Probabilistic formalism and hierarchy of models for polydispersed turbulent two-phase flows

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This paper deals with a probabilistic approach to polydispersed turbulent two-phase flows following the suggestions of Pozorski and Minier [Phys. Rev. E **59**, 855 (1999)]. A general probabilistic formalism is presented in the form of a two-point Lagrangian PDF (probability density function). A new feature of the present approach is that both phases, the fluid as well as the particles, are included in the PDF description. It is demonstrated how the formalism can be used to show that there exists a hierarchy between the classical approaches such as the Eulerian and Lagrangian methods. It is also shown that the Eulerian and Lagrangian models can be obtained in a systematic way from the PDF formalism. Connections with previous papers are discussed.

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I. INTRODUCTION

Polydispersed turbulent two-phase flows are ubiquitous in many industrial processes and natural phenomena. In these flows, a discrete phase in the form of inclusions is embedded in a turbulent fluid. The turbulent fluid is referred to as the continuous phase and the inclusions, or discrete particles, form the so-called discrete phase. These types of flows involve many aspects of physics at different scales and one may have to use simultaneously several domains such as turbulence [1], particle dispersion [2], granular matter [3], combustion and so on, to understand the basic mechanisms that come into play. There is, therefore, a real challenge to take up when one attempts to model such flows and to simulate them with modern computer technology.

The challenge might appear, at the first glance, as a pure computational one since the equations describing the dynamics of the system are known. One could solve, as in the spirit of direct numerical simulation (DNS) [4], the Navier-Stokes equations and consider the particles as moving boundaries [5]. The force exerted on each particle would be given by the surface integral of the fluid stress tensor. In practice, such an approach is not feasible since a fluid in turbulent motion has a huge number of degrees of freedom [6,7], not to mention the number of moving boundaries. Therefore, the challenge is to come up with a contracted description (a simplified model) in order to express the problem in the form of equations that contain the main physical aspects while still being tractable with modern computer technology.

Nowadays, two methods are widely used for practical numerical simulations of polydispersed turbulent two-phase flows. The *Eulerian approach*, or two-fluid model, where mean field equations are derived for both phases and the *Lagrangian approach*, or particle-tracking method, where mean field equations are solely used for the continuous phase whereas particles are tracked individually by using a set of equations describing their dynamical behavior. In the Lagrangian approach, one usually tracks stochastic particles which, hopefully, reproduce the same statistics as the real ones, i.e., real particles are replaced by stochastic particles where the time evolution of the variables of interest is described by SDEs (stochastic differential equations). The Eulerian and Lagrangian methods only differ by the level of information that is retained for the description of the discrete particles. In the Eulerian model, the discrete phase is modeled at the macroscopic level (mean field equations) whereas for the Lagrangian approach modeling is performed at a mesoscopic level (SDEs). The mesoscopic description is an intermediate level between the macroscopic description (mean field equations) and the microscopic description (exact local instantaneous equations).

It is worth emphasizing that, in both the classical Eulerian and Lagrangian methods, the fluid or continuous phase remains modeled at the macroscopic level using mean field equations. There exist, however, alternatives for the simulation of the fluid (single-phase flows) which are particularly interesting when complex physics is involved, for example, compressible reactive turbulent flows. In such flows, the classical problem of writing closure laws directly at the macroscopic level can be avoided by turning to PDF (probability density function) models that simulate explicitly local instantaneous variables [8]. In practice, PDF models appear as a good compromise between the level of information that is provided and the computational effort that is required [9]. In these methods, which are middle-of-the-road approaches between the microscopic (local instantaneous equations) and macroscopic (mean field equations) descriptions, the aim is to model and simulate the one-point PDF of the random variables that are of interest (mesoscopic description). By further contraction, one can then retrieve the mean field

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equations for single-phase flows [10].

In the present paper, the objective is to combine both the PDF approach to turbulent single-phase flows and the Lagrangian approach in order to propose a complete PDF approach to polydispersed turbulent two-phase flows. The aim of the paper is not to present new models but to introduce a formalism that contains the description of both phases. Attention is focused on a two-point PDF (one fluid point and one particle point) where one simulates the joint PDF at two different points for the variables of interest both for the fluid and for the particles. Once again, the new feature is that the present PDF description includes the two phases, that is, the fluid *and* the particle phases. Furthermore, it is shown that the present probabilistic approach is useful to highlight several points:

(i) The derivation of mean field equations: there exists a vast literature in this field and it is explained that, in the frame of the present formalism, the mean field equations are derived in a natural way.

(ii) The hierarchy between the different models: two-point PDF model, Lagrangian model, and Eulerian model.

(iii) The derivation of a closed set of mean field equations in a simplified case, the formalism is used to emphasize the level of simplification that is required by the macroscopic closures.

(iv) The connections between the present approach and previous work.

Consequently, the purpose of the present paper is not to validate or discuss the models by comparing numerical computations with experimental data. Some references to numerical computations obtained with the different approaches are, however, indicated at the end of the paper.

The paper is organized as follows. In Sec. II, the needed mathematical tools are recalled, especially the link between the trajectory and the PDF points of view for diffusion processes. Then, a probabilistic description of polydispersed turbulent two-phase flows is given in Sec. III in terms of a two-point PDF and the equivalent trajectories. After that, is it shown in Sec. IV how the corresponding closed Fokker-Planck equation is written and the mean field equations, i.e., the Eulerian model, can be derived. In Sec. V, the Lagrangian model is displayed and the hierarchy between the Eulerian and Lagrangian approaches is explained. In Sec. VI, practical trajectory models are introduced and from them, an example of a closed set of mean field equations is given in a simplified case. Finally, before concluding, connections between the present formalism and previous work are explained in Sec. VII.

II. GENERAL FORMALISM

The problem is treated with a terminology coming from the classical *N-body problem*. Let us consider an ensemble of N_f fluid particles and N_p inclusions to which p_f and p_p variables are attached, respectively. A fluid particle is defined as a small element of fluid whose characteristic length scale is much larger than the molecular mean free path but much smaller than the Kolmogorov length scale [11]. The fluid particle has a mass m, a volume V, and a velocity that equals the velocity field at the location of the particle. The dimension of the system is $d=p_fN_f+p_pN_p$. As mentioned in the Introduction, only two-point information (one fluid point and one particle point) is under investigation so that the dimension of the system is contracted to $d=p_f+p_p$. It is now assumed (see Minier and Peirano [12] for a specification of the mathematical and physical background) that the issue of modeling polydispersed turbulent two-phase flows can be successfully addressed by using stochastic diffusion processes [13] in order to mimic the evolution in time of the variables describing the physics of the flow (i.e., the p_f+p_p variables attached to a pair of particles, one fluid and one discrete particle).

When dealing with a stochastic process, there are two ways to characterize it: the time-evolution equation of the trajectories of the process or the equation satisfied in sample space by its PDF. This correspondence is particularly clear for a diffusion process and is central in the present paper. If $\mathbf{Z}(t)=(Z_1,\ldots,Z_n)$ is a diffusion process with a drift vector $\mathbf{A}=A_i$ and a diffusion matrix $\mathbf{B}=B_{ij}$, the trajectories of the process are solutions of the following SDE:

$$dZ_i(t)=A_i(t,\mathbf{Z}(t))dt+B_{ij}(t,\mathbf{Z}(t))dW_j(t), \tag{1}$$

where $\mathbf{W}(t)=(W_1,\ldots,W_n)$ is a set of independent Wiener processes [13] and $\mathbf{Z}(t)$ is the state vector (the vector containing the p_f+p_p variables). The SDEs are called *Langevin equations* in the physical literature [14]. This corresponds in sample space to the Fokker-Planck equation for the transitional PDF $p(t;\mathbf{z}|t_0;\mathbf{z}_0)$ (this equation is also verified by the PDF $p(t;\mathbf{z})$, [13]),

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial z_i}[A_i(t,\mathbf{z})p]+\frac{1}{2}\frac{\partial^2}{\partial z_i\partial z_j}[D_{ij}(t,\mathbf{z})p]. \tag{2}$$

Actually, the correspondence between the two points of view is not a strict equivalence. Indeed, the matrix $\mathbf{D}$ that enters the Fokker-Planck equation is related to the diffusion matrix of the SDEs, $\mathbf{B}$, by $\mathbf{D}=\mathbf{B}\mathbf{B}^T$ ($\mathbf{B}^T$ is the transpose of $\mathbf{B}$). Since there is not always a unique decomposition of positive definite matrices for a given matrix $\mathbf{D}$, there may exist several choices for the diffusion matrix $\mathbf{B}$. Therefore, one can have different models for the trajectories that still correspond to the same transitional PDF. In other words, there is more information in the trajectories of a diffusion process than in the solution of the Fokker-Planck equation. However, in the present work, interest is mainly focused on statistics extracted from the stochastic process (weak approach [13]). Consequently, one can consider that the different models for the trajectories belong to the same class and then speak of the equivalence between SDEs and Fokker-Planck equations.

It is now clear that the Lagrangian method, where the dynamics of the particles are described by a set of SDEs, is nothing else than a Monte Carlo simulation of an underlying PDF [15]. This correspondence (diffusion process–Fokker-Planck equation) is fundamental to the presentation of the PDF formalism.

III. PROBABILISTIC DESCRIPTION OF DISPERSED TWO-PHASE FLOWS

The next sections are slightly anticipated and an expression for the two-particle state vector (one fluid particle and one discrete particle) is directly introduced. In the case of turbulent, reactive, compressible, dispersed two-phase flows, an appropriate state vector is (see Sec. III D)

$$\mathbf{Z}=(\mathbf{x}_f,\mathbf{U}_f,\boldsymbol{\phi}_f,\mathbf{x}_p,\mathbf{U}_p,\boldsymbol{\phi}_p), \tag{3}$$

where $\boldsymbol{\phi}_f$ and $\boldsymbol{\phi}_p$ are to be specified [note that we distinguish between physical space and sample space, $\mathbf{z}=(\mathbf{y}_f,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{y}_p,\mathbf{V}_p,\boldsymbol{\psi}_p)$]. Once again, it is necessary to introduce two independent variables for the positions of the fluid and the discrete particles since the two kinds of particles are not convected by the same velocities.

A. Eulerian and Lagrangian descriptions

There are two possible points of view for the statistical description of the fluid-particle mixture. The Lagrangian one where one is interested in, at a fixed time, the probability to find a pair of particles (a fluid particle and a discrete particle) in a given state and the Eulerian description (field approach) where one seeks the probability to find, at a given time and at two fixed points in space (a ''fluid point'' $\mathbf{x}_f$ and a ''discrete-particle point'' $\mathbf{x}_p$), the fluid-particle mixture in a given state.

In the case of the Lagrangian description, let us introduce the PDF p_{fp}^L. The following notation is used: L or E as superscripts to distinguish between Lagrangian and Eulerian quantities and f and p as indices to specify if a two-point (fp), or one-point (f or p) PDF is used. The probability to find a pair of particles at time t whose positions are in the range $[\mathbf{y}_k,\mathbf{y}_k+d\mathbf{y}_k]$, whose velocities are in the range $[\mathbf{V}_k,\mathbf{V}_k+d\mathbf{V}_k]$, and whose associated quantities (scalars and other variables) are in the range $[\boldsymbol{\psi}_k,\boldsymbol{\psi}_k+d\boldsymbol{\psi}_k]$, is (where k is the phase index, either f or p)

$$p_{fp}^L(t;\mathbf{y}_f,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{y}_p,\mathbf{V}_p,\boldsymbol{\psi}_p)d\mathbf{y}_f\,d\mathbf{V}_f\,d\boldsymbol{\psi}_f\,d\mathbf{y}_p\,d\mathbf{V}_p\,d\boldsymbol{\psi}_p. \tag{4}$$

A distinction is made between the parameters and the variables by using a semicolumn to separate them. Two marginal PDFs have a clear physical meaning: the first one, p_f^L, obtained by integration over all variables of the discrete particles, is the PDF related to the fluid characteristics and the second one, p_p^L, derived by contraction over all characteristics of the fluid particles, is the PDF related to the discrete phase. The two PDFs are given by

$$p_k^L(t;\mathbf{y}_k,\mathbf{V}_k,\boldsymbol{\psi}_k)=\int p_{fp}^L(t;\mathbf{y}_f,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{y}_p,\mathbf{V}_p,\boldsymbol{\psi}_p)d\mathbf{y}_{\bar{k}}\,d\mathbf{V}_{\bar{k}}\,d\boldsymbol{\psi}_{\bar{k}}, \tag{5}$$

where $\bar{k}$ is the complement of k (for example, if $k=f$ then $\bar{k}=p$).

For the field description (Eulerian point of view), let us consider the quantity p_{fp}^E. The probability to find, at time t and at positions $\mathbf{x}_f$ and $\mathbf{x}_p$, the system in a given state in the range $[\mathbf{V}_k,\mathbf{V}_k+d\mathbf{V}_k]$ and $[\boldsymbol{\psi}_k,\boldsymbol{\psi}_k+d\boldsymbol{\psi}_k]$ is

$$p_{fp}^E(t,\mathbf{x}_f,\mathbf{x}_p;\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{V}_p,\boldsymbol{\psi}_p)d\mathbf{V}_f\,d\boldsymbol{\psi}_f\,d\mathbf{V}_p\,d\boldsymbol{\psi}_p. \tag{6}$$

p_{fp}^E is not a PDF since, in a fluid-particle mixture, one cannot always find with probability 1, at a given time and at two different locations, a fluid and a discrete particle in any state.

