\frac{\partial^2}{\partial t^2}$$

$$\times\sum e(\vec{r}\cdot\vec{n})^2+\frac{1}{6c^3R_0}\frac{\partial^3}{\partial t^3}\sum e(\vec{r}\cdot\vec{n})^3,$$

(Q=total charge).

Then, in order to evaluate the last terms in the above equation for $\vec{A}$ containing $(\vec{n}\cdot\vec{r})^2$ one has to use the following algebrical identity in the vectors $\vec{n}$, $\vec{r}$, $\vec{v}$ (which can be checked directly by anyone):

$$(\vec{n}\cdot\vec{r})^2v_\alpha=-\varepsilon_{\alpha\beta\gamma}\frac{n_\beta n_\lambda}{3}[(\vec{r}\times\vec{v})_\gamma r_\lambda+(\vec{r}\times\vec{v})_\lambda r_\gamma]$$

$$+\frac{n_\alpha n_\gamma}{5}[r_\gamma(\vec{r}\cdot\vec{v})-2r^2v_\gamma]-\frac{1}{5}[r_\alpha(\vec{r}\cdot\vec{v})$$

$$-2r^2v_\alpha]+\frac{n_\alpha}{5}\vec{n}\cdot\left[r^2\vec{v}+2\vec{r}(\vec{r}\cdot\vec{v})\right.$$

$$+\frac{n_\lambda n_\mu}{3}\left\{v_\beta\left[\delta_{\alpha\beta}\left(r_\lambda r_\mu-\frac{r^2}{5}\delta_{\lambda\mu}\right)+\delta_{\beta\lambda}\right.\right.$$

$$\left.\times\left(r_\alpha r_\mu-\frac{r^2}{5}\delta_{\alpha\mu}\right)+\delta_{\beta\mu}\left(r_\alpha r_\lambda-\frac{r^2}{5}\delta_{\alpha\lambda}\right)\right]$$

$$\left.-\frac{2}{5}(\vec{r}\cdot\vec{v})(r_\alpha\delta_{\lambda\mu}+r_\lambda\delta_{\alpha\mu}+r_\mu\delta_{\alpha\lambda})\right\};$$

it serves to put in evidence the contributions of the various new lower multipoles:

magnetic quadrupole moment

$$m_{\gamma\lambda}=\frac{1}{3c}\sum e[(\vec{r}\times\vec{v})_\gamma r_\lambda+(\vec{r}\times\vec{v})_\lambda r_\gamma],$$

toroid dipole moment

$$t_\gamma=\frac{1}{10c}\sum e[r_\gamma(\vec{r}\cdot\vec{v})-2r^2v_\gamma],$$

first mean-square radius of the electric dipole charge distribution

$$\overrightarrow{r_d^2}=\sum er^2\vec{r},$$

$$\left(\dot{\overrightarrow{r_d^2}}=\sum e[r^2\vec{v}+2\vec{r}(\vec{r}\cdot\vec{v})]\right),$$

electric octupole moment

$$Q_{\alpha\lambda\mu}=\frac{1}{6}\sum e\left[r_\alpha r_\lambda r_\mu-\frac{1}{5}r^2(r_\alpha\delta_{\lambda\mu}+r_\lambda\delta_{\alpha\mu}+r_\mu\delta_{\alpha\lambda})\right]$$

$$\left(\dot{Q}_{\alpha\lambda\mu}=\frac{1}{6}\sum e\left\{v_\beta\left[\delta_{\alpha\beta}\left(r_\lambda r_\mu-\frac{r^2}{5}\delta_{\lambda\mu}\right)\right.\right.\right.$$

$$\left.+\delta_{\beta\lambda}\left(r_\alpha r_\mu-\frac{r^2}{5}\delta_{\alpha\mu}\right)+\delta_{\beta\mu}\left(r_\alpha r_\lambda-\frac{r^2}{5}\delta_{\alpha\lambda}\right)\right]$$

$$\left.\left.-\frac{2}{5}(\vec{r}\cdot\vec{v})(r_\alpha\delta_{\lambda\mu}+r_\lambda\delta_{\alpha\mu}+r_\mu\delta_{\alpha\lambda})\right\}\right).$$

So the pieces lost in the Paragraph 71 of Ref. [21] due to the omission of the $\partial^2/\partial t^2$ terms in Eqs. (71.1′) are

$$\frac{1}{2R_0c^3}\frac{\partial^2}{\partial t^2}\int d^3r(\vec{n}\cdot\vec{r})^2j_\alpha(\vec{r},t')$$

$$=\frac{1}{2R_0c^3}\frac{\partial^2}{\partial t^2}\sum e(\vec{n}\cdot\vec{r})^2v_\alpha$$

$$=-\frac{\varepsilon_{\alpha\beta\gamma}}{2R_0^3c^2}(\vec{R}_0)_\beta(\vec{R}_0)_\lambda\ddot{m}_{\gamma\lambda}+\frac{(\vec{R}_0)_\alpha(\vec{R}_0)_\gamma}{R_0^3c^2}\ddot{t}_\gamma$$

$$-\frac{1}{R_0c^2}\ddot{t}_\alpha+\frac{(\vec{R}_0)_\alpha}{10R_0^3c^3}\vec{R}_0\cdot\dddot{\overrightarrow{r_d^2}}+\frac{(\vec{R}_0)_\lambda(\vec{R}_0)_\mu}{R_0^2c^3}\ddot{Q}_{\alpha\lambda\mu}.$$

Also note that the last two terms of φ are

$$\frac{1}{2R_0c^2}\frac{\partial^2}{\partial t^2}\int d^3r(\vec{n}\cdot\vec{r})^2\rho(\vec{r},t')$$

$$=\frac{1}{2R_0c^2}\frac{\partial^2}{\partial t^2}\sum e(\vec{n}\cdot\vec{r})^2$$

$$=\frac{n_\alpha n_\beta}{R_0c^2}\ddot{Q}_{\alpha\beta}+\frac{1}{6R_0c^2}\ddot{r}_q^{(2)},$$

$$\frac{1}{6R_0c^3}\frac{\partial^3}{\partial t^3}\int d^3r(\vec{n}\cdot\vec{r})^3\rho(\vec{r},t')$$

$$=\frac{1}{6R_0c^3}\frac{\partial^3}{\partial t^3}\sum e(\vec{n}\cdot\vec{r})^3$$

$$=\frac{n_\alpha n_\beta n_\gamma}{R_0c^3}\dddot{Q}_{\alpha\beta\gamma}+\frac{1}{10R_0c^3}\vec{n}\cdot\dddot{\overrightarrow{r_d^2}}.$$

The correct form of the vector and scalar potentials, therefore, are (coming back from now on to the notations of our paper for the observation point $\vec{R}_0\to\vec{r}$),

$$A_\alpha(\vec{r},t)=\frac{1}{rc}\dot{d}_\alpha-\frac{1}{r^2c}(\vec{r}\cdot\dot{\vec{m}})_\alpha+\frac{r_\beta}{r^2c^2}\ddot{Q}_{\alpha\beta}+\frac{r_\alpha}{6r^2c^2}\ddot{r}_q^{(2)}-\frac{\varepsilon_{\alpha\beta\gamma}}{2r^3c^2}r_\beta r_\delta\ddot{m}_{\gamma\delta}+\frac{[\vec{r}\times(\vec{r}\times\dddot{\vec{t}})]_\alpha}{r^3c^2}+\frac{r_\alpha}{10r^3c^3}\vec{r}\cdot\dddot{\vec{r}_d^2}+\frac{r_\beta r_\gamma}{r^3c^3}\dddot{Q}_{\alpha\beta\gamma},$$

$$\varphi(\vec{r},t)=\frac{1}{r}Q+\frac{1}{r^2c}\vec{r}\cdot\dot{\vec{d}}+\frac{r_\alpha r_\beta}{r^3c^2}\ddot{Q}_{\alpha\beta}+\frac{1}{6rc^2}\ddot{r}_q^{(2)}+\frac{r_\alpha r_\beta r_\gamma}{r^4c^3}\dddot{Q}_{\alpha\beta\gamma}+\frac{1}{10r^2c^3}\vec{r}\cdot\dddot{\vec{r}_d^2}.$$

The argument of all the multipoles above is $(t-r/c)$.

