Dynamics of self-trapped singular beams in an underdense plasma

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Dynamics of an intense short laser pulse with a phase singularity, propagating in an underdense cold plasma, is investigated. Such a pulse can propagate as a vortex soliton in a self-created channel. It is shown that vortices with the topological charge $m=1,2$ (and a corresponding angular momentum) are unstable against symmetry-breaking perturbations; the breakup of the original vortex leads to the formation of stable spatial solitons that steadily fly away tangentially from the initial ring of vortex distribution.

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I. INTRODUCTION

The dynamics of laser beams with phase singularities (or wave-front dislocations) is being intensely studied [1] for its possible application to all-optical signal processing and logic. The branch-point phase singularities in electromagnetic (em) field, discovered by Nye and Berry [2], appear whenever both the real and imaginary parts of the field become zero, i.e., at the singularity the phase becomes indeterminate while the field amplitude is strictly zero. The phase singularity associated with a screw dislocation (a vortex characterized by a helical wave front) is particularly interesting because the corresponding beams can be readily generated by available techniques [3]. The singular beams demonstrate unusual properties even in free-space propagation [4]; the vortices are robust with respect to perturbations. Since the laser fields with screw phase singularities carry angular momentum, these can rotate dielectric particles trapped in the vortex core [5]. The dynamics of the singular beams in nonlinear optical media is usually described by a generalized nonlinear Schrödinger equation (NSE) with a local nonlinearity. The NSE with a defocusing nonlinearity admits stable vortex soliton solutions, which are the most fundamental two-dimensional soliton solutions of NSE with an angular 2π phase ramp, and appear as local dark minima in an otherwise bright background. The vortex soliton solutions of NSE, first suggested by Pitaevskii [6] as topological excitations in an imperfect Bose gas in the superfluids, excite considerable interest because of the recent experimental demonstrations of Bose-Einstein condensation of dilute gases in traps [7]. Thus, the laser beam with nested vortices can also be used to model the dynamical behavior of vortices in the Bose gases. Self-focusing media also support localized soliton solutions with phase dislocations surrounded by one or many bright rings. These solutions are unstable against symmetry-breaking perturbations that can cause the breakup of rings into filaments. In saturating nonlinear media these filaments form stable bright solitons, which, similar to free Newtonian particles, fly off tangentially to the initial rings conserving total angular momentum [8].

To the best of our knowledge the nonlinear dynamics of singular beams in plasmas has not yet been seriously investigated, although there are several studies reported for optical media [9]. The dynamics of ultrastrong laser pulses in plasma, however, represents an important area of research both from fundamental physics point of view as well as for the practical realization of novel concepts such as particle acceleration and the "fast ignitor" projects [10]. It should be realized that the nonsingular solutions form a very limited set of the total solutions available to the NSE, for example. An investigation of the broad class of angular momentum carrying solutions is essential to the full exploitation of the possibilities contained in the model. The importance of this program is further enhanced *a posteriori* by the discovery that the singular beams turn out to be rather robust and controllable.

II. SELF-TRAPPED STATIONARY BEAMS AND THEIR STABILITY

In the present work we deal with the propagation of a relativistically strong singular laser beam (carrying a screw type of dislocation) in an underdense cold plasma. Our primary goal is to examine the possibility of beam self-trapping, and of the formation of two-dimensional stable solitonic structures. Though the interaction of relativistically strong lasers with plasmas is relatively complex, it is possible under certain conditions (Ref. [11]) to describe it by a generic NSE with both local and nonlocal nonlinearities. The pulse duration (T_l) is chosen to be short enough that ions do not respond, and long enough that Langmuir waves are not excited [$\omega_i^{-1} \gg T_l \gg \omega_e^{-1}$]. Then the dynamics of a beam with a narrow cross section [$L_\parallel (\sim cT_l) \gg L_\perp$, where $L_\perp$ and $L_\parallel$ are, respectively, the characteristic transverse and longitudinal spatial dimensions of the beam] is governed, in the comoving variables ($\tau = t - z/v_g$ and $z' = z$), by the generalized NSE, which in dimensionless form reads

$$i\frac{\partial A}{\partial z} + \nabla_\perp^2 A + \left(1 - \frac{n}{\gamma}\right) A = 0. \qquad (1)$$