Furthermore, at a given point $\mathbf{x}$ in physical space and a given time t, the sum of the probabilities to find a fluid particle or a discrete particle in any state is one, i.e., $p_{fp}^L=0$ when $\mathbf{y}_f=\mathbf{y}_p=\mathbf{y}$ and $p_{fp}^E=0$ for $\mathbf{x}_f=\mathbf{x}_p=\mathbf{x}$. Consequently, one can write, in terms of the marginals of the Eulerian distribution function,

$$\int p_f^E(t,\mathbf{x};\mathbf{V}_f,\boldsymbol{\psi}_f)d\mathbf{V}_f\,d\boldsymbol{\psi}_f+\int p_p^E(t,\mathbf{x};\mathbf{V}_p,\boldsymbol{\psi}_p)d\mathbf{V}_p\,d\boldsymbol{\psi}_p=1, \tag{7}$$

where the marginals p_k^E are defined as done in Eq. (5) for p_k^L,

$$p_k^E(t,\mathbf{x}_k;\mathbf{V}_k,\boldsymbol{\psi}_k)=\int p_{fp}^E(t,\mathbf{x}_f,\mathbf{x}_p;\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{V}_p,\boldsymbol{\psi}_p)\times d\mathbf{x}_{\bar{k}}\,d\mathbf{V}_{\bar{k}}\,d\boldsymbol{\psi}_{\bar{k}}. \tag{8}$$

Equation (7) can also be rewritten by introducing the normalization factors of p_f^E and p_p^E, namely, $\alpha_f(t,\mathbf{x})$ and $\alpha_p(t,\mathbf{x})$, respectively, to yield

$$\alpha_f(t,\mathbf{x})+\alpha_p(t,\mathbf{x})=1, \tag{9}$$

where, by definition,

$$\alpha_k(t,\mathbf{x})=\int p_k^E(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)d\mathbf{V}_k\,d\boldsymbol{\psi}_k. \tag{10}$$

$\alpha_f(t,\mathbf{x})$ represents the probability to find the fluid phase, at time t and position $\mathbf{x}$, in any state $[0\leqslant\alpha_f(t,\mathbf{x})\leqslant 1]$. This probability is not always 1 as in single-phase flows where the physical space is continuously filled by the fluid. In a fluid-particle mixture, at $(t,\mathbf{x})$ there might be some fluid or a discrete particle. Similarly, the probability to find the discrete phase at time t and position $\mathbf{x}$ in any state is $\alpha_p(t,\mathbf{x})$ $[0\leqslant\alpha_p(t,\mathbf{x})\leqslant 1]$. It has been explained above that p_{fp}^E is not a PDF but rather a distribution function (as a matter of fact, it represents a field of distribution functions): the normalization factor of p_{fp}^E is always less than or equal to 1. This can be clarified in the particular case where the fluid particles and the discrete particles represent independent events, i.e., $p_{fp}^E=p_f^E p_p^E$ (strictly speaking, this is not always possible since they cannot be located, for a given time, at the same point in physical space). Under this assumption, the normalization factor of p_{fp}^E becomes $\alpha_p(t,\mathbf{x})\ \alpha_f(t,\mathbf{x})$.

B. Mass density functions

As explained in Sec. II, a fluid particle (and also a discrete particle) is completely described by its mass, position, velocity and associated scalars, so that it is logical to introduce a mass density function (MDF) F_k^L where

$$F_k^L(t;\mathbf{y}_k,\mathbf{V}_k,\boldsymbol{\psi}_k)d\mathbf{y}_k\,d\mathbf{V}_k\,d\boldsymbol{\psi}_k \tag{11}$$

is the probable mass of fluid ($k=f$) or discrete particles ($k=p$) in an element of volume $d\mathbf{y}_k d\mathbf{V}_k d\boldsymbol{\psi}_k$. Both mass density functions are consequently normalized by the total mass M_k of the respective phases (M_f for the continuous phase and M_p for the discrete phase, which are constant in time for the sake of simplicity),

$$M_k=\int F_k^L(t;\mathbf{y}_k,\mathbf{V}_k,\boldsymbol{\psi}_k)d\mathbf{y}_k\,d\mathbf{V}_k\,d\boldsymbol{\psi}_k\,. \tag{12}$$

The mass density functions F_k^L can be expressed in terms of the respective total masses M_k and the marginal PDFs p_k^L as $F_k^L=M_k p_k^L$. A two-point fluid-particle mass density function is also defined,

$$\begin{aligned}&\mathcal{F}_{fp}^L(t;\mathbf{y}_f,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{y}_p,\mathbf{V}_p,\boldsymbol{\psi}_p)\\&\quad=M_pM_f p_{fp}^L(t;\mathbf{y}_f,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{y}_p,\mathbf{V}_p,\boldsymbol{\psi}_p),\end{aligned} \tag{13}$$

and its marginals are related to the mass density function of phase k by $\mathcal{F}_k^L=M_k\bar{F}_k^L$.

C. General relations between Eulerian and Lagrangian quantities

Since one of the aims of the present paper is the derivation of mean field equations, relations between Lagrangian and Eulerian MDFs (and PDFs) have to be found. By doing so, the partial differential equations verified by different Eulerian quantities will be written and from there, by defining an appropriate operator (expected value), mean field equations will be derived.

By generalization of the ideas of Balescu [16], the Lagrangian MDF $\mathcal{F}_{fp}^L$ can be linked to an Eulerian MDF by writing [12]

$$\begin{aligned}&\mathcal{F}_{fp}^E(t,\mathbf{x}_f,\mathbf{x}_p;\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{V}_p,\boldsymbol{\psi}_p)\\&\quad=\mathcal{F}_{fp}^L(t;\mathbf{y}_f=\mathbf{x}_f,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{y}_p=\mathbf{x}_p,\mathbf{V}_p,\boldsymbol{\psi}_p)\\&\quad=\int\mathcal{F}_{fp}^L(t;\mathbf{y}_f,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{y}_p,\mathbf{V}_p,\boldsymbol{\psi}_p)\\&\qquad\times\quad\delta(\mathbf{x}_f-\mathbf{y}_f)\,\delta(\mathbf{x}_p-\mathbf{y}_p)d\mathbf{y}_f\,d\mathbf{y}_p\,,\end{aligned} \tag{14}$$

where $\mathcal{F}_{fp}^E$ is the two-point fluid-particle Eulerian mass density function. By direct integration of the previous equation over physical space $\mathbf{x}_k$ and phase space $(\mathbf{V}_k,\boldsymbol{\psi}_k)$, the associated marginals (the one-point Eulerian mass density functions, $\mathcal{F}_k^E$) verify a similar relation, that is,

$$\begin{aligned}\mathcal{F}_k^E(t,\mathbf{x}_k;\mathbf{V}_k,\boldsymbol{\psi}_k)&=\mathcal{F}_k^L(t;\mathbf{y}_k=\mathbf{x}_k,\mathbf{V}_k,\boldsymbol{\psi}_k)\\&=\int\mathcal{F}_k^L(t;\mathbf{y}_k,\mathbf{V}_k,\boldsymbol{\psi}_k)\,\delta(\mathbf{x}_k-\mathbf{y}_k)d\mathbf{y}_k\,.\end{aligned} \tag{15}$$

By recalling that $\mathcal{F}_k^L=M_k\bar{F}_k^L$, a direct consequence of the previous equation is that $\mathcal{F}_k^E=M_k\bar{F}_k^E$. Therefore, the relations between the Eulerian mass density functions F_k^E and the Lagrangian mass density functions F_k^L are also given by Eqs. (15).

Bearing in mind the results that have been displayed so far, there are two possible strategies, yet equivalent, for the derivation of the mean field equations, i.e., for the path between Lagrangian and Eulerian MDFs, since the physical space is shared by the fluid and the particles.

(i) In the first procedure, relations between the Lagrangian and Eulerian MDFs are worked out at the two-point level. Once on the Eulerian side, information is still available at the two-point level. For a given point $(t,\mathbf{x})$ in the time-space domain, we consider

$$\begin{aligned}&\mathcal{F}_{fp}^L(t;\mathbf{y}_f=\mathbf{x},\mathbf{y}_p,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{V}_p,\boldsymbol{\psi}_p),\\&\mathcal{F}_{fp}^L(t;\mathbf{y}_f,\mathbf{y}_p=\mathbf{x},\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{V}_p,\boldsymbol{\psi}_p).\end{aligned} \tag{16}$$

Correspondence with the Eulerian MDFs is found by using Eq. (14). Then, from these two-point Eulerian MDFs both marginals at the same point in physical space can be extracted, i.e., $F_k^E(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)$.

(ii) In the second procedure, relations between the Lagrangian and Eulerian MDFs are worked out at the one-point level, that is,

$$F_k^L(t;\mathbf{y},\mathbf{V}_k,\boldsymbol{\psi}_k), \tag{17}$$

is under consideration (or $\mathcal{F}_k^L$). Contraction has been made for the two-point Lagrangian MDF before going on to the field description. By using Eq. (15), information is obtained in the form of both the one-point fluid and particle Eulerian mass density functions, $F_k^E(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)$, at the same point in physical space.

1. Two-point relations between Eulerian and Lagrangian quantities

If strategy (i) is adopted, the following relations are needed. With Eq. (14), the definitions of the two-point fluid-particle Lagrangian MDF, $\mathcal{F}_{fp}^L=M_fM_p p_{fp}^L$, and the two-point fluid-particle transitional PDF, $\hat{p}_{fp}^L$, one can write [12]

$$\begin{aligned}&\mathcal{F}_{fp}^E(t,\mathbf{x}_f,\mathbf{x}_p;\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{V}_p,\boldsymbol{\psi}_p)\\&\quad=\int\hat{p}_{fp}^L(t;\mathbf{x}_f,\mathbf{V}_f,\boldsymbol{\psi}_f,\mathbf{x}_p,\mathbf{V}_p,\boldsymbol{\psi}_p|t_0;\end{aligned}$$

$$\mathbf{x}_{f0},\mathbf{V}_{f0},\boldsymbol{\psi}_{f0},\mathbf{x}_{p0},\mathbf{V}_{p0},\boldsymbol{\psi}_{p0})$$
$$\times\mathcal{F}_{fp}^{E}(t,\mathbf{x}_{f0},\mathbf{x}_{p0};\mathbf{V}_{f0},\boldsymbol{\psi}_{f0},\mathbf{V}_{p0},\boldsymbol{\psi}_{p0})$$
$$\times d\mathbf{x}_{f0}\,d\mathbf{V}_{f0}\,d\boldsymbol{\psi}_{f0}\,d\mathbf{x}_{p0}\,d\mathbf{V}_{p0}\,d\boldsymbol{\psi}_{p0}. \tag{18}$$

This relation shows that the Eulerian MDF $\mathcal{F}_{fp}^{E}$ is ''propagated'' by the transitional PDF, or in the language of statistical physics, the transitional PDF $\hat{p}_{fp}^{L}$ is the *propagator* of an information that is the two-point fluid-particle Eulerian MDF. Consequently, the partial differential equation that is verified by the transitional PDF is also verified by the Eulerian mass density function $\mathcal{F}_{fp}^{E}$.

The definitions of the expected densities, $\langle\rho_f\rangle(t,\mathbf{x})$ and $\langle\rho_p\rangle(t,\mathbf{x})$, and the probability of presence of both phases $\alpha_f(t,\mathbf{x})$ and $\alpha_p(t,\mathbf{x})$, can be expressed in terms of the two-point Eulerian MDFs and the associated marginals. For the expected densities, one can write

$$\alpha_k(t,\mathbf{x})\langle\rho_k\rangle(t,\mathbf{x})=\frac{1}{M_k^-}\int\mathcal{F}_{fp}^{E}(t,\mathbf{x},\mathbf{x}_k^-;\mathbf{V}_k,\boldsymbol{\psi}_k,\mathbf{V}_k^-,\boldsymbol{\psi}_k^-)$$
$$\times d\mathbf{x}_k^- d\mathbf{V}_k^- d\boldsymbol{\psi}_k^- d\mathbf{V}_k\,d\boldsymbol{\psi}_k, \tag{19}$$

$$\Rightarrow\alpha_k(t,\mathbf{x})\langle\rho_k\rangle(t,\mathbf{x})=\int F_k^{E}(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)d\mathbf{V}_k\,d\boldsymbol{\psi}_k. \tag{20}$$

Similarly, α_f and α_p are defined by

$$\alpha_k(t,\mathbf{x})=\frac{1}{M_k^-}\int\frac{1}{\rho_k(\boldsymbol{\psi}_k)}\mathcal{F}_{fp}^{E}(t,\mathbf{x},\mathbf{x}_k^-;\mathbf{V}_k,\boldsymbol{\psi}_k,\mathbf{V}_k^-,\boldsymbol{\psi}_k^-)$$
$$\times d\mathbf{x}_k^- d\mathbf{V}_k^- d\boldsymbol{\psi}_k^- d\mathbf{V}_k\,d\boldsymbol{\psi}_k, \tag{21}$$

$$\Rightarrow\alpha_k(t,\mathbf{x})=\int\frac{1}{\rho_k(\boldsymbol{\psi}_k)}F_k^{E}(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)d\mathbf{V}_k\,d\boldsymbol{\psi}_k. \tag{22}$$

2. One-point relations between Eulerian and Lagrangian quantities

If strategy (ii) is adopted, the following relations are needed. Using Eq. (15), the definition of the one-point Lagrangian MDF $F_k^{L}=M_k p_k^{L}$, and introducing the one-point transitional PDF $\hat{p}_k^{L}$, one can write [12]

$$F_k^{E}(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)=\int\hat{p}_k^{L}(t;\mathbf{x},\mathbf{V}_k,\boldsymbol{\psi}_k|t_0;\mathbf{x}_{k0},\mathbf{V}_{k0},\boldsymbol{\psi}_{k0})$$
$$\times F_k^{E}(t,\mathbf{x}_0;\mathbf{V}_{k0},\boldsymbol{\psi}_{k0})d\mathbf{x}_0\,d\mathbf{V}_{k0}\,d\boldsymbol{\psi}_{k0}. \tag{23}$$

Once again, this relation shows that the Eulerian mass density function F_k^{E} is ''propagated'' by the transitional PDF, or in the language of statistical physics, the transitional PDF $\hat{p}_k^{L}$ is the *propagator* of an information which is the Eulerian mass density function F_k^{E}.