Consequently the fields, $\vec{E}$, $\vec{H}$, are

$$H_\alpha(\vec{r},t)=-\frac{1}{r^2c^2}(\vec{r}\times\ddot{\vec{d}})_\alpha+\frac{1}{r^3c^2}[\vec{r}\times(\vec{r}\times\ddot{\vec{m}})]_\alpha-\frac{\varepsilon_{\alpha\beta\gamma}r_\beta r_\delta}{r^2c^3}\dddot{Q}_{\gamma\delta}+\frac{r_\alpha r_\beta r_\gamma\dddot{m}_{\beta\gamma}}{2r^4c^3}+\frac{(\vec{r}\times\dddot{\vec{t}})_\alpha}{r^2c^3}-\frac{r_\beta\dddot{m}_{\alpha\beta}}{2r^2c^3}-\frac{\varepsilon_{\alpha\beta\gamma}r_\beta r_\delta r_\zeta}{r^4c^4}\dddot{Q}_{\gamma\delta\zeta}+\left(\text{terms of higher order than }\frac{1}{rc^4}\right),$$

$$E_\alpha(\vec{r},t)=-\frac{1}{rc^2}\ddot{d}_\alpha+\frac{r_\alpha}{r^3c^2}(\vec{r}\cdot\ddot{\vec{d}})-\frac{r_\beta}{r^2c^3}\dddot{Q}_{\alpha\beta}+\frac{r_\alpha r_\beta r_\gamma}{r^4c^3}\times\dddot{Q}_{\beta\gamma}+\frac{1}{r^2c^2}(\vec{r}\times\ddot{\vec{m}})_\alpha+\frac{\varepsilon_{\alpha\beta\gamma}}{2r^3c^3}r_\beta r_\delta\dddot{m}_{\gamma\delta}+\frac{[(\vec{r}\times\dddot{\vec{t}})\times\vec{r}]_\alpha}{r^3c^3}+\left(\text{terms of higher order than }\frac{1}{rc^3}\right).$$

Comparing our Eqs. (71.4′) with Eqs. (71.4) of Ref. [21], one sees that new multipole contributions appear, including that of the toroid dipole, which have to be considered on the same footing with, e.g., the one of the electric quadrupole moment. $\vec{E}$, $\vec{H}$, are no more the fields of a plane wave but of a spherical wave. They are the correct ones as against those of Eqs. (71.4) obtained in Ref. [21] in the plane-wave approximation. Note that to the order $1/r$ in distance $\vec{n}\cdot\vec{E}$, $\vec{n}\cdot\vec{H}$ are still vanishing, but to the next $1/r^2$ order [appearing in the correct formulas above in what has been named as (terms of higher order than $1/rc^3$) in the equation for $\vec{E}$ and as (terms of higher order than $1/rc^4$) in the equation for $\vec{H}$] are not $(\vec{n}\cdot\vec{E}\neq 0,\vec{n}\cdot\vec{H}\neq 0)$. To get the above correct expressions for $\vec{H}$, $\vec{E}$, we calculated them starting from both the retarded scalar and vector potentials developed in $\vec{r}\cdot\vec{n}/c$ to the right order, while in Ref. [21] one has used only the vector potential developed one order less in $\vec{r}\cdot\vec{n}/c$ and the plane-wave approximation Eqs. (66.3), which is insufficient.

So one loses in Paragraph 71 of Ref. [21] the toroid dipole moment alongside with other "normal" lower multipole contributions (such as $m_{\gamma\lambda}$, $\overline{\vec{r}_d^2}$), which could be (depending only on the internal dynamics of the system) of the same order of magnitude as the electric quadrupole moment contribution already retained.

Calculating the radiation intensity, recoil force, and angular momentum loss with the above fields, obtained within the framework of Paragraph 71 of Ref. [21] amended by a correct handling of all the necessary multipole terms, one recovers exactly the corresponding results from our paper, obtained by us by particularizing our exact results obtained in a general context to the lower multipole contributions.

In Paragraph 72 of Ref. [21] "The radiation field at near distances," one goes indeed outside the plane-wave approximations, one calculates (approximately) the now contributing longitudinal components $\vec{n}\cdot\vec{E}$, $\vec{n}\cdot\vec{H}$, but one limits now from the start the considerations only to the electrical dipole radiation, thus giving up from the start the radiation coming from higher source's multipoles. And so one does the same also for the calculation of the angular momentum loss presented in problem 2 at the end of Paragraph 72.

To summarize, what is the lesson following from the above considerations? The lesson is that in order to have correct expressions for I, F_α, $d\vec{M}/dt$ one has to calculate the fields $\vec{E}$, $\vec{H}$ accurately enough in a region as close to the source as needed, in order to avoid losing competing terms. The same also holds for the treatment presented by Jackson [24], where the "radiation field" is considered and studied in relation to the sources but the toroidal moments and radii, hidden in Eqs. (9.167) and (9.168) are afterwards lost in the long wavelength approximation. In our paper the fields given by our Eqs. (2.39) and (2.40) are exact *everywhere* outside the sources and correctly related to them, while those parts of the fields that remain in the $1/r$ order in distance for $r\to\infty$ [as given by our Eqs. (3.4) and (3.5)] are similar to the ones considered by Jackson. A warning in this respect on manipulating with the fields in the wave zone has been long ago given by Blatt and Weisskopf Ref. [1], Appendix B "Multipole radiation," Sec. IV. "The sources of multipole radiation; multipole moments," in the footnote 1 that warrants to be cited here wholly:

> [1]It follows from this derivation that the angular momentum is contained everywhere in the field, not merely in the near zone, as asserted by Heitler (36). This can be seen most simply by imagining that the field was emitted during the short time Δt. The field then spreads out like a spherical shell, and the angular momentum contained in the field initially has to stay within that expanding shell.

We can write (3.2) in the form

$$G=(4\pi c)^{-1}\int [(\vec{r}\cdot\vec{\mathcal{H}})\vec{\mathcal{E}}^{*}-(\vec{r}\cdot\vec{\mathcal{E}}^{*})\vec{\mathcal{H}}]dV$$
$$+\text{complex conjugate.}$$

> Heitler's argument is based on the fact that the fields are transverse in the wave zone. However, this transversality is not complete. Either $\vec{\mathcal{E}}$ or $\vec{\mathcal{H}}$ or both have radial components proportional to r^{-2}, so that either $(\vec{r}\cdot\vec{E})$ or $(\vec{r}\cdot\vec{\mathcal{H}})$ or both are proportional to r^{-1}, giving a finite contribution to the integral even in the wave zone.

People seem to have paid little attention to this.

Next we shall show explicitly where the toroid moments and radii are hidden in the treatment given in Ref. [24] and how are they subsequently lost in the long wavelength approximation.

We do this by carefully analyzing the multipole amplitude of the electric radiation $a_E(l,m)$ given by Eq. (9.167) or Ref. [24]. We set the magnetization $\vec{\mathcal{M}}=0$ in Eq. (9.167) of Ref. [24], because in our work we have only ρ and $\vec{j}$ in vacuum (unlike Ref. [24] where one has ρ, $\vec{j}$ and the magnetization $\vec{\mathcal{M}}$), which does not restrict the generality since we can describe a medium as well. So, we rewrite here Eq. (9.167) of Ref. [24] for $\vec{\mathcal{M}}=0$,

$$a_E(l,m)=\frac{4\pi k^2}{i\sqrt{l(l+1)}}\int d^3r Y^{*}_{lm}(\theta,\varphi)\left\{\rho\frac{\partial}{\partial r}[rj_l(kr)]+\frac{ik}{c}(\vec{r}\cdot\vec{j})j_l(kr)\right\}, \tag{7.1}$$

where the Bessel functions $j_l(kr)$ appearing above are those of Ref. [24], which differ from ours by a factor of $[4\pi i^l]^{-1}$,

$$j_l^{\text{Ref. [24]}}=j_l^{[\text{this paper}]}\cdot\frac{1}{4\pi i^l}. \tag{7.2}$$

Our aim is to show the important fact that the amplitude $a_E(l,m)$ as given by Eq. (7.1) in terms of ρ and $\vec{j}$ (*without* any magnetization $\vec{\mathcal{M}}$) contains two *independent parts*, the usual one related to the charge distribution ρ and another one coming from the toroidal sources, more precisely from both the toroidal moments and the corresponding toroidal mean-square radii, i.e., coming from the whole toroid multipole formfactor $T_{lm}(-k^2,t)$. In Ref. [24], only the first part is considered and related to the electric multipole moment $Q_{lm}(0,t)$ while the second one is slighted in the long wavelength approximation as being of a higher order in k.