Here A, normalized to $m_e c^2/e$, is the slowly varying amplitude of the vector potential of a circularly polarized laser beam: $\mathbf{A} = \frac{1}{2}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})A \exp[i(kz - \omega t)] + \text{c.c.}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors, ω and k are, respectively, the mean frequency and wave number of the laser beam, $\gamma = \sqrt{1 + |A|^2}$ is the

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relativistic factor associated with the high-frequency electron motion. The transverse lengths are measured in units of c/ω_e while the length in the propagation direction z is normalized by $\omega v_g/2\omega_e^2$, where v_g is the group velocity $v_g = kc^2/\omega = c(1-\omega_e^2/\omega^2)^{1/2}$ ($\approx c$ if $\omega \gg \omega_e$, i.e., in the underdense plasma). The operator $\nabla_\perp^2$ is the Laplacian in the (x,y) plane. The electron density n is normalized by the equilibrium density n_0,

$$n = 1 + \nabla_\perp^2 \gamma. \tag{2}$$

Two effects that contribute to the nonlinearity in Eq. (1) are (i) the electron mass increase in the field of strong laser radiation and (ii) the electron density variation caused by the high-frequency pressure of the field. For the latter effect, Eq. (2) indicates that at high intensity or for strongly localized (in x-y plane) beams the electron density may become zero or even negative. Zero density or the complete expulsion of electrons (electron cavitation) is possible in the region of the strong field because the charge separation electric field cannot oppose the ponderomotive force of the laser field. In the current model (widely exploited for the problem of relativistic self-focusing of the laser beams), however, the occurrence of nonphysical negative values for the electron density cannot be prevented. This failure of the cold hydrodynamical plasma model [used for deriving Eqs. (1) and (2)] is generally corrected by putting $n=0$ in the entire spatial region where $n<0$ [11,12]. In this paper, for the vortex soliton solution described below we consider the domain of parameters when the condition $(1+\nabla_\perp^2\gamma)>0$ holds, i.e., the electron cavitation does not take place.

Let us now consider the solitary wave solutions carrying vortices. On assuming that solutions in polar coordinates are of the form $A = \hat{A}(r)\exp(im\theta + i\beta z)$, where $r = \sqrt{x^2+y^2}$ and θ is the polar angle, Eqs. (1) and (2) reduce to an ordinary differential equation for the real valued amplitude $\hat{A}$,

$$\frac{d^2\hat{A}}{dr^2} + \frac{1}{r}\frac{d\hat{A}}{dr} - \beta\hat{A} - \frac{m^2}{r^2}\hat{A} + \hat{A}\left(1 - \frac{n}{\sqrt{1+\hat{A}^2}}\right) = 0, \tag{3}$$

where, for $n>0$, we have

$$n = 1 + \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\sqrt{1+\hat{A}^2}. \tag{4}$$

Here β is a propagation constant and $m(\neq 0)$ is an integer known as the topological charge of the vortex. Note that the total phase of the field is $\psi = (k+\beta)z - \omega t + m\theta$. The wave front forms a helicoidal surface given by $m\theta + (k+\beta)z = \text{const}$, rotating in time with angular frequency ω/m, thus, justifying the terminology—the screw-type dislocation.

We are looking for localized solutions of Eq. (3) with boundary conditions $\hat{A} \to 0$ for $r \to (0,\infty)$. The localized solution of Eq. (3) can exist if $\beta > 0$ with the following asymptotic behavior: $\hat{A}_{r\to 0} \to A_0 r^{|m|}$ and $\hat{A}_{r\to\infty} \to A_\infty \exp(-r\sqrt{\beta})/\sqrt{r}$, where A_0 and A_∞ are constants.