Integration of Eq. (15) over $\mathbf{x}=\mathbf{x}_k,\mathbf{V}_k,\boldsymbol{\psi}_k$ gives the total mass of phase k, M_k, which means that the integral of F_k^{E} over phase space $(\mathbf{V}_k,\boldsymbol{\psi}_k)$ is the expected density of phase k at $(t,\mathbf{x})$ (the probable mass of phase k in a given state per unit volume). The expected density, denoted $\langle\rho_k\rangle(t,\mathbf{x})$, is

$$\alpha_k(t,\mathbf{x})\langle\rho_k\rangle(t,\mathbf{x})=\int\rho_k(\psi_k)p_k^{E}(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)d\mathbf{V}_k\,d\boldsymbol{\psi}_k, \tag{24}$$

where the Eulerian mass density function F_k^{E} is

$$F_k^{E}(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)=\rho_k(\psi_k)p_k^{E}(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k), \tag{25}$$

and $\alpha_k(t,\mathbf{x})$ is of course defined as the normalization factor of p_k^{E}, see Eq. (10). As mentioned at the beginning of the section, $\alpha_k(t,\mathbf{x})$ represents the probability to find phase k, at time t and position $\mathbf{x}$, in any state $[0\leqslant\alpha_k(t,\mathbf{x})\leqslant 1]$. Integration of F_k^{L} over phase space $(\mathbf{V}_k,\boldsymbol{\psi}_k)$ yields

$$p_k^{L}(t;\mathbf{x})=\frac{1}{M_k}\alpha_k(t,\mathbf{x})\langle\rho_k\rangle(t,\mathbf{x}), \tag{26}$$

and therefore the conditional expectation $p_k^{L}(t;\mathbf{V}_k,\boldsymbol{\psi}_k|\mathbf{x})$ is given by

$$p_k^{L}(t;\mathbf{V}_k,\boldsymbol{\psi}_k|\mathbf{x})=\frac{\rho_k(\psi_k)}{\alpha_k(t,\mathbf{x})\langle\rho_k\rangle(t,\mathbf{x})}p_k^{E}(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k). \tag{27}$$

Thus, in a compressible flow, the one-point fluid Lagrangian PDF conditioned by the position is not the one-point fluid Eulerian distribution function but the density-weighted one-point fluid Eulerian PDF, p_f^{E}/α_f.

D. Trajectory point of view

The trajectory point of view is now chosen (see Sec. II) and the construction of the trajectory of a pair of particles is briefly explained with no emphasis on the models, and this for the sake of generality. Indeed, as specified in the Introduction, the purpose of the present paper is to present a general formalism and not to introduce and discuss models used in numerical simulations. Practical models will be displayed in Sec. VI.

From now on, the study is limited to nonreactive polydispersed turbulent two-phase flows with two-way coupling, i.e., particles are dispersed by the turbulent fluid and at the same time they modify the turbulent state of the fluid. The collisional mechanisms between discrete particles are neglected. Furthermore, both phases have a constant density with $\rho_f\ll\rho_p$ (heavy particles). These restrictions are made for the sake of simplicity and extension of the present formalism to reactive flows is straightforward (this is precisely one of the main interests of PDF models), provided a proper introduction of the relevant scalar variables in $\boldsymbol{\phi}_k$. The treatment of collisions is a more complex issue that is outside the scope of the present paper but some proposal for a possible approach can be found in Ref. [12].

In the particular case of heavy particles, the force exerted on a rigid sphere in a turbulent fluid reduces to the sum of the drag force and possible external force fields [12]. The acceleration $\mathbf{A}_p$ reads

$$\mathbf{A}_p = \frac{d\mathbf{U}_p}{dt} = \frac{1}{\tau_p}(\mathbf{U}_s - \mathbf{U}_p) + \mathbf{F}_E, \tag{28}$$

where $\mathbf{U}_s = \mathbf{U}(\mathbf{x}_p(t),t)$ is the fluid velocity seen, i.e., the fluid velocity sampled along the particle trajectory $\mathbf{x}_p(t)$, not to be confused with the fluid velocity $\mathbf{U}_f = \mathbf{U}(\mathbf{x}_f(t),t)$ denoted by the subscript *f*. These velocities are indeed different since, due to particle inertia and external force fields, a fluid and a discrete particle located at nearby positions at time *t* do not follow the same trajectories under a time interval Δt [2,12] (this drift is often referred to as the *crossing trajectory effect* in the literature [2]). In Eq. (28), τ_p is the particle relaxation time given by $\tau_p = (4\rho_p d_p)/(3\rho_f C_D|\mathbf{U}_r|)$ where $\mathbf{U}_r = \mathbf{U}_s - \mathbf{U}_p$ is the local instantaneous relative velocity. C_D, the drag coefficient, is a nonlinear function of the particle-based Reynolds number, $\mathrm{Re} = \rho_p|\mathbf{U}_r|/\nu_f$ (in fact, C_D is a complex nonlinear function of the discrete particle diameter, d_p) [17].

1. Trajectory of a fluid particle

Kolmogorov theory [11] (for Lagrangian statistics) tells us that the acceleration of a fluid particle is a fast variable for a time scale *dt* belonging to the inertial range. This variable can be eliminated by fast variable elimination techniques (see [12] for a detailed proof). A general diffusion process is then used to simulate the time rate of change of $\mathbf{Z}_f = (\mathbf{x}_f, \mathbf{U}_f)$,

$$dx_{f,i} = U_{f,i}\,dt, \tag{29a}$$

$$dU_{f,i} = A_{f,i}\,dt + B_{f,ij}\,dW_j, \tag{29b}$$

where the drift vector $\mathbf{A}_f$ and the diffusion matrix $\mathbf{B}_f$ are functions of *t* and $\mathbf{Z}_f$ but also of the moments of $\mathbf{Z}_f$ ($\langle \mathbf{Z}_f \rangle, \langle \mathbf{Z}_f \mathbf{Z}_f \rangle, \ldots$). In Eqs. (29), the local instantaneous equations (the Navier-Stokes equations in Lagrangian form) have been replaced by SDEs, that is, real fluid particles are replaced by stochastic particles, which reproduce the same statistics.

2. Trajectory of a discrete particle

Let us assume for the moment that, at each point in the time-space domain, the properties of the fluid are known in terms of mean fields, i.e., in terms of the moments of $\mathbf{Z}_f$. In the case of discrete particles, the extension of Kolmogorov theory is not straightforward. The choice of the variables for the construction of the discrete particle state vector is still subject to some debate [12,18,19]. One hint can be found, however, if the limit case of particles having small inertia is considered (particles nearly behave as fluid elements). In this case, Kolmogorov theory indicates that fluid-particle accelerations are governed by small scales which have a better chance of showing some universal characteristics whereas fluid-particle velocities are more likely to be problem or flow dependent. Building from the fluid case, it appears preferable to include fluid velocities in the state vector, i.e., the fluid velocity seen $\mathbf{U}_s$. It is then possible to generalize Kolmogorov theory and derive results that suggest to use a diffusion process (Langevin equation) for the simulation of $\mathbf{U}_s$ [20,21] (see [12] for a detailed explanation). A general diffusion process is then introduced to simulate the time rate of change of $\mathbf{Z}_p = (\mathbf{x}_p, \mathbf{U}_p, \mathbf{U}_s, d_p)$,

$$dx_{p,i} = U_{p,i}dt, \tag{30a}$$

$$dU_{p,i} = A_{p,i}dt, \tag{30b}$$

$$dU_{s,i} = A_{s,i}\,dt + B_{s,ij}\,dW_j. \tag{30c}$$

The drift vector $\mathbf{A}_s$ and the diffusion matrix $\mathbf{B}_s$ are functions of *t*, $\mathbf{Z}_f$ and $\mathbf{Z}_p$ but also of the moments of $\mathbf{Z}_f$ and $\mathbf{Z}_p$. By writing Eqs. (30), one merely wants to mimic the local instantaneous behavior of the real discrete particles by stochastic particles whose dynamical behavior can be described by Langevin equations.

3. Trajectory of a pair of particles

The path that is adopted here is to gather the preceding results that have just been derived for the time increments of the fluid velocity seen along discrete particle trajectories and for the time increments of the fluid velocity along fluid particle trajectories. The system of SDEs is, however, supplemented by two terms (accelerations), namely, $\mathbf{A}_{p\to f}$ that reflects the influence of the discrete particles on the fluid and $\mathbf{A}_{p\to s}$ that accounts for the influence of the discrete particles on the statistics of the fluid velocity sampled along the trajectory of a discrete particle. These terms are a simple consequence of Newton's third law: the fluid exerts a force $\mathbf{F}_{f\to p}$ on the discrete particles and, in return, the particles exert a force $\mathbf{F}_{p\to f} = -\mathbf{F}_{f\to p}$ on the fluid. The trajectory of a pair of particles is simulated by resorting to a general diffusion process. The time rate of change of $\mathbf{Z} = (\mathbf{Z}_f, \mathbf{Z}_p)$ is given by

$$dx_{f,i} = U_{f,i}\,dt, \tag{31a}$$

$$dU_{f,i} = A_{f,i}\,dt + A_{p\to f,i}\,dt + B_{f,ij}\,dW'_j, \tag{31b}$$

$$dx_{p,i} = U_{p,i}\,dt, \tag{31c}$$

$$dU_{p,i} = A_{p,i}\,dt, \tag{31d}$$

$$dU_{s,i} = A_{s,i}\,dt + A_{p\to s,i}\,dt + B_{s,ij}\,dW_j, \tag{31e}$$

where the expression for the drift vectors $\mathbf{A}_f, \mathbf{A}_p, \mathbf{A}_s$ and the diffusion matrices $\mathbf{B}_f, \mathbf{B}_s$, can be found by simple identification with Eqs. (29) and (30). By assuming that the trajectories of a pair of particles can be obtained in such a way, the following approximations have been made.

(i) Two different Wiener processes are used for the velocity increments of the fluid and the velocity increments of the fluid seen. Consequently, the correlation between the fluid acceleration at location $\mathbf{x}_f$ and the time rate of change of $\mathbf{U}_s$ along discrete particle trajectories at location $\mathbf{x}_p$ is neglected. At two nearby locations, when particle inertia becomes small ($\tau_p \to 0$), the present approximation is not accurate (see Kolmogorov theory) but as soon as inertia is not negligible the two accelerations are not necessarily correlated. This imperfection is bearable in the frame of our work where our real

objective is not a two-point description for the fluid but a two-point description for two-phase flows from which mean field equations can be extracted.

(ii) In Eqs. (31) an additional term should be present in the form of a short-range interaction since at a given point in the time-space domain only one phase can be present as mentioned previously (this term should also enter the Fokker-Planck equation verified by p^L_{fp}). The form of this short-range interaction is not discussed here.

4. Treatment of the coupling terms

In the exact local instantaneous equations for the fluid (the Navier-Stokes equations), a formal treatment of the force exerted on the fluid by the discrete particles implies the use of a distribution (or density of force) acting on the fluid located in the neighborhood of the discrete particles in order to express the resulting acceleration on nearby fluid particles. This accurate treatment, which would result in a multipoint treatment of the discrete phase, is outside the scope of the present paper. Here, in the frame of the one-point approach, the influence of the discrete particles on the fluid is expressed directly in the SDEs, Eqs. (31), with stochastic tools.