First, we note that the toroidal moments are hidden only in the middle $(\vec{r}\cdot\vec{j})$ term of Eq. (9.167) of Ref. [24] [i.e., in the second term of Eq. (7.1) above] since the toroid moments are related to the current density $\vec{j}$ and not to the charge density ρ. However, the middle $(\vec{r}\cdot\vec{j})$ term in Eq. (9.167) of Ref. [24] contains also the nonradiating electric mean-square radii, as it is seen from Eq. (1.3) of this paper (expressing the correct multipole expansion of the current density $\vec{j}$). The nonradiating part from the $(\vec{r}\cdot\vec{j})$ term (the middle term in Eq. (9.167) of Ref. [24] or the second term in Eq. (7.1) above) is exactly compensated by parts of the first term $\rho\,\partial/\partial_r[rj_l(kr)]$. Here, for simplicity, we shall substantiate what we have said above only in the simpler dipole case $l=1$, but the conclusions will be valid for any l. The point is that in the harmonic approach developed in Ref. [24], the appearing factors of k multiplying ρ in Eq. (9.167) [or in its transcription (7.1) above for $\vec{\mathcal{M}}=0$], complicate the picture because in the harmonic case one has

$$\rho(\vec{r},t)=\rho(\vec{r})e^{-i\omega t},\quad \omega=kc, \tag{7.3}$$

and, therefore, ρ is related in this case, to its time derivative $\dot{\rho}$

$$\rho=\frac{i}{\omega}\dot{\rho}=\frac{i}{kc}\dot{\rho}, \tag{7.4}$$

and in turn, by the continuity relation

$$\dot{\rho}+\vec{\nabla}\cdot\vec{j}=0 \tag{7.5}$$

(expressing the charge conservation), one has, in the harmonic case considered in Ref. [24], the following connection between ρ and $\vec{j}$:

$$\rho=-\frac{i}{kc}\vec{\nabla}\cdot\vec{j}. \tag{7.6}$$

Therefore, the k factors in Eq. (9.167) of Ref. [24] or Eq. (7.1) above are tricky and one should be careful. Now we start analyzing in detail the dipole ($l=1$) case and see concretely what happens.

So writing Eq. (7.1) for $l=1$, $m=0$ and using Eq. (7.4) one has

$$a_E(l=1,m=0)=\frac{4\pi k}{\sqrt{2}c}\int d^3r Y^{*}_{10}\left\{\dot{\rho}\frac{\partial}{\partial r}[rj_1(kr)]+k^2(\vec{r}\cdot\vec{j})j_1(kr)\right\}. \tag{7.7}$$

Setting in Eq. (7.7) above the concrete expressions of the spherical function Y^{*}_{10} and of the Bessel function $j_1(kr)$,

$$Y^{*}_{10}=\frac{1}{2}\frac{\sqrt{3}}{\sqrt{\pi}}\frac{z}{r}, \tag{7.8}$$

$$j_1(kr)=\frac{\sin kr}{k^2r^2}-\frac{\cos kr}{kr} \tag{7.9}$$

and calculating further $\partial/\partial r[rj_1(kr)]$, one finds

$$a_E(l=1,m=0)=\sqrt{6\pi}\frac{k}{c}\int d^3r\,\frac{z}{r}\left\{\dot\rho\left[\frac{1}{kr}\cos(kr)-\frac{1}{k^2r^2}\times\sin(kr)+\sin(kr)\right]+k^2(\vec r\cdot\vec j)\left[\frac{1}{k^2r^2}\sin(kr)-\frac{1}{kr}\cos(kr)\right]\right\}. \tag{7.10}$$

Developing now the trigonometric functions in series,

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots,$$

$$\cos x=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots,$$

and retaining only terms to the order k^5 inclusively, one finds for the first square bracket in the rhs of Eq. (7.10),

$$\left[\frac{1}{kr}\cos(kr)-\frac{1}{k^2r^2}\sin(kr)+\sin(kr)\right]=\frac{2}{3}kr-\frac{2}{15}k^3r^3+\frac{1}{140}k^5r^5+\cdots, \tag{7.11}$$

and for the second square bracket in the rhs of Eq. (7.10),

$$\left[\frac{1}{k^2r^2}\sin(kr)-\frac{1}{kr}\cos(kr)\right]=\frac{1}{3}kr-\frac{1}{30}k^3r^3+\frac{1}{840}k^5r^5+\cdots. \tag{7.12}$$

Inserting these developments into the rhs of Eq. (7.10) and retaining only terms to the order k^5 inclusively, one finds for $a_E(l=1,m=0)$,

$$a_E(l=1,m=0)=\sqrt{6\pi}\,\frac{k}{c}\int d^3r\,\frac{z}{r}\left\{\dot\rho\,\frac{2}{3}\,kr-\dot\rho\,\frac{2}{15}\,k^3r^3+\dot\rho\,\frac{1}{140}\,k^5r^5+k^2(\vec r\cdot\vec j)\times\frac{1}{3}\,kr-k^3(\vec r\cdot\vec j)\,\frac{1}{30}\,k^3r^3+k^2(\vec r\cdot\vec j)\,\frac{1}{840}\,k^5r^5\right\}+\cdots. \tag{7.13}$$

Taking outside a general factor of k^2 one gets further:

$$a_E(l=1,m=0)=\sqrt{6\pi}\frac{k^2}{c}\left\{\frac{2}{3}\int d^3r\dot\rho z-\frac{2}{15}k^2\int d^3r\dot\rho r^2z+\frac{k^2}{3}\int d^3r(\vec r\cdot\vec j)z+\frac{k^4}{140}\int d^3r\dot\rho r^4z-\frac{k^4}{30}\int d^3r(\vec r\cdot\vec j)r^2z\right\}+\cdots. \tag{7.14}$$

The second and the fourth integrals in the rhs of Eq. (7.14) above represent the first- and the second-order mean-square radii of the electric dipole distributions, respectively. They are exactly compensated by the same radii appearing in the third and fifth integrals containing $(\vec r\cdot\vec j)$.

Now we want to put in evidence inside the rhs of Eq. (7.14) quantities with known well defined multipole content. We observe that the first integral inside the braces from the rhs of Eq. (7.14) is just $\dot d_z$, i.e., the time derivative of the z component of the electric dipole moment,

$$\vec d=\int\rho\vec r d^3r, \tag{7.15}$$

so

$$\int d^3r\dot\rho z=\dot d_z, \tag{7.16}$$

where

$$\dot d_z=-i\omega d_z. \tag{7.17}$$

Since the next two integrals, containing $\dot\rho$, in the rhs of Eq. (7.14) cannot be related to electric multipole moments, we shall use the continuity relation Eq. (7.5) and shall put $\dot\rho=-\vec\nabla\cdot\vec j$ into them. By part integration we put these next two integrals containing $\dot\rho$ into the form

$$\int d^3r\dot\rho r^2z=-\int d^3r(\vec\nabla\cdot\vec j)r^2z=\int d^3r\vec j\cdot\vec\nabla(r^2z)=\int d^3r\vec j\cdot(2z\vec r+\vec k r^2)=\int d^3r[2z(\vec r\cdot\vec j)+r^2j_z], \tag{7.18}$$

and

$$\int d^3r\dot\rho r^4z=-\int d^3r(\vec\nabla\cdot\vec j)r^4z=\int d^3r\vec j\cdot\vec\nabla(r^4z)=\int d^3rr^2\vec j(4z\vec r+\vec k r^2)=\int d^3rr^2[4z(\vec r\cdot\vec j)+r^2j_z]. \tag{7.19}$$

Returning to the expression Eq. (7.14) for $a_E(l=1,m=0)$, and inserting the expressions from the rhs of Eqs. (7.18) and (7.19) into it, one finds

$$a_E(l=1,m=0)=\sqrt{6\pi}\frac{k^2}{c}\left\{\frac{2}{3}\dot{d}_z+k^2\left[-\frac{2}{15}\int d^3r(2z(\vec{r}\cdot\vec{j})+r^2j_z)+\frac{1}{3}\int d^3r(\vec{r}\cdot\vec{j})z\right]_{(1)}+k^4\left[\frac{1}{140}d^3rr^2(4z(\vec{r}\cdot\vec{j})+r^2j_z)-\frac{1}{30}\int d^3r(\vec{r}\cdot\vec{j})r^2z\right]_{(2)}\right\}. \tag{7.20}$$

The first square bracket in the rhs of Eq. (7.20) is

$$[\cdots]_{(1)}=\frac{2}{30}\int d^3r[z(\vec{r}\cdot\vec{j})-2r^2j_z], \tag{7.21}$$

while the second one is

$$[\cdots]_{(2)}=\frac{2}{3\times 280}\int d^3r[3r^2j_z-2(\vec{r}\cdot\vec{j})z]. \tag{7.22}$$