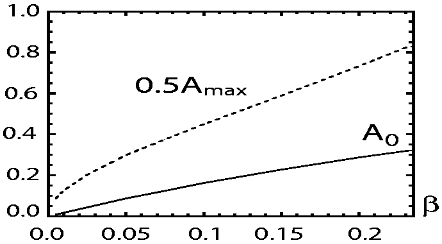

FIG. 1. The spatial derivative of the field at the origin A_0 and the normalized amplitude $A_m/2$ versus the propagation constant β for $m=1$.

Multiplying Eq. (3) by $\hat{A}^*$, and making simple manipulations, we derive the inequality

$$-\int d\mathbf{r}|\boldsymbol{\nabla}_\perp A|^2 > [\beta - \max(1 - n/\sqrt{1+|A|^2})]N, \tag{5}$$

where $N = \int d\mathbf{r}|A|^2$ is the so-called "photon number" or the beam power. Taking into account Eq. (2) and the condition that $n>0$ one can conclude that the propagation constant or the eigenvalue $\beta<1$; it lies between zero and unity.

We have used numerical methods to determine the localized solutions of Eq. (3). It is possible to map the equation in the $(\hat{A},\hat{A}_r)$ plane (phase plane), and show (through the analogy between the resulting equation with that of the nonconservative motion) that for the eigenvalue lying in the range $0<\beta<1$, Eq. (3) admits an infinity of discrete bound states $\hat{A}_j(r)$ $(j=1,2,\dots)$ where j denotes the finite r zeros of the eigenfunction. In what follows we consider only the lowest-order (lowest radial eigenmode) solution of Eq. (3) $(j=1)$. For nonzero m, the "ground state" solution is positive, has a node at the origin $r=0$, reaches a maximum, and then monotonically decreases with increasing r. A shooting code was employed to numerically solve Eq. (3) for $m=1$. The results of the simulation are shown in Fig. 1 where we display A_0 (the measure of the slope of $\hat{A}$ at the origin) and the field amplitude A_m (normalized for convenience to 2) versus the propagation constant β. One can see that for $\beta\to 0$ the solitary wave amplitude is nonrelativistic, but quickly becomes ultrarelativistic ($A_m>1$) as β becomes larger. In Fig. 2(a) we display a typical eigenmode $\hat{A}(r)$ along with the corresponding electron density $n(r)$. In this plot the propagation constant $\beta=0.1$. Thus, the radial profile of the stationary mode corresponds to a bright ring of field, which may be viewed as the "compact support" of the vortex. As the propagation constant increases the field amplitude becomes larger, and for $\beta>\beta_c\approx 0.24$ the electron cavitation takes place. Thus, noncavitating vortex solitary waves exist in the range $0<\beta<\beta_c$. The plot in Fig. 2(b), similar to Fig. 2(a), but for $\beta=0.2$ clearly shows that the electron density is considerably reduced in the large amplitude region a little away from the vortex core.

The laser beam power N trapped in the solitary mode is a fast growing function of β $(dN/d\beta>0)$. For $\beta\to 0$, $N\to N_{cr}\approx 97.5$. Thus, the necessary condition to form the vortex soliton is that the laser beam power must exceed N_{cr},

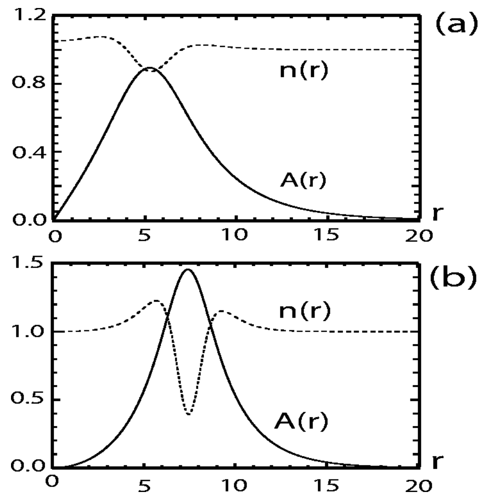

FIG. 2. Stationary vortex solutions for $\beta=0.1$ (a) and for $\beta=0.2$ (b).