The force exerted by a discrete particle on the neighboring fluid corresponds to the drag force and it can be written as

$$\mathbf{F}_{p\to f} = -m_p \mathbf{A}_p^D = -m_p \frac{\mathbf{U}_s - \mathbf{U}_p}{\tau_p}, \tag{32}$$

and obviously the variables entering $\mathbf{F}_{p\to f}$ are variables attached only to the discrete particles, namely, $\mathbf{U}_p$, $\mathbf{U}_s$, and d_p. As a consequence, the influence of the particles on the fluid seen can be expressed directly as a function of these variables. Let us consider a local model where, at location $\mathbf{x}_p$, the force due to one particle is given by Eq. (32). The total force acting on the fluid element surrounding a discrete particle is then the sum of all elementary forces, $\mathbf{F}_{p\to f}$, due to all neighboring discrete particles,

$$\mathbf{A}_{p\to s} = -\frac{\alpha_p \rho_p}{\alpha_f \rho_f} \frac{\mathbf{U}_s - \mathbf{U}_p}{\tau_p}. \tag{33}$$

Here, it is implicitly assumed that all neighboring particles have the same acceleration term $\mathbf{A}_p^D$. This acceleration is multiplied by the expected particle mass at $\mathbf{x}_p$, $\alpha_p \rho_p$, divided by the expected mass of fluid, $\alpha_f \rho_f$, since the total force is distributed only on the fluid phase. This simple model is only a first proposal and work remains to be done to improve the closure of this term.

In the case of the reverse force in the equation of a fluid particle, the situation is more delicate. Indeed, a local model for $\mathbf{A}_{p\to f}$ at location $\mathbf{x}_f$ cannot be expressed directly in terms of the local instantaneous variables attached to the discrete element that is located at $\mathbf{x}_p$. At time t and for a fluid particle located at $\mathbf{x}_f = \mathbf{x}$, $\mathbf{A}_{p\to f}$ is modeled as a random variable that is defined by $\mathbf{A}_{p\to f} = 0$ with probability $1 - \alpha_p(t, \mathbf{x}_f)$ and $\mathbf{A}_{p\to f} = \boldsymbol{\pi}_p$ with probability $\alpha_p(t, \mathbf{x}_f)$. $\boldsymbol{\pi}_p$ is a random variable that plays the role of an ersatz of the Eulerian random variable that is formed from the discrete particles at location $\mathbf{x}_p = \mathbf{x}$,

$$\Pi_p \equiv \frac{\rho_p}{\rho_f} \frac{\mathbf{U}_p - \mathbf{U}_s}{\tau_p}. \tag{34}$$

In other words, from the stochastic models for the discrete particles, or from the one-point particle PDF value at location $\mathbf{x} = \mathbf{x}_f$, the random variables $\boldsymbol{\pi}_p$ are formed with the same distribution. This random term mimics the reverse forces due to the discrete particles and is only nonzero where the fluid particle is in the close neighborhood of a discrete particle. At the location $\mathbf{x}$ considered, $\prod_p$ is defined as a random acceleration term in the equation of $\mathbf{U}_f$, correlated with $\mathbf{U}_f$ so that one has

$$\langle \Pi_{p,i} \rangle = -\frac{\rho_p}{\rho_f} \langle A^D_{p,i} \rangle, \tag{35a}$$

$$\langle \Pi_{p,i} U_{f,j} \rangle = -\frac{\rho_p}{\rho_f} (A^D_{p,i} U_{s,j}). \tag{35b}$$

E. Fokker-Planck equation

According to Sec. II, the two-point model given by Eqs. (31) is equivalent to a Fokker-Planck equation given in closed form for the transitional PDF $\hat{p}^L_{fp}$. This Fokker-Planck equation is also verified by the two-point fluid-particle Lagrangian PDF p^L_{fp} and by the fluid-particle Eulerian mass density function $\mathcal{F}^E_{fp}$, as seen in Sec. III C. The Fokker-Planck equation is, for p^L_{fp},

$$\begin{aligned}
&\frac{\partial p^L_{fp}}{\partial t} + V_{f,i} \frac{\partial p^L_{fp}}{\partial y_{f,i}} + V_{p,i} \frac{\partial p^L_{fp}}{\partial y_{p,i}} \\
&\quad = -\frac{\partial}{\partial V_{f,i}} ([A_{f,i} + \langle A_{p\to f,i} | \mathbf{y}_f, \mathbf{V}_f \rangle] p^L_{fp}) \\
&\qquad - \frac{\partial}{\partial V_{p,i}} (A_{p,i} p^L_{fp}) - \frac{\partial}{\partial V_{s,i}} ([A_{s,i} \\
&\qquad + \langle A_{p\to s,i} | \mathbf{y}_p, \mathbf{V}_p, \boldsymbol{\psi}_p \rangle] p^L_{fp}) \\
&\qquad + \frac{1}{2} \frac{\partial^2}{\partial V_{f,i} \partial V_{f,j}} ([B_f B_f^T]_{ij} p^L_{fp}) \\
&\qquad + \frac{1}{2} \frac{\partial^2}{\partial V_{s,i} \partial V_{s,j}} ([B_s B_s^T]_{ij} p^L_{fp}).
\end{aligned} \tag{36}$$

All tools, which are needed to write mean field equations, have now been gathered, i.e., (i) the correspondence between a SDE and a Fokker-Planck equation and (ii) the relations between Eulerian and Lagrangian tools.

IV. MEAN FIELD EQUATIONS

The partial differential equations (PDEs) satisfied by different mean fields $\langle f(\mathbf{Z}) \rangle (t, \mathbf{x})$, which are expectations of a

given polynomial function f of $\mathbf{Z}$, are now derived. Here, the study is limited to the expected values of $\mathbf{Z}$, $\langle \mathbf{Z} \rangle$, and the second-order moments, $\langle \mathbf{ZZ} \rangle$. In the literature, the system of mean field equations for $\langle \mathbf{Z} \rangle$ and $\langle \mathbf{ZZ} \rangle$ is often referred to as the *Eulerian model* or sometimes *two-fluid model*. As a matter of fact, the system of equations formed by the mean field equations should rather be called *two-field model*. Indeed, the spirit of the approach is to derive field equations for both phases by using arguments from statistical physics. Let us investigate how such equations are derived (for incompressible turbulent flows carrying discrete particles of constant density but different diameters).

Let us recall momentarily the Lagrangian and Eulerian tools that were defined in the previous section as well as the relations between them, see Fig. 1. A two-point fluid-particle Lagrangian PDF, p_{fp}^L, (extracted from the transitional PDF $\hat{p}_{fp}^L$) has been introduced and from it separate information on each phase was obtained in the form of the marginals p_k^L. Associated MDFs (F_k^L) were defined and for both of them correspondence with the field (Eulerian) description could be made (this crucial step is indicated with dashed arrows in Fig. 1). It was then found that each Eulerian mass density function F_k^E is propagated by the corresponding transitional PDF, $\hat{p}_k^L$. The Fokker-Planck equations verified by F_k^E can then be directly derived from the Fokker-Planck equations satisfied by the transitional PDFs $\hat{p}_k^L$ or from the Fokker-Planck equation verified by the transitional PDF $\hat{p}_{fp}^L$. There is another, yet equivalent, way to go from the transitional PDF $\hat{p}_{fp}^L$ to the Eulerian MDFs F_k^E, see Fig. 1. One can keep the joint (one fluid point–one particle point) information by treating the two-point fluid-particle Eulerian MDF, $\mathcal{F}_{fp}^E$. As indicated in Fig. 1, by direct integration, the Fokker-Planck equations verified by the marginals $\mathcal{F}_k^E$ can be obtained from the Fokker-Planck equation verified by $\mathcal{F}_{pf}^E$ which is, in its turn, obtained from the PDF verified by the transitional PDF $\hat{p}_{fp}^L$. The latter equations are also verified by F_k^E.

To sum up, it is now known how the Fokker-Planck equations verified by the Eulerian MDFs F_k^E can be derived. Let us show how mean field equations are obtained from the Fokker-Planck equations verified by F_k^E.

A. Fluid and discrete particle expectations

In the case of discrete particles of constant density but variable diameter carried by an incompressible fluid, all information is contained in the distribution functions $p_k^E(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)$ [with $\boldsymbol{\psi}_p=(\mathbf{V}_s,\delta_p)$ for the discrete phase and $\boldsymbol{\psi}_f=\varnothing$ for the fluid]. However, the definition of the expected values and the derivation of the mean field equations will be addressed, for both phase, in terms of the MDFs $F_k^E(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)=\rho_k p_k^E(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)$ (the reason for this will shortly be explained).

The mathematical definition of the expected *Eulerian* value of a function $\mathcal{H}(\mathbf{V}_k,\boldsymbol{\psi}_k)$ (a sufficiently smooth function attached to a given particle, i.e., a fluid or a discrete particle) is

$$\alpha_k(t,\mathbf{x})\langle\rho_k\rangle(t,\mathbf{x})\langle\mathcal{H}_k\rangle(t,\mathbf{x}) = \int \mathcal{H}(\mathbf{V}_k,\boldsymbol{\psi}_k)F_k^E(t,\mathbf{x};\mathbf{V}_k,\boldsymbol{\psi}_k)d\mathbf{V}_k\,d\boldsymbol{\psi}_k. \quad (37)$$

Therefore, in the present formalism, all expected values must be understood as *mass-weighted mean values*. The fluctuating component of the variables attached to the discrete particles are, for the velocity of the discrete particles $\mathbf{u}_p=\mathbf{U}_p-\langle\mathbf{U}_p\rangle$ with $\langle\mathbf{u}_p\rangle=0$, for the fluid velocity seen $\mathbf{u}_s=\mathbf{U}_s-\langle\mathbf{U}_s\rangle$ with $\langle\mathbf{u}_s\rangle=0$, and for the diameter of the discrete particles $d_p'=d_p-\langle d_p\rangle$ with $\langle d_p'\rangle=0$. Similarly, the fluctuating velocity for a fluid particle is given by $\mathbf{u}_f=\mathbf{U}_f-\langle\mathbf{U}_f\rangle$.

For the moments of the discrete phase, a general definition is introduced, that is a moment of order $n+m+q$ (with $n+m+q>1$),

$$\alpha_p(t,\mathbf{x})\rho_p\langle(d_p')^n u_{s,i_1}\cdots u_{s,i_m}u_{p,j_1}\cdots u_{p,j_q}\rangle(t,\mathbf{x}) = \int (\delta_p')^n\prod_{k=1}^{m} v_{s,i_k}\prod_{l=1}^{q} v_{p,j_l}F_p^E(t,\mathbf{x};\mathbf{V}_p,\boldsymbol{\psi}_p)d\mathbf{V}_p\,d\boldsymbol{\psi}_p, \quad (38)$$

where $(i_k,j_l)\in\{1,2,3\}^2$, $\forall(k,l)$. Different moments can then be obtained by choosing the appropriate values for (n,m,q). In the present paper, information is limited to the second-order moments, i.e., $n+m+q=2$. At last, the moments of order n for the fluid phase are given by

$$\alpha_f(t,\mathbf{x})\rho_f\langle u_{f,i_1}\cdots u_{f,i_n}\rangle(t,\mathbf{x}) = \int\prod_{k=1}^{n} v_{f,i_k}F_f^E(t,\mathbf{x};\mathbf{V}_f)d\mathbf{V}_f. \quad (39)$$

All second-order moments are listed in Table I. Note that the dimension of the space associated to these moments is already 34, and this gives a foretaste, first, of the level of complexity when one formulates mean field equations for polydispersed turbulent two-phase flows, and second, of the amount of computational effort needed to solve such a system of equations (when it is finally closed).

It is now necessary to clarify the correspondence between the mathematical expectations, Eq. (37), and Monte Carlo estimations drawn from a finite ensemble of particles. With Eq. (15) and by approximating $\delta(\mathbf{x}_k-\mathbf{y}_k)$ as $1/\delta\mathcal{V}_x$ where $\delta\mathcal{V}_x$ is a small-volume around point $\mathbf{x}_k$, it is straightforward to write Eq. (37) as

$$\alpha_k(t,\mathbf{x})\langle\rho_k\rangle(t,\mathbf{x})\langle\mathcal{H}_k\rangle \simeq \frac{1}{\delta\mathcal{V}_x}\sum_{i=1}^{N_x^k} m_k^i\mathcal{H}(\mathbf{U}_k^i(t),\phi_k^i(t)). \quad (40)$$

Here N_x^k is the number of fluid ($k=f$) or discrete ($k=p$) particles in volume $\delta\mathcal{V}_x$ and m_k^i is the mass of a fluid or discrete particle indexed i. The preceding equation can be written by supposing that (i) all particles in $\delta\mathcal{V}_x$ represent different realizations, (ii) space homogeneity is fulfilled in $\delta\mathcal{V}_x$, and (iii) N_x^k is sufficiently large so that the ensemble

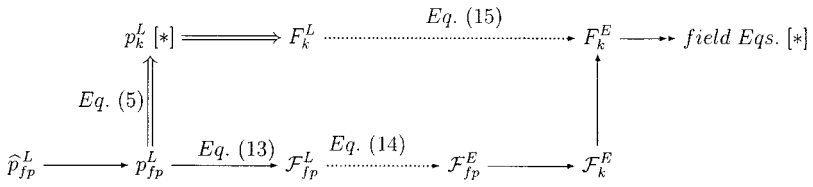

FIG. 1. Derivation of the mean field equations from the two-point fluid-particle Eulerian mass density function (→) or derivation of the mean field equations from the marginal Lagrangian PDFs (⇒). The Eulerian approach corresponds to the field equations whereas the Lagrangian approach is indicated by the symbol [*]

average is a good estimation of the mathematical expectation. Furthermore, by making the following approximation

$$\alpha_k(t,\mathbf{x})\langle\rho_k\rangle(t,\mathbf{x}) \simeq \frac{1}{\delta\mathcal{V}_x}\sum_{i=1}^{N_x^k} m_k^i, \tag{41}$$

one has

$$\langle\mathcal{H}_k\rangle \simeq \sum_{i=1}^{N_x^k} m_k^i \mathcal{H}(\mathbf{U}_k^i(t),\phi_k^i(t)) \Big/ \sum_{i=1}^{N_x^k} m_k^i, \tag{42}$$

which is indeed the discrete form of the Eulerian mass-weighted mean value of a given function.