Inserting the rhs of Eqs. (7.21) and (7.22) into the rhs of Eq. (7.20) one finally finds for the $m=0$ component of the electric dipole coefficient $a_E(l=1,m=0)$ considered in Ref. [24] and usually used in the literature, the following form in terms of quantities with well defined multipole content:

$$a_E(l=1,m=0)=\frac{2\sqrt{6\pi}}{3}\frac{k^2}{c}\left\{-i\omega d_z+k^2\frac{1}{10}\int d^3r[z(\vec{r}\cdot\vec{j})-2r^2j_z]+k^4\frac{1}{280}\int d^3r[3r^2j_z-2(\vec{r}\cdot\vec{j})z]\right\}+\cdots. \tag{7.23}$$

Indeed, in Eq. (7.23) above, the formerly hidden toroidal contributions have made their appearance and all the nonradiating electric mean-square radii coming from both the first (ρ) term and the second ($\vec{r}\cdot\vec{j}$) term in Eq. (7.1) have exactly canceled, as it must: the first integral in the rhs of Eq. (7.23) is just the z component of the toroidal dipole moment,

$$\vec{t}=\frac{1}{10c}\int[\vec{r}(\vec{r}\cdot\vec{j})-2r^2\vec{j}]d^3r, \tag{7.24}$$

while the second integral expresses just the (radiating) first mean-square radius of the toroid dipole distribution in the Cartesian basis,

$$\overrightarrow{R^2}=\frac{1}{28c}\int d^3rr^2[3r^2\vec{j}-2\vec{r}(\vec{r}\cdot\vec{j})] \tag{7.25}$$

(see Appendix C for the relation between the spherical and Cartesian expressions of the lowest order multipoles considered in our paper, as well as for a discussion of $\overrightarrow{R^2}$ expressed in the Cartesian basis). The exact cancellation of the first mean-square radii of the electric dipole in the rhs of Eq. (7.14) can be easily seen noting the identity,

$$\int d^3rz(\vec{r}\cdot\vec{j})=2ct_z+\frac{2}{5}(\dot{\overline{r_d^2}})_z, \tag{7.26}$$

where $\overrightarrow{r_d^2}$ is the first mean-square radius of the electric dipole distribution,

$$\overrightarrow{r_d^2}=\int d^3r\rho r^2\vec{r}.$$

Similarly one can see the exact cancellation of the second-order mean-square radius of the electric dipole distribution.

So finally it is seen that $a_E(l=1,m=0)$ can be expressed as

$$a_E(l=1,m=0)=\frac{2\sqrt{6\pi}}{3}\frac{k^2}{c}\left[-i\omega d_z+k^2ct_z+\frac{k^4c}{10}\overline{(R^2)_z}\right]+(\text{terms of higher order than }k^4). \tag{7.27}$$

Noting the relations between the spherical and Cartesian components of the toroidal dipole moment and the first toroidal dipole radius given in Appendix C,

$$t_z=T_{10},\overline{R_z^2}=-(\overline{R^2})_{l=1,m=0}, \tag{7.28}$$

as well as Eqs. (1.37) expressing the numerical factors entering the connections between the derivatives of the formfactors and the mean-square radii,

$$\overline{R_{lm}^{2n}}(t)=\frac{2^n(2l+2n+1)!!}{(2l+1)!!}\frac{d^n}{d(-k^2)^n}\times T_{lm}(-k^2,t)|_{k^2=0},\ [\text{Eq. }(1.37)]$$

which in our case $(l=1,m=0,n=1)$ becomes

$$\overline{R_{10}^2}(\omega)=10\frac{d}{d(-k^2)}T_{10}(-k^2,\omega)|_{k^2=0}, \tag{7.29}$$

one can write the coefficient $a_E(l=1,m=0)$ in the form

$$a_E(l=1,m=0)=\frac{2\sqrt{6\pi}}{3}\frac{k^2}{c}[-i\omega Q_{10}(0,\omega)+k^2cT_{10}(-k^2,\omega)]. \tag{7.30}$$

The same kind of relation holds also in general, for any l,

$$a_E(l,m)=\frac{2\sqrt{6\pi}}{3}\frac{k^2}{c}[-i\omega Q_{lm}(0,\omega)+k^2cT_{lm}(-k^2,\omega)]. \tag{7.31}$$

$Q_{lm}(0,\omega)$ are the (complex) Fourier components of the electric multipole moments and $T_{lm}(-k^2,\omega)$ the (complex) Fourier components of the toroidal form factors considered in our paper. We recall that for $k^2=0$, the form factors $T_{lm}(-k^2,\omega)$ reduce to the toroid moments $T_{lm}(0,\omega)$. Unfortunately, in Ref. [24] the toroidal contribution in Eq. (7.31) is lost as being of a higher order in k, when making the approximations that lead to the electric multipole coefficient in Eq. (9.169).

So, from Eq. (7.31) it is seen that the electric multipole coefficient $a_E(l,m)$ from Eq. (7.1) contains two different independent pieces, the electric moment and the also radiating toroidal formfactor. Both these pieces are independent from the third term of Eq. (9.167) from Ref. [24] related to the magnetization $\mathcal{M}$ (which leads there to the so called "induced electric moments" Q'_{lm}). The *El* radiation coming from the toroidal moment is two orders in ω higher that the one from the electric moment and shifted in phase with $\pi/2$. To disentangle the radiation coming from the toroidal formfactor from the one coming from the corresponding electric moment one needs a spectral analysis of the radiation. As noted in Ref. [4], one may also extract the toroidal moments and radii by flowing an electric current through the source.

To have the fields exact everywhere outside the sources and subsequently exactly calculate the physical quantities I, F_α, $d\vec{M}/dt$ that are quadratic in the fields (what we did in this paper), one needs first a *complete* parametrization of the sources in terms of *independent* multipole parameters, up to the ultimate ones, the multipole radii of various types (electric, magnetic, toroid), any multipolarity l and any order. To our knowledge, such a multipole parametrization is the one from Refs. [4,5] which essentially (with some minor errors corrected in our paper) has been used by us to find correctly I, F_α, $d\vec{M}/dt$. There are mathematical rules that must be obeyed if one wants exact results, but which may be diverse, may be disagreeable, and even disputable on their economical aspect. The point is that here, nevertheless, is something more. There is a physical subtlety that makes our approach unavoidable: the magnetic and toroid mean-square radii of various orders *do* radiate, while the electric (charge) radii *do not*. Only the electric radius of order zero (i.e., the electric *moment*) of any multipolarity does radiate. One of the main virtues of the multipole analysis given in Refs. [4], [5] was to achieve not only an exact mathematical expansion of $\rho(\vec{r},t)$, $\vec{j}(\vec{r},t)$ (related by $\partial\rho/\partial t+\nabla\vec{j}=0$) into a complete set of independent multipole parameters (the radii), but to do this in such a way as to disentangle properly, without any approximation, the *nonradiating* electric radii from the radiating magnetic and toroid ones. That is why we *had* to go to radii and not restrict ourselves to multipole moments as in most textbooks, including Jackson's, not because of unnecessary mathematical arguments, but because here physics is involved with its powerful demands; and rewards, perhaps—e.g., the somewhat easy way of solving the not so trivial problem of the arbitrary time dependence of the sources.

Some of the final results of this work have been reported in Ref. [25].