which for the $m=1$ case is a few times larger than the critical power $N_{cr}^0=23.4$ needed for the fundamental $m=0$ solitons [11] that do not carry any topological charge. Note that the critical power for the fundamental soliton in dimensional units reads as $N_{cr}^0\approx1.6\times10^{10}(\omega/\omega_e)^2$ W. The higher power requirement is quite understandable, because the confining power has to contend against the centrifugal barrier present for all solitons carrying angular momentum.

Are these solitonlike solutions stable? We begin answering this question by first performing a linear stability analysis. The total electric field amplitude of the laser beam may be split as

$$A=[\hat{A}(r)+a_+(z,r)\exp(il\theta)+a_-(z,r)\exp(-il\theta)]\times\exp(i\beta z+im\theta), \tag{6}$$

where the complex valued amplitude of the perturbation is small $\hat{A}\gg|a_\pm(z,r)|$, and l is an integer. Substituting Eq. (6) into Eqs. (1) and (2) and linearizing, we obtain two coupled differential equations in terms of a_+ and a_-,

$$\left[\frac{\partial}{\partial z}+i\frac{1}{r}\frac{\partial}{\partial r}\left(rM\frac{\partial}{\partial r}\right)+iR-i\frac{(m\pm l)^2}{r^2}\right]a_\pm=-i\left[\frac{1}{r}\frac{\partial}{\partial r}\left((M-1)\frac{\partial}{\partial r}\right)+W\right]a_\mp^*. \tag{7}$$

Here $M=(1-\hat{A}^2/2\gamma_0^2)$, $\gamma_0=(1+\hat{A}^2)^{1/2}$, and the operators W and R are defined by

$$W=\frac{1}{2}\left[\frac{\hat{A}^2}{\gamma_0^3}+\frac{\hat{A}^2\Delta_{rr}\gamma_0}{\gamma_0^3}-\frac{\hat{A}}{\gamma_0}\Delta_{rr}\left(\frac{\hat{A}}{\gamma_0}\right)+\frac{\hat{A}^2l^2}{\gamma_0^2r^2}\right], \tag{8}$$

$$R=\left[1-\frac{1}{\gamma_0}-\frac{\Delta_{rr}\gamma_0}{\gamma_0}\right]+W, \tag{9}$$

with $\Delta_{rr}=(1/r)[\partial_r(r\partial_r)]$ introduced for convenience.

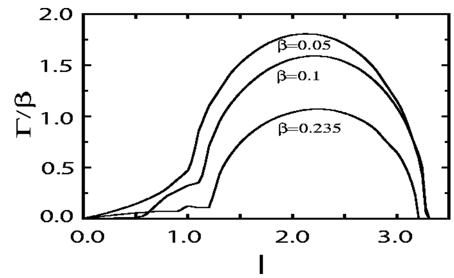

FIG. 3. Normalized growth rate Γ/β as a function of l for different values of propagation constant β and $m=1$.

The preceding complicated set of equations is solved numerically using standard procedure (see, for instance, Ref. [8]). The functions $a_\pm$ are chosen to vanish at $r=0$ and $r\to\infty$. After we integrate Eqs. (7) along z until the perturbation growth rate $\Gamma=||a||^{-1}\partial||a||/\partial z$ [where $||a||=\int_0^r dr(|a_+|^2+|a_-|^2)$] becomes stationary.