In the particular case of an incompressible fluid, since fluid particles have then a constant mass, the preceding expression is of course simplified to yield the local ensemble average. For discrete particles of constant density such a simplification is not relevant. Indeed, such particles may have different diameters and therefore different masses. The natural averaging operator is therefore the mass-weighted average.

B. Mean field equations for the fluid phase

In order to obtain the mean field equations for the mean fluid velocity $\langle U_{f,i}\rangle$ and the second-order velocity moment $\langle u_{f,i}u_{f,j}\rangle$, a standard procedure is used in analogy with the derivations that can be found in kinetic theory [22,23]. This procedure is general and can be followed to obtain the mean field equations verified by any moment. The expected value of a function $\mathcal{H}_f(\mathbf{V}_f)$ is defined by Eq. (37) ($\boldsymbol{\psi}_f$ is omitted since the flow is incompressible). With Eqs. (23) and (36), and the explanations of Fig. 1, it is straightforward to write the Fokker-Planck equation verified by $F_f^E(t,\mathbf{x};\mathbf{V}_f)$,

$$\frac{\partial F_f^E}{\partial t} + V_{f,i}\frac{\partial F_f^E}{\partial x_i} = -\frac{\partial}{\partial V_{f,i}}(A_{f,i}F_f^E) + \frac{1}{2}\frac{\partial^2}{\partial V_{f,i}\partial V_{f,j}}([B_jB_f^T]_{ij}F_f^E) - \frac{\partial}{\partial V_{f,i}}(\langle A_{p\to f,i}|\mathbf{x},\mathbf{V}_f\rangle F_f^E), \tag{43}$$

and if one multiplies Eq. (43) by $\mathcal{H}_f$ and applies the $\langle\cdot\rangle$ operator, one can write after some algebra

$$\frac{\partial}{\partial t}(\alpha_f\rho_f\langle\mathcal{H}_f\rangle) + \frac{\partial}{\partial x_i}(\alpha_f\rho_f\langle V_{f,i}\mathcal{H}_f\rangle) = \alpha_f\rho_f\left\langle A_{f,i}\frac{\partial\mathcal{H}_f}{\partial V_{f,i}}\right\rangle + \frac{1}{2}\alpha_f\rho_f\left\langle (B_fB_f^T)_{ij}\frac{\partial^2\mathcal{H}_f}{\partial V_{f,i}\partial V_{f,j}}\right\rangle + \int (A_{p\to f,i}|\mathbf{x},\mathbf{V}_f)\frac{\partial\mathcal{H}_f}{\partial V_f}F_f^E(t,\mathbf{x};\mathbf{V}_f)d\mathbf{V}_f. \tag{44}$$

In this derivation, it has been supposed that $\mathbf{A}_f$, $\mathbf{B}_f$, and $\mathcal{H}_f$ are sufficiently smooth so that all generalized integrals converge (by construction, F_f^E and $\partial F_f^E/\partial V_{f,i}$ converge to zero when, at least, one component of the fluid velocity goes to infinity, $V_{f,i}\to\pm\infty$). By replacing $\mathcal{H}_f$ by $\mathcal{H}_f=1$, $\mathcal{H}_f=V_{f,i}$ and $\mathcal{H}_f=V_{f,i}V_{f,j}$, the continuity equation the momentum equations (mean field equations for $\mathbf{U}_f$), and the Reynolds-stress equations (mean field equations for $\langle\mathbf{u}_f\mathbf{u}_f\rangle$) are obtained, respectively. The Reynolds-stress equations can also be derived using another route, i.e., by making a change of coordinates in sample space (this procedure will be outlined in the next section). The continuity and momentum equations are

$$\frac{\partial}{\partial t}(\alpha_f\rho_f) + \frac{\partial}{\partial x_i}(\alpha_f\rho_f\langle U_{f,i}\rangle) = 0, \tag{45}$$

$$\alpha_f\rho_f\frac{D_f}{Dt}\langle U_{f,i}\rangle = -\frac{\partial}{\partial x_j}(\alpha_f\rho_f\langle u_{f,i}u_{f,j}\rangle) + \alpha_f\rho_f\langle A_{f,i}\rangle + I_{f,i}^M \tag{46}$$

and after some algebra, the Reynolds-stress equations are given by

$$\alpha_f\rho_f\frac{D_f}{Dt}\langle u_{f,i}u_{f,j}\rangle = -\frac{\partial}{\partial x_k}(\alpha_f\rho_f\langle u_{f,i}u_{f,j}u_{f;k}\rangle) - \alpha_f\rho_f\langle u_{f,i}u_{f,k}\rangle\frac{\partial\langle U_{f,j}\rangle}{\partial x_k} - \alpha_f\rho_f\langle u_{f,j}u_{f,k}\rangle\frac{\partial\langle U_{f,i}\rangle}{\partial x_k} + \alpha_f\rho_f\langle A_{f,i}v_{f,j} + A_{f,j}v_{f,i}\rangle + \alpha_f\rho_f\langle (B_fB_f^T)_{ij}\rangle + I_{f,ij}^R, \tag{47}$$

where the Eulerian derivative along the path of a fluid particle is denoted D_f/Dt with $D_f/Dt = \partial/\partial t + \langle U_{f,m}\rangle\partial/\partial x_m$. In both the momentum and Reynolds-stress equations, there is a

TABLE I. Definition of the second-order moments: all moments attached to the discrete particles are calculated with Eq. (38) whereas the Reynolds stresses are obtained from Eq. (39).

Second-order moment	n	m	p	Variable
Reynolds stresses	2			$\langle u_{f,i}u_{f,j}\rangle$
Second-order particle velocity moment	0	0	2	$\langle u_{p,i}u_{p,j}\rangle$
Second-order fluid velocity seen moment	0	2	0	$\langle u_{s,i}u_{s,j}\rangle$
Fluid-particle velocity correlation tensor	0	1	1	$\langle u_{p,i}u_{s,j}\rangle$
Diameter-particle velocity correlation tensor	1	0	1	$\langle d_p' u_{p,i}\rangle$
Diameter-fluid velocity seen correlation tensor	1	1	0	$\langle d_p' u_{s,i}\rangle$
Second-order diameter moment	2	0	0	$\langle (d_p')^2\rangle$

term ($I^M_{f,i}$ for the momentum equations and $I^R_{f,ij}$ for the Reynolds-stress equations) that accounts for the influence of the discrete particles on the fluid phase. For both terms, further considerations are necessary before their final form can be obtained. Let us start with $I^M_{f,i}$.

The term $I^M_{f,i}$ is expressed as a function of the conditional expectation of a random variable (as it has been suggested in Sec. III D 4)

$$I^M_{f,i}=\frac{\alpha_p}{\alpha_f}\int \langle \Pi_{p,i}|\mathbf{V}_f\rangle F^E_f(t,\mathbf{x};\mathbf{V}_f)d\mathbf{V}_f, \tag{48}$$

where the ratio α_p/α_f expresses the probability to have a random force conditioned on the fact that there is a fluid particle at $\mathbf{x}$. Let $\mathbf{\Upsilon}_p$ be the sample-space value for the random variable $\mathbf{\Pi}_p$ at time t and location $\mathbf{x}$. It is then possible to rewrite the previous equation as [where $p(\mathbf{\Upsilon}_p|\mathbf{V}_f)$ is the PDF of $\mathbf{\Pi}_p$ conditioned upon $\mathbf{U}_f$]

$$I^M_{f,i}=\frac{\alpha_p}{\alpha_f}\int \Upsilon_{p,i}p(\mathbf{\Upsilon}_p|\mathbf{V}_f)F^E_f(t,\mathbf{x};\mathbf{V}_f)d\mathbf{V}_f\,d\mathbf{\Upsilon}_p. \tag{49}$$

Let us introduce the joint PDF of $\mathbf{\Pi}_p$ and $\mathbf{U}_f$, $p(\mathbf{\Upsilon}_p,\mathbf{V}_f)$, and the definition of the conditional PDF $p(\mathbf{\Upsilon}_p,\mathbf{V}_f)=p(\Upsilon_{p,i}|\mathbf{V}_f)p(\mathbf{V}_f)$, where $p(\mathbf{V}_f)$ denotes the normalized PDF of $\mathbf{U}_f$ at location $\mathbf{x}$. From the relations given in Sec. III C 2 one has directly $\alpha_f\rho_f p(\mathbf{V}_f)=F^E_f(t,\mathbf{x};\mathbf{V}_f)$ and therefore the term $I^M_{f,i}$ can be written as

$$I^M_{f,i}=\left(\frac{\alpha_p}{\alpha_f}\right)\alpha_f\rho_f\langle\Pi_{p,i}\rangle=-\alpha_p\rho_p\langle A^D_{p,i}\rangle. \tag{50}$$

The term $I^R_{f,ij}$ that enters the Reynolds-stress equations is expressed by

$$I^R_{f,ij}=(I^E_{f,ij}+I^E_{f,ji})-(\langle U_{f,i}\rangle I^M_{f,j}+\langle U_{f,j}\rangle I^M_{f,i}), \tag{51}$$

where

$$I^E_{f,ij}=\int \langle A_{p\to f,i}|\mathbf{x},\mathbf{V}_f\rangle V_{f,j}F^E_f(t,\mathbf{x};\mathbf{V}_f)d\mathbf{V}_f. \tag{52}$$

This form of $I^R_{f,ij}$ is easily found as follows: the PDE, which is written for the local instantaneous second-order moment $\langle U_{f,i}U_{f,j}\rangle$, is linearly combined with the PDE for $\langle U_{f,i}\rangle\langle U_{f,j}\rangle$. The latter PDE is obtained by developing $\langle U_{f,i}\rangle\mathcal{N}(\langle U_{f,j}\rangle)+\langle U_{f,j}\rangle\mathcal{N}(\langle U_{f,i}\rangle)=0$, where the operator $\mathcal{N}$ symbolizes Eq. (46). Using the same reasoning as in the case of the momentum equation and applying the model of Sec. III D 4 and the results of Sec. III C 2, the term $I^E_{f,ij}$ is expressed as

$$I^E_{f,ij}=\left(\frac{\alpha_p}{\alpha_f}\right)\alpha_f\rho_f\int \Upsilon_{p,i}V_{f,j}\,p(\Upsilon_{p,i},\mathbf{V}_f)d\mathbf{V}_f\,d\mathbf{\Upsilon}_p \tag{53}$$

that is,

$$I^E_{f,ij}=-\alpha_p\rho_p\langle A^D_{p,i}U_{s,j}\rangle, \tag{54}$$

and finally

$$I^R_{f,ij}=-\alpha_p\rho_p\langle A^D_{p,i}u_{s,j}+A^D_{p,j}u_{s,i}\rangle+\alpha_p\rho_p(\mathcal{U}_{d,i}\langle A^D_{p,j}\rangle + \mathcal{U}_{d,j}\langle A^D_{p,i}\rangle). \tag{55}$$

The expression of $I^R_{f,ij}$ is written in a form where the difference $\langle U_{f,j}\rangle-\langle U_{s,j}\rangle$, denoted $\mathcal{U}_{d,i}$, explicitly appears. This quantity represents, at a given time t and a given location $\mathbf{x}$, the difference between the expected fluid velocity and the expected fluid velocity seen by the discrete particles.

Finally, in Table II, a list of the terms to be closed is given. A distinction is made between the unclosed terms and the third-order moments that appear naturally in the PDE verified by $\langle \mathbf{u}_f\mathbf{u}_f\rangle$. The closure issue will be addressed in Sec. VI where a closed two-field model is derived.

TABLE II. List of the unknown terms in the mean field equations of the continuous phase.