APPENDIX A: CONVENTIONS FOR BESSEL AND HANKEL FUNCTIONS

The cylindrical Bessel functions of first species are

$$J_\nu(x)=\left(\frac{x}{2}\right)^\nu\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,\Gamma(j+\nu+1)}\left(\frac{x}{2}\right)^{2j}.$$

For $\nu=m=$ integer: $J_{-m}(x)=(-1)^m J_m(x)$. The cylindrical Bessel functions of second species are

$$N_\nu=\frac{J_\nu(x)\cos(\nu\pi)-J_{-\nu}(x)}{\sin(\nu\pi)}.$$

Cylindrical Hankel functions are

$$H_\nu^{(\pm)}(x)=J_\nu(x)\pm iN_\nu(x).$$

Spherical Bessel functions of first species are

$$j_l(x)=(2\pi)^{3/2}i^l\frac{J_{l+1/2}(x)}{\sqrt{x}}.$$

Spherical Bessel functions of second species are

$$n_l(x)=(2\pi)^{3/2}i^l\frac{N_{l+1/2}(x)}{\sqrt{x}}.$$

Spherical Hankel functions are

$$h_l^{(\pm)}(x)=j_l(x)\pm in_l(x),$$

$$j_l(x)=4\pi i^l(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin x}{x};$$

$$n_l(x)=-4\pi i^l(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos x}{x}.$$

Low and high x behavior:

$$j_l(x)\underset{x\to 0}{\sim}4\pi i^l\frac{x^l}{(2l+1)!!};\quad j_l(x)\underset{x\to\infty}{\sim}4\pi i^l\frac{\sin\left(x-\frac{l\pi}{2}\right)}{x};$$

$$n_l(x)\underset{x\to 0}{\sim}-4\pi i^l\frac{(2l-1)!!}{x^{l+1}};\quad n_l(x)\underset{x\to\infty}{\sim}-4\pi i^l\frac{\cos\left(x-\frac{l\pi}{2}\right)}{x};$$

$$h_l^{(\pm)}(x)\underset{x\to\infty}{\sim}\mp 4\pi i^{l+1}\frac{e^{\pm i(x-(l\pi/2))}}{x}.$$

One has

$$\frac{d}{dz}j_l(z)=\frac{i}{2l+1}[lj_{l-1}(z)+(l+1)j_{l+1}(z)],$$

$$j_l(z)=\frac{iz}{2l+1}[j_{l-1}(z)-j_{l+1}(z)].$$

APPENDIX B: WORKING FORMULAS

We list below some working formulas often used in this paper [22]:

$$\vec{\nabla}\times[\phi(r)\vec{Y}_{llm}]=i\left(\frac{d}{dr}-\frac{1}{r}\right)\phi(r)\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{ll+1m}+i\left(\frac{d}{dr}+\frac{l+1}{r}\right)\phi(r)\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{ll-1m},$$

$$\vec{\nabla}\times[\phi(r)\vec{Y}_{ll+1m}]=i\left(\frac{d}{dr}+\frac{l+2}{r}\right)\phi(r)\frac{\sqrt{l}}{\sqrt{2l+1}}\vec{Y}_{llm},$$

$$\vec{\nabla}\times[\phi(r)\vec{Y}_{ll-1m}]=i\left(\frac{d}{dr}-\frac{l-1}{r}\right)\phi(r)\frac{\sqrt{l+1}}{\sqrt{2l+1}}\vec{Y}_{llm},$$

$$\vec{\nabla}[\phi(r)Y_{lm}]=-\frac{\sqrt{l+1}}{\sqrt{2l+1}}\left(\frac{d}{dr}-\frac{l}{r}\right)\phi(r)\vec{Y}_{ll+1m}+\frac{\sqrt{l}}{\sqrt{2l+1}}\left(\frac{d}{dr}+\frac{l+1}{r}\right)\phi(r)\vec{Y}_{ll-1m}.$$

APPENDIX C: CONNECTION BETWEEN SPHERICAL AND CARTESIAN COMPONENTS OF THE FIRST MULTIPOLES

Since, in this paper, we have completed some formulas given in known books with contributions coming from the toroid family of multipoles (previously either simply not considered or neglected as small corrections or sometimes obscurely included in so called ‘‘retardation effects,’’ we give below in the first relevant cases the connections between the spherical (as used in this paper) and the (usually employed) Cartesian components of the first multipoles.

1. *Electric (charge) dipole*: For the electric multipole moments we have used

$$Q_{lm}(t)=\frac{\sqrt{4\pi}}{\sqrt{2l+1}}\int r^{l}Y^{*}_{lm}\left(\frac{\vec{r}}{r}\right)\rho(\vec{r},t)d^3r.$$

With

$$\vec{d}=\int \rho(\vec{r},t)\vec{r}d^3r,$$

one has

$$Q_{10}=d_z,\quad Q_{11}=\frac{1}{\sqrt{2}}(-d_x+id_y),\quad Q_{1-1}=\frac{1}{\sqrt{2}}(d_x+id_y).$$

2. *Magnetic dipole*: For the magnetic multipole moments we had

$$M_{lm}(t)=-\frac{i}{c}\frac{\sqrt{4\pi l}}{\sqrt{(2l+1)(l+1)}}\int d^3r r^{l}\vec{Y}^{*}_{llm}\left(\frac{\vec{r}}{r}\right)\vec{j}(\vec{r},t).$$

With

$$\vec{m}=\frac{1}{2c}\int d^3r[\vec{r}\times\vec{j}(\vec{r},t)],$$

one has

$$M_{11}=\frac{1}{\sqrt{2}}(m_x-im_y),$$

$$M_{1-1}=-\frac{1}{\sqrt{2}}(m_x+im_y),$$

$$M_{10}=-m_z.$$

3. *Toroid dipole*: The toroid multipole moments were defined as

$$T_{lm}=-\frac{\sqrt{\pi l}}{c(2l+1)}\int r^{l+1}\left[\vec{Y}^{*}_{ll-1m}\left(\frac{\vec{r}}{r}\right)+\frac{2}{(2l+3)}\frac{\sqrt{l}}{\sqrt{l+1}}\vec{Y}^{*}_{ll+1m}\left(\frac{\vec{r}}{r}\right)\right]\vec{j}(\vec{r},t)d^3r.$$

With

$$\vec{t}=\frac{1}{10c}\int[\vec{r}(\vec{r}\cdot\vec{j})-2\vec{r}^2\vec{j}]d^3r,$$

one has

$$T_{11}=-\frac{1}{\sqrt{2}}(t_x-it_y),$$

$$T_{1-1}=\frac{1}{\sqrt{2}}(t_x+it_y),$$

$$T_{10}=t_z.$$

4. *Electric (charge) quadrupole*: For $l=2$

$$Q_{2m}(t)=\frac{\sqrt{4\pi}}{\sqrt{5}}\int d^3r r^2\rho(\vec{r},t)Y^{*}_{2m}\left(\frac{\vec{r}}{r}\right).$$

With

$$Q_{ij}=\frac{1}{2}\int \rho(\vec{r},t)\left(r_ir_j-\frac{1}{3}\delta_{ij}\vec{r}^2\right)d^3r,$$

one has

$$Q_{22}=\frac{\sqrt{6}}{2}(Q_{xx}-2iQ_{xy}-Q_{yy}),$$

$$Q_{2-2}=\frac{\sqrt{6}}{2}(Q_{xx}+2iQ_{xy}-Q_{yy}),$$

$$Q_{21}=\sqrt{6}(-Q_{xz}+iQ_{yz}),\quad Q_{2-1}=\sqrt{6}(Q_{xz}+iQ_{yz}),$$

$$Q_{20}=3Q_{zz}.$$

5. *Magnetic quadrupole*: For $l=2$,

$$M_{2m}(t)=-\frac{i}{c}\frac{\sqrt{8\pi}}{\sqrt{15}}\int d^3r r^2\vec{Y}^*_{22m}\left(\frac{\vec{r}}{r}\right)\vec{j}(\vec{r},t).$$

With

$$m_{ij}=\frac{1}{3c}\int d^3r[(\vec{r}\times\vec{j})_i r_j+(\vec{r}\times\vec{j})_j r_i],$$

one has

$$M_{21}=\sqrt{\frac{3}{2}}(m_{xz}-im_{yz}),\quad M_{2-1}=-\sqrt{\frac{3}{2}}(m_{xz}+im_{yz}),$$

$$M_{22}=-\frac{\sqrt{6}}{4}(m_{xx}-2im_{xy}-m_{yy}),$$

$$M_{2-2}=-\frac{\sqrt{6}}{4}(m_{xx}+2im_{xy}-m_{yy}),$$

$$M_{20}=-\frac{3}{2}m_{zz}.$$

6. *Toroid quadrupole*: For $l=2$,

$$T_{2m}(t)=-\frac{\sqrt{2\pi}}{5c}\int r^3\left[\vec{Y}^*_{21m}\left(\frac{\vec{r}}{r}\right)+\frac{2\sqrt{2}}{7\sqrt{3}}\vec{Y}^*_{23m}\left(\frac{\vec{r}}{r}\right)\right]\vec{j}(\vec{r},t)d^3r.$$