Though l appearing in Eqs. (7) is an integer, we will treat it as an arbitrary positive number as was done in Ref. [8]. In Fig. 3 we plot the normalized growth rate Γ/β as a function of l for $m=1$ and for different values of the propagation constant. We further follow Ref. [8] in averaging the growth rate over z for small l for which it tends to be oscillatory. One can see that the growth rates vanish for $l=0$ and attain their maximum value somewhere between $l=2$ and $l=3$. The vortex solitons are, thus, stable against radial perturbation while they are unstable against azimuthal, symmetry-breaking perturbation. We also find that for larger propagation constant instability development is slower. It is natural to expect that azimuthal modulation instability will break the solitons into several filaments.

Subsequently, highly nonlinear stage of instability can be studied by direct simulations of Eqs. (1) and (2). Some insight into the problem can, however, be gained by analyzing the integrals of motion of the system. It has been shown in Ref. [12] that for $n>0$, the system, in addition to the photon number, conserves the Hamiltonian as

$$H=\int[|\nabla_\perp A|^2-(\gamma-1)^2-(\nabla_\perp\gamma)^2]d\mathbf{r}_\perp. \tag{10}$$

Any initial field distributions must conserve these integrals during their evolution. From the work of Zakharov *et al.* [13], it follows that if the Hamiltonian is negative there is no diffraction, since the maximum value of the field intensity has a (z-independent) lower bound $|A|^2_{\max}>4|H|/N$. For $H<0$ the laser beam can be trapped in a self-created waveguide. Due to the "saturating character" of nonlinearity (related to the possibility of electron cavitation as well as to the fact that at high intensity the electrons become heavier and cannot respond to the field) the development of wave collapse is prevented in this model [12].

For the laser field that carries a nonzero topological charge, the system of Eqs. (1) and (2) admits one more, important integral of motion, the angular momentum

$$M=\frac{i}{2}\int d\mathbf{r}_\perp\left[x\left(A^*\frac{\partial A}{\partial y}-\text{c.c.}\right)-y\left(A^*\frac{\partial A}{\partial x}-\text{c.c.}\right)\right]. \tag{11}$$

Equation (11) for the angular momentum is the paraxial approximation for the orbital angular momentum $M=\int d\mathbf{r}_\perp[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]_z$, of the laser field. One can easily show that for steady state solutions, the angular momentum is determined by the topological charge and the photon number, $M=mN$.

For the solitonic solutions of Eq. (3) the Hamiltonian is negative for the entire allowed range of the propagation constant ($0<\beta<\beta_c$) insuring that the solitons are in the self-trapped regime. Consequently, neither the azimuthal nor the radial modulation instability can lead to either diffraction or collapse. However, the modulation instability may cause the beam to break into multiple filaments. In order to conserve the total angular momentum $M=mN$, these filaments (forbidden to fuse or join together for topological reasons) can eventually spiral about each other or fly off tangentially to the initial ring generating bright solitonic structures similar to what was found for the index saturation nonlinearity [8]. Our time-dependent numerical simulations give evidence of a quickly developing instability. We solve the system of Eqs. (1) and (2) using finite difference methods. Initial input is the solution of Eq. (3) for different $\beta<\beta_c$. This choice guarantees that the initial profile is not augmented by electron cavitation. The beam breaks into filaments running away tangentially without spiraling. Thus, the mutual interaction between filaments does not lead to their stable spiraling; the radial force in the present problem is repulsive (at the center the field is always zero, and the effective refractive index of the plasma is minimal). This is to be contrasted with the stable spiraling reported in Ref. [14] where the effective force is attractive. During evolution the amplitude of the filaments grows till cavitation takes place in the regions of filament localization. To prevent the appearance of negative density we, following Refs. [11,12,15], replace n by zero.