Equation	Variable	Unclosed term	Third-order moment
Eq. (46)	$\langle U_f\rangle$	$I^M_{pf,i}\langle A_{f,i}\rangle$	
Eq. (47)	$\langle u_{f,i}u_{f,j}\rangle$	$I^R_{pf,ij}\langle A_f,u_{f,j}\rangle\langle (B_fB_f^T)_{ij}\rangle$	$\langle u_{f,i}u_{f,j}u_{f,k}\rangle$

C. Mean field equations for the discrete phase

The mean field equations for the discrete phase are now obtained following the procedure presented in Sec. IV B when deriving mean field equations for the fluid phase. The expectation of a given function $\mathcal{H}_p(\mathbf{V}_p,\boldsymbol{\psi}_p)$ is defined by Eq. (37). Using Eqs. (23) and (36), and the explanations of Fig. 1, it is straightforward to prove that $F_p^E(t,\mathbf{x};\mathbf{V}_p,\boldsymbol{\psi}_p)$ verifies the following Fokker-Planck equation:

$$\frac{\partial F_p^E}{\partial t}+V_{p,i}\frac{\partial F_p^E}{\partial x_i}=-\frac{\partial}{\partial V_{p,i}}(A_{p,i}F_p^E)-\frac{\partial}{\partial V_{s,i}}(A_{s,i}F_p^E)+\frac{1}{2}\frac{\partial^2}{\partial V_{s,i}\partial V_{s,j}}[(B_sB_s^T)_{ij}F_p^E]-\frac{\partial}{\partial V_{s,i}}(\langle A_{p\to s,i}|\mathbf{x},\mathbf{V}_p,\boldsymbol{\psi}_p\rangle F_p^E). \quad (56)$$

As it was done in the previous section, Eq. (56) is multiplied by $\mathcal{H}_p$ and the $\langle\cdot\rangle$ operator is applied. Then, as done in Sec. IV B, $\mathbf{A}_s$, $\mathbf{A}_p$, $\mathbf{B}_s$, and $\mathcal{H}_p$ are assumed to be sufficiently smooth so that all generalized integrals converge (F_p^E, $\partial F_p^E/\partial V_{s,i}$, and $\partial F_p^E/\partial V_{p,i}$ converge to zero when, at least, one component of $\mathbf{V}_s$ or $\mathbf{V}_p$ goes to infinity). After some derivations, one can write

$$\begin{aligned}&\frac{\partial}{\partial t}(\alpha_p\rho_p\langle\mathcal{H}_p\rangle)+\frac{\partial}{\partial x_i}(\alpha_p\rho_p\langle V_{p,i}\mathcal{H}_p\rangle)\\&=\alpha_p\rho_p\left\langle A_{p,i}\frac{\partial\mathcal{H}_p}{\partial V_{p,i}}\right\rangle+\alpha_p\rho_p\left\langle A_{s,i}\frac{\partial\mathcal{H}_p}{\partial V_{s,i}}\right\rangle\\&\quad+\frac{1}{2}\alpha_p\rho_p\left\langle(B_sB_s^T)_{ij}\frac{\partial^2\mathcal{H}_p}{\partial V_{s,i}\partial V_{s,j}}\right\rangle\\&\quad+\int\langle A_{p\to s,i}|\mathbf{x},\mathbf{V}_p,\boldsymbol{\psi}_p\rangle\frac{\partial\mathcal{H}_p}{\partial V_{s,i}}F_p^E(t,\mathbf{x};\mathbf{V}_p,\boldsymbol{\psi}_p)d\mathbf{V}_p d\boldsymbol{\psi}_p.\end{aligned} \quad (57)$$

The PDEs for the specified discrete particle expectations can now be derived, simply by choosing the right function for $\mathcal{H}_p$. $\mathcal{H}_p=1,V_{p,i},V_{s,i},\delta_p$ gives the continuity equation, the momentum equation, the PDE verified by the expected fluid velocity seen, and the PDE for the mean diameter, respectively. These equations have the form

$$\frac{\partial}{\partial t}(\alpha_p\rho_p)+\frac{\partial}{\partial x_i}(\alpha_p\rho_p\langle U_{p,i}\rangle)=0, \quad (58a)$$

$$\alpha_p\rho_p\frac{D_p}{Dt}\langle U_{p,i}\rangle=-\frac{\partial}{\partial x_j}(\alpha_p\rho_p\langle u_{p,i}u_{p,j}\rangle)+\alpha_p\rho_p\langle A_{p,i}\rangle, \quad (58b)$$

$$\alpha_p\rho_p\frac{D_p}{Dt}\langle U_{s,i}\rangle=-\frac{\partial}{\partial x_j}(\alpha_p\rho_p\langle u_{s,i}u_{p,j}\rangle)+\alpha_p\rho_p\langle A_{s,i}\rangle-\alpha_p\rho_p\chi\langle A_{p,i}^D\rangle, \quad (58c)$$

$$\alpha_p\rho_p\frac{D_p}{Dt}\langle d_p\rangle=-\frac{\partial}{\partial x}(\alpha_p\rho_p\langle d_p'u_{p,i}\rangle), \quad (58d)$$

where $\chi=(\alpha_p\rho_p)/(\alpha_f\rho_f)$ and where the Eulerian derivative along the path of a discrete particle is denoted D_p/Dt with $D_p/Dt=\partial/\partial t+\langle U_{p,m}\rangle\partial/\partial x_m$.

The mean field equations verified by the second-order moments [$n+m+q=2$ in Eq. (38)] can be derived in a way that requires less algebra than the procedure that has been outlined so far by using Eq. (57) and the right function for $\mathcal{H}_p$. By introducing a change of coordinates in sample space, $\mathbf{v}=\mathbf{V}-\langle\mathbf{U}\rangle(t,\mathbf{x})$ and $\delta_p'=\delta_p-\langle d_p\rangle(t,\mathbf{x})$ (where $\mathbf{v}$ stands for $\mathbf{v}_p$ or $\mathbf{v}_s$), it is straightforward to write the Fokker-Planck equation verified by $F_p^E(t,\mathbf{x};\mathbf{v}_p,\mathbf{v}_s,\delta_p')$. Then using the usual technique, the PDE verified by a given function $\mathcal{H}_p(\mathbf{v}_p,\mathbf{v}_s,\delta_p)$ is derived in the same fashion as for Eq. (57) where similar conditions for the convergence of the generalized integrals are required. This procedure is not detailed here but it can be found with all necessary derivations in Ref. [12]. After some algebra, one finds for $\langle u_{p,i}u_{p,j}\rangle$,

$$\begin{aligned}\alpha_p\rho_p\frac{D_p}{Dt}\langle u_{p,i}u_{p,j}\rangle=&-\frac{\partial}{\partial x_k}(\alpha_p\rho_p\langle u_{p,i}u_{p,j}u_{p,k}\rangle)\\&-\alpha_p\rho_p\langle u_{p,i}u_{p,k}\rangle\frac{\partial\langle U_{p,j}\rangle}{\partial x_k}\\&-\alpha_p\rho_p\langle u_{p,j}u_{p,k}\rangle\frac{\partial\langle U_{p,i}\rangle}{\partial x_k}\\&+\alpha_p\rho_p\langle A_{p,i}v_{p,j}+A_{p,j}v_{p,i}\rangle,\end{aligned} \quad (59)$$

for $\langle u_{p,i}u_{s,j}\rangle$,

$$\begin{aligned}\alpha_p\rho_p\frac{D_p}{Dt}\langle u_{s,i}u_{p,j}\rangle=&-\frac{\partial}{\partial x_k}(\alpha_p\rho_p\langle u_{s,i}u_{p,j}u_{p,k}\rangle)\\&-\alpha_p\rho_p\langle u_{s,i}u_{p,k}\rangle\frac{\partial\langle U_{p,j}\rangle}{\partial x_k}\\&-\alpha_p\rho_p\langle u_{p,j}u_{p,k}\rangle\frac{\partial\langle U_{s,i}\rangle}{\partial x_k}\\&+\alpha_p\rho_p\langle A_{s,i}v_{p,j}\rangle+\alpha_p\rho_p\langle A_{p,j}v_{s,i}\rangle\\&-\alpha_p\rho_p\chi\langle A_{p,i}^Du_{p,j}\rangle,\end{aligned} \quad (60)$$

and for $\langle u_{s,i}u_{s,j}\rangle$,

$$\begin{aligned}\alpha_p\rho_p\frac{D_p}{Dt}\langle u_{s,i}u_{s,j}\rangle=&-\frac{\partial}{\partial x_k}(\alpha_p\rho_p\langle u_{s,i}u_{s,j}u_{s,k}\rangle)-\alpha_p\rho_p\langle u_{s,i}u_{s,k}\rangle\frac{\partial\langle U_{s,j}\rangle}{\partial x_k}\\&-\alpha_p\rho_p\langle u_{s,j}u_{s,k}\rangle\frac{\partial\langle U_{s,i}\rangle}{\partial x_k}+\alpha_p\rho_p\langle A_{s,j}v_{s,i}+A_{s,i}v_{s,j}\rangle\\&+\alpha_p\rho_p\langle(B_sB_s^T)_{ij}\rangle-\alpha_p\rho_p\chi\langle A_{p,j}^Du_{s,i}+A_{p,i}^Du_{s,j}\rangle.\end{aligned} \quad (61)$$

After some calculus, one finds for $\langle d_p'u_{p,i}\rangle$,

$$\alpha_p\rho_p\frac{D_p}{Dt}\langle d_p'u_{p,i}\rangle = -\frac{\partial}{\partial x_j}(\alpha_p\rho_p\langle d_p'u_{p,i}u_{p,j}\rangle) - \alpha_p\rho_p\langle d_p'u_{p,j}\rangle\frac{\partial\langle U_{p,i}\rangle}{\partial x_j} - \alpha_p\rho_p\langle u_{p,i}u_{p,j}\rangle\frac{\partial\langle d_p\rangle}{\partial x_j} + \alpha_p\rho_p\langle A_{p,i}d_p'\rangle, \tag{62}$$

for $\langle d_p'u_{s,i}\rangle$,

$$\alpha_p\rho_p\frac{D_p}{Dt}\langle d_p'u_{s,i}\rangle = -\frac{\partial}{\partial x_j}(\alpha_p\rho_p\langle d_p'u_{s,i}u_{p,j}\rangle) - \alpha_p\rho_p\langle d_p'u_{p,j}\rangle\frac{\partial\langle U_{s,i}\rangle}{\partial x_j} - \alpha_p\rho_p\langle u_{s,i}u_{p,j}\rangle\frac{\partial\langle d_p\rangle}{\partial x_j} + \alpha_p\rho_p\langle A_{s,i}d_p'\rangle - \alpha_p\rho_p\chi\langle A_{p,i}^D d_p'\rangle, \tag{63}$$

and for $\langle (d_p')^2\rangle$,

$$\alpha_p\rho_p\frac{D_p}{Dt}\langle (d_p')^2)\rangle = -\frac{\partial}{\partial x_i}(\alpha_p\rho_p\langle (d_p')^2u_{p,i}\rangle) - 2\alpha_p\rho_p\langle d_p'u_{p,i}\rangle\frac{\partial\langle d_p\rangle}{\partial x_i}, \tag{64}$$

which concludes the set of mean field equations for the second-order moments related to the discrete phase. A first glance at the amount of terms that are unknown, not to mention the third-order moments, gives an insight into the intricate work which is left, that is to provide suitable closures in order to use the model in practical computations. Table III gives a list of all unknown terms that appear in the mean field equations for the discrete phase. This information has to be supplemented by Table II to obtain the total amount of unknown terms. The wise reader has already realized that a tremendous work is waiting and it is already clear that such an approach (mean field equations) can only be used in practice when further contractions are possible (the dimension of the system is 46 with 19 unknown terms, which is nearly intractable), that is in a simplified case. This is the subject of Sec. VI where an example of a practical model is given in the case where there is no size distribution for the discrete particles, i.e., the particle diameter is eliminated from the state vector. As mentioned in the Introduction, there are alternatives to the mean field equations, for example, the Lagrangian approach. It is now shown that, with the Lagrangian approach, a great deal of the difficulties, created by the treatment of the discrete phase with mean field equations, can be eliminated.

V. MEAN FIELD–PDF APPROACH

In the previous section, the derivation of the mean field equations (up to the second-order moments) for both phases has been presented. It has been explained, in both Tables II and III, that apart from the third-order moments, writing mean field equations is a move that generates, from a closed mesoscopic model, unclosed terms at the macroscopic level. Indeed, one has to express the expectations of complex nonlinear expressions of the basic variables, for example, terms such as $\langle A_i^D\rangle$ or $\langle A_i^D u_{s,i}\rangle$.

There is a natural way to avoid most of these difficulties, which is the Lagrangian approach. In fact, in such an approach, mean field equations are used solely for the fluid phase whereas for the discrete phase information is still available at the mescocopic level, see Fig. 1. Actually, the term ''Lagrangian approach'' is rather misleading. Here the approach is called from now on *mean field*–PDF *approach* since mean fields are used for the description of the fluid and a PDF is employed for the discrete phase. It is now clear that such a method *contains* the mean field approach. Although the description of the fluid is identical for the two-field model and the mean field–PDF model, the treatment of the discrete phase is different. In the mean field–PDF approach, information is available for the local instantaneous values of any variable attached to the discrete phase whereas in the two-field approach a contraction has been made (only the two first moments of the variables of interest are available). This distinction is apparently often missed in the literature. The two models are frequently compared without specifying that they do not correspond to the same level of information. As a matter of fact, if such a comparison is attempted, one should not only look at the computational effort and the easiness of implementation but also at the information that can be extracted from the model. If one is interested in, for instance, discrete-particle residence time or conditional statistics, the mean field–PDF approach can provide such information but the two-field model cannot.

The mean field–PDF model equations are, for the fluid, Eqs. (45) to (47), and for the particles Eqs. (30). The closure problem has been greatly simplified since the exhaustive list of Table III disappears. Closure is now limited to Table II.