With

$$t_{ik}=\frac{1}{28c}\int[4r_ir_k(\vec{r}\vec{j})-5r^2(r_ij_k+r_kj_i)+2r^2(\vec{r}\vec{j})\delta_{ik}]d^3r,$$

one has

$$T_{21}=-\frac{\sqrt{2}}{\sqrt{3}}(t_{xz}-it_{yz}),\quad T_{2-1}=\frac{\sqrt{2}}{\sqrt{3}}(t_{xz}+it_{yz}),$$

$$T_{22}=\frac{1}{\sqrt{6}}(t_{xx}-2it_{xy}-t_{yy}),\quad T_{2-2}=\frac{1}{\sqrt{6}}(t_{xx}+2it_{xy}-t_{yy}),$$

$$T_{20}=t_{zz}.$$

7. *Electric (charge) octupole*: For $l=3$,

$$Q_{3m}(t)=\sqrt{\frac{4\pi}{7}}\int d^3r\rho(\vec{r},t)r^3Y^*_{3m}\left(\frac{\vec{r}}{r}\right).$$

With

$$Q_{ikl}(t)=\frac{1}{6}\int d^3r\rho(\vec{r},t)\left[r_ir_kr_l-\frac{1}{5}r^2(r_i\delta_{kl}+r_k\delta_{il}+r_l\delta_{ik})\right],$$

one has

$$Q_{30}=15Q_{zzz},$$

$$Q_{31}=-\frac{15\sqrt{3}}{2}(Q_{zzx}+iQ_{yyy}+iQ_{xxy}),$$

$$Q_{32}=-3\sqrt{\frac{15}{2}}(Q_{zzz}+2Q_{yyz}+i2Q_{xyz}),$$

$$Q_{33}=-\frac{3\sqrt{5}}{2}(Q_{xxx}-3Q_{yyx}+iQ_{yyy}-i3Q_{xxy}),$$

$$Q_{3,-m}=(-1)^mQ^*_{3m}.$$

8. *Magnetic octupole*: For $l=3$,

$$M_{3m}(t)=-\frac{i}{c}\sqrt{\frac{3\pi}{7}}\int d^3r r^3\vec{Y}^*_{33m}\cdot\vec{j}(\vec{r},t).$$

With

$$m_{ijk}=\frac{15}{2c}\int d^3r\left[r_ir_j(\vec{r}\times\vec{j})_k+r_ir_k(\vec{r}\times\vec{j})_j+r_jr_k(\vec{r}\times\vec{j})_i+\frac{\delta_{ij}}{5}(\vec{r}\times\vec{r}\times\vec{r}\times\vec{j})_k+\frac{\delta_{ik}}{5}(\vec{r}\times\vec{r}\times\vec{r}\times\vec{j})_j+\frac{\delta_{jk}}{5}(\vec{r}\times\vec{r}\times\vec{r}\times\vec{j})_i\right],$$

one has

$$M_{30}=-\frac{1}{12}m_{zzz},$$

$$M_{31}=\frac{1}{8\sqrt{3}}(m_{zzx}+im_{yyy}+im_{xxy}),$$

$$M_{32}=\frac{\sqrt{2}}{8\sqrt{15}}(m_{zzz}+2m_{yyz}+i2m_{xyz}),$$

$$M_{33}=\frac{1}{24\sqrt{5}}(m_{xxx}-3m_{yyx}+im_{yyy}-i3m_{xxy}),$$

$$M_{3,-m}=(-1)^mM^*_{3m}.$$

9. *Electric (charge) hexadecapole*: For $l=4$,

$$Q_{4m}(t)=\frac{2\sqrt{\pi}}{3}\int d^3r\rho(\vec{r},t)r^4Y^*_{4m}\left(\frac{\vec{r}}{r}\right).$$

With

$$q_{iklm}(t)=2\int d^3r\rho(\vec{r},t)\bigg[r_ir_kr_lr_m-\frac{1}{7}r^2(\delta_{ik}r_lr_m+\delta_{il}r_kr_m+\delta_{im}r_kr_l+\delta_{kl}r_ir_m+\delta_{km}r_ir_l+\delta_{lm}r_ir_k)+\frac{1}{35}r^4(\delta_{ik}\delta_{lm}+\delta_{il}\delta_{km}+\delta_{im}\delta_{kl})\bigg],$$

one has

$$Q_{40}=\frac{35}{16}q_{zzzz},$$

$$Q_{41}=-\frac{7\sqrt{5}}{8}(q_{xzzz}-iq_{yzzz}),$$

$$Q_{42}=\frac{7}{8}\sqrt{\frac{5}{2}}(q_{xxzz}-q_{yyzz}-i2q_{xyzz}),$$

$$Q_{43}=-\frac{\sqrt{35}}{8}(q_{xxxz}-3q_{xyyz}+iq_{yyyz}-i3q_{xxyz}),$$

$$Q_{44}=\frac{35\sqrt{70}}{32}(q_{xxxx}-6q_{xxyy}+q_{yyyy}+i4q_{xyyy}-4q_{xxxy}),$$

$$Q_{4,-m}=(-1)^mQ_{4m}^{*}.$$

10. *First mean-square radius of the charge*: For $l=0$,

$$m=0,\overline{r_{00}^2}(t)=\int d^3rr^2\rho(\vec{r},t).$$

With the Cartesian definition

$$\overline{r_q^{(2)}}(t)=\int d^3rr^2\rho(\vec{r},t),$$

one has

$$\overline{r_q^{(2)}}(t)=\overline{r_{00}^2}(t).$$

11. *First mean-square radius of the electric (charge) dipole*: For $l=1$,

$$\overline{r_{1m}^2}(t)=\sqrt{\frac{4\pi}{3}}\int d^3rr^3\rho(\vec{r},t)Y_{1m}^{*}\left(\frac{\vec{r}}{r}\right).$$

With the Cartesian definition

$$\overline{\vec{r}_d^{\,2}}(t)=\int d^3r\vec{r}r^2\rho(\vec{r},t),$$

one has

$$\overline{r_{10}^2}=(\overline{r_d^2})z,$$

$$\overline{r_{11}^2}=-\frac{1}{\sqrt{2}}[(\overline{r_d^2})_x-i(\overline{r_d^2})_y],$$

$$\overline{r_{1-1}^2}=\frac{1}{\sqrt{2}}[(\overline{r_d^2})_x+i(\overline{r_d^2})_y].$$

12. *First order mean-square radius of the magnetic dipole distribution*: For $l=1$,

$$\overline{\rho_{1m}^2}=\frac{1}{ic}\frac{\sqrt{2\pi}}{\sqrt{3}}\int d^3rr^3\vec{Y}_{11m}^{*}\left(\frac{\vec{r}}{r}\right)\cdot\vec{j}(\vec{r},t).$$

With

$$\overline{\rho_i^2}=\frac{1}{2c}\varepsilon_{ijk}\int d^3rr^2r_jj_k(\vec{r},t),$$

$$\left(\overline{\vec{\rho}^{\,2}}=\frac{1}{2c}\int d^3rr^2(\vec{r}\times\vec{j})\right),$$

one has

$$\overline{\rho_{10}^2}=-\overline{\rho_z^2},$$

$$\overline{\rho_{11}^2}=\frac{1}{\sqrt{2}}(\overline{\rho_x^2}-i\overline{\rho_y^2}),$$

$$\overline{\rho_{1-1}^2}=\frac{1}{\sqrt{2}}(-\overline{\rho_x^2}-i\overline{\rho_y^2}).$$

13. *First mean-square radius of the toroidal dipole*: For $l=1$,

$$\overline{R_{1m}^2}(t)=-\frac{\sqrt{2\pi}}{3c}\int d^3rr^4\left[\frac{1}{7}\vec{Y}_{12m}^{*}\left(\frac{\vec{r}}{r}\right)+\frac{1}{2\sqrt{2}}\vec{Y}_{10m}^{*}\left(\frac{\vec{r}}{r}\right)\right]\cdot\vec{j}(\vec{r},t),$$

and in the Cartesian basis it results for the first mean-square radius of the toroid dipole distribution the form

$$\overline{\vec{R}^{2}}(t)=\frac{1}{28c}\int d^3rr^2\{3r^2\vec{j}(\vec{r},t)-2\vec{r}[\vec{r}\cdot\vec{j}(\vec{r},t)]\}.$$

Note that the square bracket multiplying r^2 under the integral above is not the toroid dipole density in the Cartesian basis. In the electric and magnetic cases this particularity for radii in the Cartesian basis does not occur.