Subsequently, the filaments steadily go apart from one another. All filaments carry zero topological charge and remain stable. In Figs. 4(a1)–4(c1) we present the results of a typical simulations for which the initial condition corresponds to the stationary solution for $\beta=0.1$. The weakly relativistic initial field with $A_m\approx0.8$, grows as it propagates to ultrarelativistic filament amplitudes ($A_m\approx1.5$). Note that the splitting takes place about $z_s\approx60$ while the corresponding diffraction length is $z_D\approx(\Delta r_\perp)^2\approx\beta^{-1}=10$ yielding the ratio $z_s/z_D\approx6$, that is, the vortex solitary wave decays after covering six diffraction lengths. However, for larger β (for relativistic amplitudes of the vortex solitary wave) the splitting distance can be larger, for instance, if $\beta\approx0.2$ the ratio $z_s/z_D\approx13$. Thus, for a high amplitude solution, the instability develops slowly allowing a lifetime spanning tens of its periods. This fact can be explained by the considerable reduction of plasma density at the solitary wave's field maximum. Indeed, we can see from Fig. 2(b) that in the spatial region where the field maximum is localized, $n\to0$; the determining equation (1) becomes linear, thus, partially suppressing the development of the azimuthal instability. Note also that due to the mutual attraction of the filaments, the complete breakup of the vortex soliton is remarkably delayed in comparison with the estimates, which follows from the results of linear stability analysis.

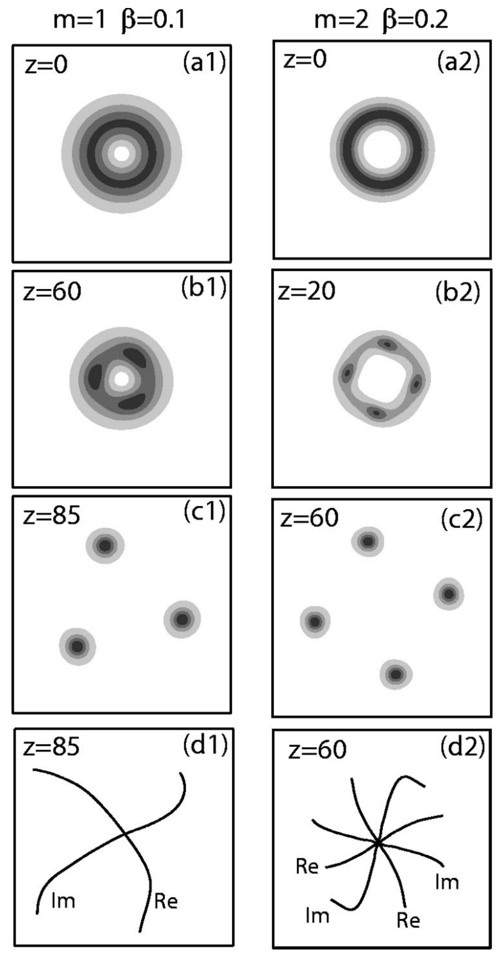

FIG. 4. Contours of constant intensity (a)–(c), and the lines on which Re A and Im A are zero (d) for $m=1$ and $m=2$ structures. In (d1),(d2), the region around $r=0$ is expanded. The topological singularity lies where the lines intersect ($r=0$). The topological charge is determined by the number of intersecting line pairs.

Subsequently dynamics of the filaments is stabilized by cavitation. However, our model equations (1) and (2) do not conserve all of the integrals of motion after $z>z_c$, where z_c is the distance when cavitation takes place in any of the filaments. Note that $z_c>z_s$; for the case presented in Fig. 4 ($m=1,\beta=0.1$), $z_c\approx78$. Our numerical methods faithfully reproduce the invariants of the differential equation, the photon number, the momentum, and the Hamiltonian before cavitation and only the first two after cavitation.

It is expected that thermal effects will eliminate the nonphysical region associated with negative density. We are in the process of deriving and solving the system that will be valid for relativistically high temperatures. The qualitative nature of the results presented in this paper, however, is not likely to change (see, for instance, Ref. [16]).