VI. PRESENTATION AND HIERARCHY OF PRACTICAL MODELS

So far, a probabilistic approach to polydispersed turbulent two-phase flows has been presented. From it, mean field equations have been written for the expectations and the second-order moments of the variables that form the state vector $\mathbf{Z}$. The SDEs, proposed for the trajectories of $\mathbf{Z}$, and the mean field equations, derived for $\langle\mathbf{Z}\rangle$ and $\langle\mathbf{ZZ}\rangle$, are limited to the case of nonreacting turbulent polydispersed two-phase flows where the fluid is incompressible and the particles are noncolliding hard spheres. Yet, the mean field equations, to be used in practical simulations, present an intricate challenge: the dimension of the problem is very large and the amount of information that has to be closed is far from being negligible. It is possible, however, in practice, to use the mean field equations for computations if further as-

TABLE III. List of the unknown terms in the mean field equations of the discrete phase.

Equation	Variable	Unclosed term	Third-order moment
Eq. (58b)	$\langle U_p \rangle$	$\langle A_{p,i} \rangle$	
Eq. (58c)	$\langle U_s \rangle$	$\langle A^D_{p,i} \rangle \langle A_{s,i} \rangle$	
Eq. (58d)	$\langle d_p \rangle$		
Eq. (59)	$\langle u_{p,i} u_{p,j} \rangle$	$\langle A_{p,i} u_{p,j} \rangle$	$\langle u_{p,i} u_{p,j} u_{p,k} \rangle$
Eq. (61)	$\langle u_{s,i} u_{s,j} \rangle$	$\langle A_{s,j} u_{s,i} \rangle \langle A^D_{p,i} u_{s,j} \rangle \langle (B_s B_s^T)_{ij} \rangle$	$\langle u_{s,i} u_{s,j} u_{s,k} \rangle$
Eq. (60)	$\langle u_{s,i} u_{p,j} \rangle$	$\langle A_{p,j} u_{s,i} \rangle \langle A_{s,j} u_{p,i} \rangle \langle A^D_{p,j} u_{p,i} \rangle$	$\langle u_{s,i} u_{p,j} u_{p,k} \rangle$
Eq. (62)	$\langle d'_p u_{p,i} \rangle$	$\langle A_{p,i} d'_p \rangle$	$\langle d'_p u_{p,i} u_{p,j} \rangle$
Eq. (63)	$\langle d'_p u_{s,i} \rangle$	$\langle A_{s,i} d'_p \rangle$	$\langle d'_p u_{s,i} u_{p,j} \rangle$
Eq. (64)	$\langle (d'_p)^2 \rangle$		$\langle (d'_p)^2 u_{p,i} \rangle$

sumptions are made on the nature of the flow, i.e., if further contractions are made [24].

Before presenting the form of the mean field equations that are used in simulations, let us show an example of a two-point PDF model from which practical Lagrangian and Eulerian models can be extracted. The presentation follows the hierarchy of models that has been displayed in the previous sections: (i) a two-point PDF model where information is available at the mesoscopic level for both phases, (ii) a mean-field–PDF model where a first contraction is made for the fluid and (iii) a two-field model where further contraction is performed (information is given solely for the two first moments of both phases).

A. Example of a two-point PDF model

From now on, the external force field $\mathbf{F}_E$ is reduced to gravity $\mathbf{g}$. In the single-phase flow case, one can use the RSM (Reynolds-stress model) equations since there is a direct connection between a Langevin equation and a RSM equation [10]. However, another route can be chosen using arguments from statistical physics [25]. The model is given by

$$A_{f,i} = -\frac{1}{\rho_f}\frac{\partial \langle P \rangle}{\partial x_i} + g_i - \frac{U_{f,i} - \langle U_{f,i} \rangle}{T_L} + G^a_{ij}(U_{f,j} - \langle U_{f,j} \rangle), \tag{65}$$

$$B_{f,ij} = \sqrt{C_0 \langle \epsilon \rangle}\, \delta_{ij}, \tag{66}$$

where $\langle P \rangle(t, \mathbf{x}_f)$ is the mean pressure field and $\langle \epsilon \rangle(t, \mathbf{x}_f)$ represents the mean dissipation rate of turbulent kinetic energy, $2k_f = \langle \mathbf{u}_f^2 \rangle(t, \mathbf{x}_f)$. T_L is the fluid integral Lagrangian time scale, i.e., $T_L = Ck_f / \langle \epsilon \rangle$, where $1/C = 1/2 + 3C_0/4$ and C_0 is a constant (given by Kolmogorov theory). G^a_{ij} is an anisotropy matrix (whose precise form is not important here) that depends *solely* on the moments of $\mathbf{Z}_f$.

The construction of the SDE for the simulation of the time increments of the fluid velocity seen is an open question [12,21] and the form that is proposed here belongs to a set of possible solutions. The present model is expressed by [12]

$$A_{s,i} = -\frac{1}{\rho_f}\frac{\partial \langle P \rangle}{\partial x_i} + g_i - G_{ij}(U_{s,j} - \langle U_{s,j} \rangle) + (\langle U_{p,j} \rangle - \langle U_{s,j} \rangle)\frac{\partial \langle U_{f,i} \rangle}{\partial x_j}, \tag{67}$$

$$B^2_{s,ii} = \langle \epsilon \rangle \left[C_0 b_i \frac{\tilde{k}_f}{k_f} + \frac{2}{3}\left(b_i \frac{\tilde{k}_f}{k_f} - 1 \right) \right], \tag{68}$$

where $B_{s,ij}$ is expressed in its simplest form (as a diagonal and nonisotropic matrix) and G_{ij} is given by $G_{ij} = \delta_{ij}/T^*_{L,i}$. $T^*_{L,i}$ is the fluid integral Lagrangian time scale sampled along the trajectories of the discrete particles. This time scale can be evaluated by a formula due to Csanady [26] as (if gravity is aligned with the first coordinate labeled 1)

$$T^*_{L,i} = T_L \Big/ \left(1 + \mathcal{C}_i \frac{|\langle \mathbf{U}_r \rangle|^2}{2k_f/3} \right)^{1/2}, \tag{69}$$

where $C_1 = \beta^2$, $C_2 = C_3 = 4\beta^2$ and β is the ratio of the fluid Lagrangian and Eulerian integral time scales, $\beta = T_L/T_E$. The formula of Csanady implies that, even in the simplest case, G_{ij} is a nonisotropic diagonal matrix. To complete the definition of Eq. (68),

$$\tfrac{2}{3}\tilde{k}_f = \sum_{i=1}^{3} b_i \langle u_i^2 \rangle \Big/ \sum_{i=1}^{3} b_i, \tag{70}$$

where $b_i = T_L/T^*_{L,i}$. In practice, the PDF approach given by Eqs. (31) is not self-sufficient since one has to compute the mean pressure field $\langle P \rangle(t, \mathbf{x}_f)$ and the mean dissipation of turbulent kinetic energy, $\langle \epsilon \rangle(t, \mathbf{x}_f)$. It is possible, following a procedure outlined by Pope [9], to derive for each phase a Poisson equation verified by the mean pressure. This Poisson equation, for each phase, guarantees that the continuity equation of each phase is satisfied. The sum of both equations is a Poisson equation that expresses the mass conservation of the mixture. For $\langle \epsilon \rangle$, a model, based on Kolmogorov's third hypothesis [$\ln(\epsilon)$ is normally distributed in homogeneous turbulence], has been developed [27]. It consists in writing a Langevin equation for the frequency rate $\omega = \epsilon/k_f$ along the trajectories of the fluid particles,

$$d\omega=[A^{\omega}+A^{\omega}_{p\to f}]dt+B^{\omega}dW, \tag{71}$$

where the drift and diffusion coefficients, A^{ω} and B^{ω} respectively, are specified in Ref. [27]. The new state vector becomes $\mathbf{Z}=(\mathbf{Z}_f,\mathbf{Z}_p,\omega)$ and A^{ω} and B^{ω} are functions of t, $\mathbf{Z}$ and its moments. Equation (71) has been supplemented by a term $A^{\omega}_{p\to f}$ accounting for the influence of the particles on the local instantaneous frequency. A possible model for this term reads

$$A^{\omega}_{p\to f}=G^{\omega}\mathbf{\Pi}_{p,i}V_{f,i}, \tag{72}$$

which means that the influence of the particles on ω is related to the local instantaneous work performed by the action of the particles on the fluid multiplied by a coefficient that is a function of the moments of $\mathbf{Z}$ *only*. This subject is not dwelt upon here and only the reduced state vector $\mathbf{Z}=(\mathbf{Z}_f,\mathbf{Z}_p)$ is under investigation. For the complete state vector, the Fokker-Planck equation, Eq. (36), can easily be extended by adding the additional terms generated by the additional dimension (ω).

We do not dwell upon the different models that can be derived for the drift vectors and the diffusion matrices and we try instead to retain the main features that are of importance for the derivation of the mean field equations. Detailed discussions and proofs of the modeling choices can be found in Ref. [12].

B. Derivation of a practical mean field-PDF model

In the case of the fluid, the only difficulty (if one accepts a gradient hypothesis for the third-order moments) when writing mean field equations is the coupling terms both in the momentum equations and the RSM equations. In the case of the discrete phase, a closed mesoscopic model, in the form of Langevin equations, has already been provided, Eqs. (67) and (68).

When the PDEs for the fluid are solved by classical methods (for example, finite volume methods) the coupling terms in a computational cell can then be calculated in a natural way by making the classical hypotheses, (i) homogeneity in space, and (ii) the approximation of the expectation by ensemble averaging when the number of samples in the cell is sufficient. For example, for the coupling term in the momentum equations, let $\mathcal{V}_f$ be the volume of the computational cell that contains N_p discrete particles. The total force in the cell due to the N_p particles is (with N_p large enough)

$$\mathbf{F}_{p\to f}=\sum_{n=1}^{N_p}\rho_p\mathcal{V}_p^n\mathbf{A}_p^{D,n}, \tag{73}$$

where $\mathcal{V}_p^n$ is the volume of the discrete particle labeled n. The previous equation is indeed the mean momentum exchange [using $\alpha_p\simeq(\Sigma_{n=1}^{N_p}\mathcal{V}_p^n)/\mathcal{V}_f$]

$$\langle\mathbf{F}_{p\to f}\rangle=\alpha_p\rho_p\mathcal{V}_f\langle\mathbf{A}_p^D\rangle. \tag{74}$$

One can, therefore, see that complicated nonlinear terms such as $\langle\mathbf{A}_p^D\rangle$ are computed in a simple and natural way. The same procedure can be applied to the coupling term in the RSM equations, $\langle F_{p\to f,i}U_{f,j}\rangle$.

The level of simplicity of the mean field-PDF model (in comparison to the simplified two-field model that is going to be presented in Sec. VI C) and its ability to compute complicated terms in the polydispersed case are noteworthy. It is a school example of the reason why PDF approaches should be preferred to mean field approaches in cases where the physics of the flow becomes complex. Examples of practical computations performed with the mean field-PDF approach can be found in Ref. [12].

C. Derivation of a practical two-field model

The derivation of the model is treated in two different steps. First, a list of hypotheses that make it possible to reduce the dimension d of the system is given and second, it is shown how to close the n unclosed terms by making further hypotheses. It is recalled that, in the frame of our study that is limited to second-order moments, $d=46$ and $n=12$ [seven third-order moments and now five unclosed terms, the other ones can be directly closed, see Eqs. (65)–(70) and Tables II and III].

1. Reduction of the dimension of the system

All hypotheses needed to reduce the dimension of the original system are now given.

(i) Let us consider the case where the distribution in diameter of the discrete particles is ''narrow'' enough so that the statistics involving the diameter can roughly be approximated as constant in time and space (this hypothesis is equivalent to state that the suspension is monodispersed and therefore segregation effects cannot be quantified anymore). The sample space is only reduced by one dimension but eight PDEs have already been eliminated (all PDEs involving the particle diameter), that is the new dimension of the system is $d=38$.

(ii) Furthermore, it is assumed that there is no statistical bias between the statistics of the second-order moments for the fluid velocity seen and for the fluid velocity, i.e., $\langle\mathbf{u}_s\mathbf{u}_s\rangle\simeq\langle\mathbf{u}_f\mathbf{u}_f\rangle$.

So far, with two hypotheses, which severely limit the number of problems which can be treated, the dimension of the problem has been reduced to $d=32$ but many unclosed terms remain $n=9$ (four third-order moments and five unclosed terms).

2. Treatment of the unclosed terms

Let us now enumerate the hypotheses that are necessary to make in order to treat the unclosed terms (the hypotheses are numbered in continuation with the previous subsection and this for the sake of clarity).

(iii) It is assumed that all third-order tensors $\langle Z_iZ_jZ_k\rangle$ can be expressed as a function of what is known (or solved for), that is, $\langle Z_i\rangle$ and $\langle Z_iZ_j\rangle$ (a macroscopic closure is made with all the difficulties and the risks that such a move entails). This is done in analogy with classical turbulence results where one seeks macroscopic relations of the type (k is the phase index)

$$-\langle u_{k,i}u_{k,j}u_{k,m}\rangle = D_{k,mn}(\langle \mathbf{Z}\rangle,\langle \mathbf{ZZ}\rangle)\frac{\partial}{\partial x_n}\langle u_{k,i}u_{k,j}\rangle, \tag{75}$$

and

$$-\langle u_{s,i}u_{p,j}u_{p,m}\rangle = D_{sp,mn}(\langle \mathbf{Z}\rangle,\langle \mathbf{ZZ}\rangle)\frac{\partial}{\partial x_n}\langle u_{s,i}u_{p,j}\rangle, \tag{76}$$

where $D_{k,ij}$ and $D_{sp,ij}$ are often called ''turbulent diffusion tensors.'' Their possible forms are not given here but can be found, for example, in Ref. [28].