One has

$$\overline{R_z^2}=-\overline{R_{10}^2},$$

$$\overline{R_x^2}=\frac{1}{\sqrt{2}}(\overline{R_{11}^2}-\overline{R_{1-1}^2}),$$

$$\overline{R_y^2}=\frac{i}{\sqrt{2}}(\overline{R_{11}^2}+\overline{R_{1-1}^2}).$$

14. *First mean-square radius of the electric (charge) quadrupole*: For $l=2$,

$$\overline{r_{2m}^2}(t)=\sqrt{\frac{4\pi}{5}}\int d^3r r^4 \rho(\vec{r},t) Y_{2m}^*.$$

With

$$\overline{r_{ij}^{(2)}}(t)=\frac{1}{2}\int d^3r r^2\left(r_i r_j-\frac{1}{3}\delta_{ij}r^2\right)\rho(\vec{r},t),$$

one has

$$\overline{r_{20}^2}=3\overline{r_{zz}^{(2)}},$$

$$\overline{r_{21}^2}=-\sqrt{6}(\overline{r_{xz}^{(2)}}-i\overline{r_{yz}^{(2)}}),$$

$$\overline{r_{22}^2}=\sqrt{\frac{3}{2}}(\overline{r_{xx}^{(2)}}-\overline{r_{yy}^{(2)}}-2i\overline{r_{xy}^{(2)}}),$$

$$\overline{r_{2,-m}^2}=(-1)^m(\overline{r_{2m}^2})^*.$$

APPENDIX D: CARTESIAN COMPONENTS OF $\vec{Y}_{ll'm}$ ($l'=l,l\pm 1$)

The vector spherical harmonics $\vec{Y}_{ll'm}(\vec{n})$ ($\vec{n}=\vec{r}/r$) are defined in the spherical basis as

$$[\vec{Y}_{ll'm}(\vec{n})]_\mu=\sum_{m'} C^{l',1,l}_{m',\mu,m} Y_{l'm'}(\vec{n}), \quad \mu=-1,0,+1,$$

in terms of the usual spherical harmonics $Y_{l'm'}$ and the Clebsch-Gordan coefficients. Since we used them extensively in the course of this work, we list below the expressions of the Cartesian components of $\vec{Y}_{ll'm}$ ($l'=l,l\pm 1$),

$$(\vec{Y}_{llm})_x=-\frac{c_1}{\sqrt{2}}Y_{l,m-1}+\frac{c_3}{\sqrt{2}}Y_{l,m+1},$$

$$(\vec{Y}_{llm})_y=-\frac{ic_1}{\sqrt{2}}Y_{l,m-1}-\frac{ic_3}{\sqrt{2}}Y_{l,m+1},$$

$$(\vec{Y}_{llm})_z=c_2 Y_{l,m},$$

where

$$c_1=-\frac{\sqrt{(l+m)(l-m+1)}}{\sqrt{l(2l+2)}},$$

$$c_2=\frac{m}{\sqrt{l(l+1)}},$$

$$c_3=\frac{\sqrt{(l-m)(l+m+1)}}{\sqrt{l(2l+2)}},$$

$$(\vec{Y}_{ll-1m})_x=-\frac{c_1}{\sqrt{2}}Y_{l-1,m-1}+\frac{c_3}{\sqrt{2}}Y_{l-1,m+1},$$

$$(\vec{Y}_{ll-1m})_y=-\frac{ic_1}{\sqrt{2}}Y_{l-1,m-1}-\frac{ic_3}{\sqrt{2}}Y_{l-1,m+1},$$

$$(\vec{Y}_{ll-1m})_z=c_2 Y_{l-1,m},$$

where

$$c_1=\frac{\sqrt{(l+m-1)(l+m)}}{\sqrt{2l(2l-1)}},$$

$$c_2=\frac{\sqrt{(l-m)(l+m)}}{\sqrt{l(2l-1)}},$$

$$c_3=\frac{\sqrt{(l-m-1)(l-m)}}{\sqrt{2l(2l-1)}}.$$

$$(\vec{Y}_{ll+1m})_x=-\frac{c_1}{\sqrt{2}}Y_{l+1,m-1}+\frac{c_3}{\sqrt{2}}Y_{l+1,m+1},$$

$$(\vec{Y}_{ll+1m})_y=-\frac{ic_1}{\sqrt{2}}Y_{l+1,m-1}-\frac{ic_3}{\sqrt{2}}Y_{l+1,m+1},$$

$$(\vec{Y}_{ll+1m})_z=c_2 Y_{l+1,m},$$

where

$$c_1=\frac{\sqrt{(l-m+1)(l-m+2)}}{\sqrt{(2l+2)(2l+3)}},$$

$$c_2=-\frac{\sqrt{(l+m+1)(l-m+1)}}{\sqrt{(l+1)(2l+3)}},$$

$$c_3=\frac{\sqrt{(l+m+1)(l+m+2)}}{\sqrt{(2l+2)(2l+3)}}.$$

In the calculation of the recoil force (Sec. VI) use has been made of the Cartesian components of $\vec{Y}_{ll'm}$ expressed with the aid of the $3j$ Wigner symbols, rather than Clebsch-Gordan coefficients as above, so we give below for convenience the equivalent relations as well:

$$(\vec{Y}_{llm})_x=(-1)^{l+m-1}\sqrt{2l+1}\times\left[-\frac{1}{\sqrt{2}}\begin{pmatrix} l & 1 & l \\ m-1 & 1 & -m\end{pmatrix}Y_{l,m-1}+\frac{1}{\sqrt{2}}\begin{pmatrix} l & 1 & l \\ m+1 & -1 & -m\end{pmatrix}Y_{l,m+1}\right],$$

$$(\vec{Y}_{llm})_y=(-1)^{l+m-1}\sqrt{2l+1}\times\left[-\frac{i}{\sqrt{2}}\begin{pmatrix} l & 1 & l\\ m-1 & 1 & -m\end{pmatrix}Y_{l,m-1}-\frac{i}{\sqrt{2}}\begin{pmatrix} l & 1 & l\\ m+1 & -1 & -m\end{pmatrix}Y_{l,m+1}\right],$$

$$(\vec{Y}_{llm})_z=(-1)^{l+m-1}\sqrt{2l+1}\begin{pmatrix} l & 1 & l\\ m & 0 & -m\end{pmatrix}Y_{l,m};$$

$$(\vec{Y}_{ll-1m})_x=(-1)^{l+m}\sqrt{2l+1}\times\left[-\frac{1}{\sqrt{2}}\begin{pmatrix} l-1 & 1 & l\\ m-1 & 1 & -m\end{pmatrix}Y_{l-1,m-1}+\frac{1}{\sqrt{2}}\begin{pmatrix} l-1 & 1 & l\\ m+1 & -1 & -m\end{pmatrix}Y_{l-1,m+1}\right],$$

$$(\vec{Y}_{ll-1m})_y=(-1)^{l+m}\sqrt{2l+1}\times\left[-\frac{i}{\sqrt{2}}\begin{pmatrix} l-1 & 1 & l\\ m-1 & 1 & -m\end{pmatrix}Y_{l-1,m-1}-\frac{i}{\sqrt{2}}\begin{pmatrix} l-1 & 1 & l\\ m+1 & -1 & -m\end{pmatrix}Y_{l-1,m+1}\right],$$

$$(\vec{Y}_{ll-1m})_z=(-1)^{l+m}\sqrt{2l+1}\begin{pmatrix} l-1 & 1 & l\\ m & 0 & -m\end{pmatrix}Y_{l-1,m};$$

$$(\vec{Y}_{ll+1m})_x=(-1)^{l+m}\sqrt{2l+1}\times\left[-\frac{1}{\sqrt{2}}\begin{pmatrix} l+1 & 1 & l\\ m-1 & 1 & -m\end{pmatrix}Y_{l+1,m-1}+\frac{1}{\sqrt{2}}\begin{pmatrix} l+1 & 1 & l\\ m+1 & -1 & -m\end{pmatrix}Y_{l+1,m+1}\right],$$

$$(\vec{Y}_{ll+1m})_y=(-1)^{l+m}\sqrt{2l+1}\times\left[-\frac{i}{\sqrt{2}}\begin{pmatrix} l+1 & 1 & l\\ m-1 & 1 & -m\end{pmatrix}Y_{l+1,m-1}-\frac{i}{\sqrt{2}}\begin{pmatrix} l+1 & 1 & l\\ m+1 & -1 & -m\end{pmatrix}Y_{l+1,m+1}\right],$$

$$(\vec{Y}_{ll+1m})_z=(-1)^{l+m}\sqrt{2l+1}\begin{pmatrix} l+1 & 1 & l\\ m & 0 & -m\end{pmatrix}Y_{l+1,m}.$$

We recall that the $3j$ symbols are related to the Clebsch-Gordan coefficients as follows [22]:

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}=(-1)^{j_1-j_2-m_3}\frac{1}{\sqrt{2j_3+1}}C^{j_1,j_2,j_3}_{m_1,m_2,-m_3}.$$