We would like to emphasize that the vortices have a topological sense as the branch points, where both the real and imaginary parts of the field become strictly zero, and the topological charge represents the number of intersecting

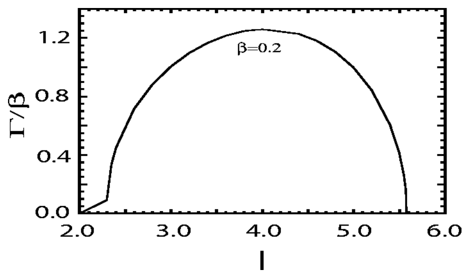

FIG. 5. The growth rate versus l for $m=2$ and $\beta=0.2$.

pairs of zero lines of the real and imaginary parts of the field A [17]. Applying the Madelung transformation $A=\sqrt{\rho}\exp(i\psi)$ to Eq. (1) we can convert it to a set of fluid hydrodynamic equations with a fluid "density" ρ and fluid "velocity" $\mathbf{v}=\boldsymbol{\nabla}_{\perp}\psi$. According to Kelvin's theorem the velocity circulation is conserved along the closed path moving with the local fluid velocity. The velocity circulation $\oint \mathbf{v}d\mathbf{l}=2\pi m$, where the integration path encloses the branch point. As a consequence of the topological conservation of the frozen-in law, the vortex nested in the EM beam cannot disappear even when the EM beam undergoes a structural change. [This simple argument is valid despite the concrete (but physical) model equations describing the plasma density response.] To illustrate this statement we plot in Fig. 4(d1) the zero lines of the EM field distribution at $z=85$ after the field has undergone splitting. One can see that the zero lines of the field's real and imaginary parts (one pair for $m=1$) still intersect at the center implying that the vortex has survived structural changes of the field, and remains at the center of structure.

Though in this paper we mainly concentrated on the EM singular beam with $m=1$, it is natural to expect that the EM beam with higher topological charge will exhibit similar properties. Indeed in Figs. 4(a2)–4(c2) we present the breaking of the EM vortex solitary wave with $m=2$. The beam intensity distribution, indeed follows a path similar to that of $m=1$. In Fig. 5 we plot the instability growth rate versus l for the vortex soliton with $m=2$ and $\beta=0.2$. The growth rate reaches its maximum at $l=4$. Thus, the beam breaks up, in accordance with the linear stability analysis into four filaments that run away tangentially to the initial field-intensity distribution ring. At a certain stage, the collapse of the field into filaments is stabilized by electron cavitation, and the formation of steady moving solitonic structures take place. The vortex position still remains unaffected [see Fig. 4(d2)]. We would like to inform the reader that our linear stability analysis leaves out the possibility of an algebraic (secular) instability. We have, however, tested the topologically allowed possibility of the multicharged vortices breaking into single charged ones; in our simulations we did not observe any such breakup. We believe that, though the multicharged vortex is topologically unstable, the breaking up may be a very slow process and special care will have to be taken to observe this effect in practice.

III. CONCLUSION

In conclusion, we have shown that the azimuthal perturbations that break up a moderately relativistic vortex solitary wave become considerably less menacing for the waves that attain ultrarelativistic amplitudes; the cavitation (expulsion of electrons) in the high field region strongly suppresses the instability growth rate allowing the localized structures to live for many periods. The dynamics of the filaments is ruled by the conservation of angular momentum (topological charge) forcing the filaments to form stable bright solitons, which fly off tangentially to the initial rings conserving total angular momentum. The vortex, nested in the beam, appears to be robust and survives and can reach the target as an exact zero of the field-intensity distribution even though the beam itself undergoes dramatic structural changes.

It seems reasonable to conclude that, in contrast to the nonsingular beams, the filament dynamics for singular beams with nonzero topological charge (angular momentum) is much more robust and predictable. These moderately stable, long-lived, and compact bundles of ultrahigh electromagnetic fields must be seriously considered in delineating the dynamics of plasmas in ultrastrong laser fields, for instance, in the laser target interaction experiments. In particular, the singular beam can be used to produce multiple localized beams with controllable positions at the target.

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