APPENDIX E: COMPONENTS OF $\vec{Y}_{ll'm}$ $(l'=l,l\pm1)$ IN SPHERICAL CORDINATES

In Appendix D, we have listed the Cartesian components of $\vec{Y}_{ll'm}$ $(l'=l,l\pm1)$ we have worked with. Here we shall give the components of $\vec{Y}_{ll'm}$ in the spherical coordinates (r, θ, φ) and, by using some recursion formulas, put them into a slightly different form that provides some relations between various such components with $(l'=l,l\pm1)$ that we have used in the calculations. One has in terms of Legendre associated functions $P_l^m(\cos\theta)$,

$$(\vec{Y}_{llm})_r=0,$$

$$(\vec{Y}_{llm})_\theta=\frac{(-1)^{m+1}}{2}\sqrt{\frac{(2l+1)(l-m)!}{4\pi l(l+1)(l+m)!}}\times[(l+m)(l-m+1)(\cos\theta)P_l^{m-1}(\cos\theta)+(\cos\theta)P_l^{m+1}(\cos\theta)+2m(\sin\theta)\times P_l^m(\cos\theta)]e^{im\varphi},$$

$$(\vec{Y}_{llm})_\varphi=\frac{i(-1)^{m-1}}{2}\sqrt{\frac{(2l+1)(l-m)!}{4\pi l(l+1)(l+m)!}}[(l+m)\times(l-m+1)P_l^{m-1}(\cos\theta)-P_l^{m+1}(\cos\theta)]e^{im\varphi};\tag{E1}$$

$$(\vec{Y}_{ll-1m})_r=\frac{(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi l(l+m)!}}[(l+m)(l+m-1)\times(\sin\theta)P_{l-1}^{m-1}(\cos\theta)-(\sin\theta)P_{l-1}^{m+1}(\cos\theta)+2(l+m)(\cos\theta)P_{l-1}^m(\cos\theta)]e^{im\varphi},$$

$$(\vec{Y}_{ll-1m})_\theta=\frac{(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi l(l+m)!}}[(l+m)(l+m-1)\times(\cos\theta)P_{l-1}^{m-1}(\cos\theta)-(\cos\theta)P_{l-1}^{m+1}(\cos\theta)-2(l+m)(\sin\theta)P_{l-1}^m(\cos\theta)]e^{im\varphi},$$

$$(\vec{Y}_{ll-1m})_\varphi=\frac{i(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi l(l+m)!}}[(l+m)\times(l+m-1)P_{l-1}^{m-1}(\cos\theta)+P_{l-1}^{m+1}(\cos\theta)]e^{im\varphi};\tag{E2}$$

$$(\vec{Y}_{ll+1m})_r=\frac{(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi(l+1)(l+m)!}}[(l-m+1)\times(l-m+2)(\sin\theta)P_{l+1}^{m-1}(\cos\theta)-(\sin\theta)P_{l+1}^{m+1}(\cos\theta)-2(l-m+1)\times(\cos\theta)P_{l+1}^m(\cos\theta)]e^{im\varphi},$$

$$(\vec{Y}_{ll+1m})_\theta=\frac{(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi(l+1)(l+m)!}}[(l-m+1)\times(l-m+2)(\cos\theta)P_{l+1}^{m-1}(\cos\theta)-(\cos\theta)P_{l+1}^{m+1}(\cos\theta)+2(l-m+1)\times(\sin\theta)P_{l+1}^{m}(\cos\theta)]e^{im\varphi},$$

$$(\vec{Y}_{ll+1m})_\varphi=\frac{i(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi(l+1)(l+m)!}}[(l-m+1)\times(l-m+2)P_{l+1}^{m-1}(\cos\theta)+P_{l+1}^{m+1}(\cos\theta)]e^{im\varphi}. \tag{E3}$$

Note that our P_l^m is $(-1)^m$ the P_l^m from [23]. Using the following recursion formulas for the Legendre associated functions $P_\nu^\mu(x)$,

$$P_\nu^{\mu+2}(x)-2(\mu+1)\frac{x}{\sqrt{1-x^2}}P_\nu^{\mu+1}(x)+(\nu-\mu)\times(\nu+\mu+1)P_\nu^\mu(x)=0,$$

$$xP_\nu^\mu(x)-P_{\nu+1}^\mu(x)+(\nu+\mu)\sqrt{1-x^2}P_\nu^{\mu-1}(x)=0,$$

$$(2\nu+1)xP_\nu^\mu(x)=(\nu-\mu+1)P_{\nu+1}^\mu(x)+(\nu+\mu)P_{\nu-1}^\mu(x),$$

$$\sqrt{1-x^2}P_\nu^{\mu+1}(x)=\frac{(\nu+\mu)(\nu+\mu+1)}{(2\nu+1)}P_{\nu-1}^\mu(x)-\frac{(\nu-\mu)(\nu-\mu+1)}{(2\nu+1)}P_{\nu+1}^\mu(x),$$

$$\sqrt{1-x^2}P_\nu^{\mu-1}(x)=\frac{1}{(2\nu+1)}P_{\nu+1}^\mu(x)-\frac{1}{(2\nu+1)}P_{\nu-1}^\mu(x),$$

one can bring Eqs. (E1)–(E3) to the useful form

$$(\vec{Y}_{llm})_r=0,$$

$$(\vec{Y}_{llm})_\theta=\frac{(-1)^{m-1}}{2}\sqrt{\frac{(l-m)!}{4\pi l(l+1)(2l+1)(l+m)!}}\times\{l[(l-m+1)(l-m+2)P_{l+1}^{m-1}(\cos\theta)+P_{l+1}^{m+1}(\cos\theta)]+(l+1)[(l+m)\times(l+m-1)P_{l-1}^{m-1}(\cos\theta)+P_{l-1}^{m+1}(\cos\theta)]\}e^{im\varphi},$$

$$(\vec{Y}_{llm})_\varphi=\frac{i(-1)^{m-1}}{2}\sqrt{\frac{(2l+1)(l-m)!}{4\pi l(l+1)(l+m)!}}[(l+m)\times(l-m+1)P_l^{m-1}(\cos\theta)-P_l^{m+1}(\cos\theta)]e^{im\varphi}; \tag{E4}$$

$$(\vec{Y}_{ll-1m})_r=(-1)^m\sqrt{\frac{l(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\varphi},$$

$$(\vec{Y}_{ll-1m})_\theta=\frac{(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi l(l+m)!}}[-(l+m)\times(l-m+1)P_l^{m-1}(\cos\theta)+P_l^{m+1}(\cos\theta)]e^{im\varphi},$$

$$(\vec{Y}_{ll-1m})_\varphi=\frac{i(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi l(l+m)!}}[(l+m)\times(l+m-1)P_{l-1}^{m-1}(\cos\theta)+P_{l-1}^{m+1}(\cos\theta)]e^{im\varphi}; \tag{E5}$$

$$(\vec{Y}_{ll+1m})_r=-(-1)^m\sqrt{\frac{(l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\varphi},$$

$$(\vec{Y}_{ll+1m})_\theta=\frac{(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi(l+1)(l+m)!}}[(l-m+1)\times(l+m)P_l^{m-1}(\cos\theta)-P_l^{m+1}(\cos\theta)]e^{im\varphi},$$

$$(\vec{Y}_{ll+1m})_\varphi=\frac{i(-1)^m}{2}\sqrt{\frac{(l-m)!}{4\pi(l+1)(l+m)!}}[(l-m+1)\times(l-m+2)P_{l+1}^{m-1}(\cos\theta)+P_{l+1}^{m+1}(\cos\theta)]e^{im\varphi}. \tag{E6}$$

From Eqs. (E4)–(E6) one can immediately derive relations of the type

$$(\vec{Y}_{llm})_\theta=\frac{i}{\sqrt{2l+1}}[\sqrt{l}(\vec{Y}_{ll+1m})_\varphi+\sqrt{1+1}(\vec{Y}_{ll-1m})_\varphi],$$

$$(\vec{Y}_{ll-1m})_\theta=i\sqrt{\frac{l+1}{2l+1}}(\vec{Y}_{llm})_\varphi,$$

$$(\vec{Y}_{ll+1m})_\theta=i\sqrt{\frac{l}{2l+1}}(\vec{Y}_{llm})_\varphi,$$

$$-\sqrt{l}(\vec{Y}_{ll-1m})_\varphi+\sqrt{l+1}(\vec{Y}_{ll+1m})_\varphi=0, \tag{E7}$$

which we have used in the calculations.

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