(iv) It is assumed that, for heavy particles, a good approximation of $\mathbf{A}_p$ is [Eq. (28) is supplemented with the mean pressure gradient]

$$A_{p,i} = -\frac{1}{\rho_p}\frac{\partial\langle P\rangle}{\partial x_i} + A^D_{p,i} + g_i, \tag{77}$$

where $\mathbf{A}^D_p$ is defined by Eq. (32). In this expression, the influence of pressure fluctuations has explicitly been neglected. Indeed, the pressure that should be involved in the gradient is the local instantaneous pressure seen, $P_s(t,x_p(t))$, along the path of the discrete particle, which is different from the pressure field $P(t,x_f(t))$ seen around the fluid particles. Therefore, it has been assumed that the field $P_s-\langle P\rangle$ has no influence on the motion of the discrete particles.

(v) Let us suppose that all terms involving $\mathbf{A}^D_p$ can be linearized as follows (where, in our case, $\mathcal{G}(\mathbf{u})$ is a linear function of $\mathbf{u}=\mathbf{1},\mathbf{u}_s,\mathbf{u}_p$):

$$\langle \mathbf{A}^D_p\mathcal{G}(\mathbf{u})\rangle = \frac{1}{\langle\tau_p\rangle}\langle(\mathbf{U}_s-\mathbf{U}_p)\mathcal{G}(\mathbf{u})\rangle, \tag{78}$$

$$\langle\tau_p\rangle = \tau_p(\langle \mathbf{Z}\rangle,\langle \mathbf{ZZ}\rangle). \tag{79}$$

This final assumption allows us to close all terms involving $\mathbf{A}^D_p$ and more generally $\mathbf{A}_p$. After a subsequent number of hypotheses, it now possible to finalize our task and give a simplified, but still quite intricate ($d=32$), closed two-field model.

3. Finalization of the model

It is now straightforward to write the set of continuity and momentum equations for both phases. The continuity equations are given by Eqs. (45) and (58a). For the momentum equations, one has

$$\alpha_k\rho_k\frac{D_k}{Dt}\langle U_{k,i}\rangle = -\alpha_k\frac{\partial\langle P\rangle}{\partial x_i} - \frac{\partial}{\partial x_j}(\alpha_k\rho_k\langle u_{k,i}u_{k,j}\rangle) + I_{k,i} + \alpha_k\rho_k g_i, \tag{80}$$

where the interaction term $l_{k,i}$ is simply given by linearization of $\mathbf{A}^D_p$ as explained in hypothesis (v) (the precise form of the calculation of $\langle\tau_p\rangle$ is not given here but can be found in Ref. [28]),

$$I_{k,i} = \mathcal{I}_k\frac{\alpha_p\rho_p}{\langle\tau_p\rangle}(\langle U_{s,i}\rangle - \langle U_{p,i}\rangle). \tag{81}$$

Here, $\mathcal{I}_k$ is equal to 1 if $k=p$ and to -1 if $k=f$. Bearing in mind the models presented above and hypotheses (iii)–(v), the remaining equations can now be written. For the expectation of the fluid velocity seen, $\langle\mathbf{U}_s\rangle$, it is found that the sum of the terms to close read [the two last terms on the right-hand side of Eq. (58c)]

$$\alpha_p\rho_p\left[-\frac{1}{\rho_f}\frac{\partial\langle P\rangle}{\partial x_i} + g_i + (\langle U_{p,j}\rangle - \langle U_{s,j}\rangle)\frac{\partial\langle U_{f,i}\rangle}{\partial x_j}\right] - \chi I_{p,i}, \tag{82}$$

where, as mentioned before, $\chi=\alpha_p\rho_p/\alpha_f\rho_f$. For the RSM equations, as done for the expected fluid velocity seen, only the sum of the terms to close is given, and this for the sake of clarity. The closure of the third-order moments is immediate by resorting to hypothesis (iii). Using hypothesis (ii), the sum of the three last terms on the right-hand side of Eq. (59) is

$$\begin{aligned}&-\frac{\alpha_f\rho_f}{T}[\langle u_{f,i}u_{f,j}\rangle - \tfrac{2}{3}k_f\delta_{ij}] + \alpha_f\rho_f[G^a_{ik}\langle u_{f,j}u_{f,k}\rangle \\ &+ G^a_{jk}\langle u_{j,i}u_{f,k}\rangle] + \frac{\alpha_p\rho_p}{\langle\tau_p\rangle}[-2\langle u_{f,i}u_{f,j}\rangle + M_{ij} + \mathcal{U}_{d,i}\langle U_{r,j}\rangle \\ &+ \mathcal{U}_{d,j}\langle U_{r,i}\rangle] - \alpha_f\rho_f\tfrac{2}{3}\langle\epsilon\rangle\delta_{ij},\end{aligned} \tag{83}$$

where $1/T=(1+3C_0/2)\langle\epsilon\rangle/k_f$ and where M_{ij} is a symmetric tensor given by $M_{ij}=\langle u_{s,i}u_{p,j}\rangle+\langle u_{s,j}u_{p,i}\rangle$. It is easily seen that, apart from the supplementary terms that arise from the two-phase flow formulation ($\alpha_f\rho_f$) and the influence of the discrete particles on the fluid, Eq. (47) supplemented by Eq. (83) is equivalent to the Rotta model when $G^a_{ij}=0$. This illustrates perfectly the correspondence between RSM models and SDEs for fluid particles. Depending on the chosen form of the SDEs, different RSM formulations can be obtained.

The term to close in Eq. (59) reads [once again the closure of the third-order moments is immediate by resorting to hypothesis (iii)]

$$\alpha_p\rho_p\langle A_{p,i}u_{p,j} + A_{p,j}u_{p,i}\rangle = -\frac{\alpha_p\rho_p}{\langle\tau_p\rangle}[2\langle u_{p,i}u_{p,j}\rangle - M_{ij}], \tag{84}$$

and for the fluid-particle velocity moment, the sum of the three last terms on the right-hand side of Eq. (60) is

$$\begin{aligned}&\frac{\alpha_p\rho_p}{\langle\tau_p\rangle}[\langle u_{f,i}u_{f,j}\rangle - (1+\chi)\langle u_{s,i}u_{p,j}\rangle + \chi\langle u_{p,i}u_{p,j}\rangle] \\ &+ \alpha_p\rho_p G_{ik}\langle u_{s,k}u_{p,j}\rangle.\end{aligned} \tag{85}$$

This last closure completes the formulation of the ''simplified'' two-field model. The form of the equations given here, especially the ones where $\mathbf{A}_s$ plays a part, should not be

taken as final since the exact formulation of the acceleration of the fluid along the trajectories of discrete particles is still an open question.

4. Further reduction of the system

As mentioned in the Introduction, one of the underlying goals, when one attempts to write mean field equations to describe a physical phenomenon, is the use of modern computer technology to obtain numerical solutions by, for example, control volume methods in combination with fractional step algorithms [24]. Here, it is shown, by further reduction of the closed system of PDEs of the previous section, that our approach is in line with the two-fluid models that can be encountered in research and industrial softwares [24]. Let us point out that the dimension of the system $d=32$ is still too large to allow practical simulations with conventional techniques for the resolutions of PDEs. The system can be further reduced by making two additional hypotheses.

(vi) Let us consider flows where the level of anisotropy (for both phases) is low so that all second-order tensors, $\langle \mathbf{u}_p \mathbf{u}_p \rangle$ and $\langle \mathbf{u}_f \mathbf{u}_f \rangle$, can be contracted and expressed by their trace $\langle \mathbf{Z}^2 \rangle$ as stated in the Boussinesq-like approximation. For both phases, the Boussinesq-like approximation is given by

$$\langle u_{k,i} u_{k,j} \rangle = \tfrac{1}{3} \langle u_k^2 \rangle \delta_{ij} - 2\nu_k(\langle \mathbf{Z} \rangle, \langle \mathbf{Z}^2 \rangle) \hat{S}_{k,ij}(\langle \mathbf{U}_k \rangle), \quad (86)$$

where ν_k are viscositylike coefficients whose specific form is not given but possible expressions can be found in Ref. [28]. $\hat{S}_{ij}$ is the deviatoric part of the strain rate tensor $S_{k,ij}$. In the Boussinesq-like approximation, it is implicitly assumed that the characteristic time scale of the fluctuating motion is much smaller than the time scale of the mean flow, a constraint that is not always true in practice since, in many flows, this separation of scales is not always verified.

(vii) Let us make a similar assumption as hypothesis (vi) for the tensor $\langle u_{s,i} u_{p,j} \rangle$. It is supposed that a general macroscopic law exists, that is,

$$\langle u_{s,i} u_{p,j} \rangle = \tfrac{1}{3} \langle u_{s,i} u_{p,i} \rangle \delta_{ij} + f_{ij}(\langle \mathbf{Z} \rangle, \langle \mathbf{Z}^2 \rangle), \quad (87)$$

where we do not describe the exact form of the function f_{ij}, see Refs. [28] and [29] for possible laws.

If hypotheses (vi) and (vii) can be applied, the dimension of the system becomes "reasonable" ($d=14$). Let k_k denote the turbulent kinetic energy of both phases, $k_k = \langle \mathbf{u}_k^2 \rangle / 2$ and k_{fp} the fluid-particle velocity covariance, $k_{fp} = \langle u_{s,i} u_{p,i} \rangle$. Equations (47), (59), and (60), where the closures given by Eqs. (83)–(85) have been inserted, can than be replaced by three scalar PDEs of the form

$$\alpha_f \rho_f \frac{D_f k_f}{Dt} = \frac{\partial}{\partial x_k}\left[\alpha_f \rho_f D_f \frac{\partial k_f}{\partial x_k} \right] - \alpha_f \rho_f \langle u_{f,i} u_{f,j} \rangle \frac{\partial \langle U_{f,i} \rangle}{\partial x_j} + \frac{\alpha_p \rho_p}{\langle \tau_p \rangle}[-2k_f + k_{pf} + \mathcal{U}_{d,i} \langle U_{r,i} \rangle] - \alpha_f \rho_f \langle \epsilon \rangle, \quad (88)$$

$$\alpha_p \rho_p \frac{D_p k_p}{Dt} = \frac{\partial}{\partial x_k}\left[\alpha_p \rho_p D_p \frac{\partial k_p}{\partial x_k} \right] - \alpha_p \rho_p \langle u_{p,i} u_{p,j} \rangle \frac{\partial \langle U_{p,i} \rangle}{\partial x_j} - \frac{\alpha_p \rho_p}{\langle \tau_p \rangle}[2k_p - k_{fp}], \quad (89)$$

and

$$\alpha_p \rho_p \frac{D_p k_{fp}}{Dt} = \frac{\partial}{\partial x_k}\left[\alpha_p \rho_p D_{sp} \frac{\partial k_{fp}}{\partial x_k} \right] - \alpha_p \rho_p \langle u_{s,i} u_{p,j} \rangle \frac{\partial \langle U_{p,i} \rangle}{\partial x_j} - \alpha_p \rho_p \langle u_{p,j} u_{p,i} \rangle \frac{\partial \langle U_{s,i} \rangle}{\partial x_j} + \frac{\alpha_p \rho_p}{\langle \tau_p \rangle}[2k_f - (1+\chi) k_{fp} + 2\chi k_p] + \alpha_p \rho_p G_{ij} \langle u_{s,i} u_{p,i} \rangle. \quad (90)$$

Here, it has of course been assumed that $\mathbf{G}^a = \mathbf{0}$, $D_{k,mn} = 2D_k \delta_{nm}$ and $D_{sp,mn} = D_{sp} \delta_{mn}$. Equations (88)–(90) supplemented by the continuity equations, Eqs. (45) and (58a), the momentum equations, Eq. (80), and Eq. (58a) in combination with Eq. (82), are often used in the literature for computations in different applications [24]. Given the amount of contraction and the needed hypotheses, one can wonder whether it is suitable or not to use such a model or if a more detailed description, such as the mean field-PDF model, should be used.

For numerical reasons [24], the coupling term between the two momentum equations is often written in terms of the so-called drift velocity as

$$\mathcal{I}_k \frac{\alpha_p \rho_p}{(\tau_p)}(\langle U_{f,i} \rangle - \langle U_{p,i} \rangle - \mathcal{U}_{d,i}), \quad (91)$$

so that both momentum equations are coupled. One has then to give the PDE satisfied by the drift velocity that is simply obtained by using Eqs. (45), (58a), (58c), and (82), that is, after some algebra,

$$\alpha_p \rho_p \frac{D_p}{Dt} \mathcal{U}_{d,i} = \frac{\partial}{\partial x_j}(\alpha_p \rho_\rho \langle u_{s,i} u_{p,j} \rangle) - \chi \frac{\partial}{\partial x_j}(\alpha_f \rho_f \langle u_{f,i} u_{f,j} \rangle) - \alpha_p \rho_p \mathcal{U}_{d,j} \frac{\partial \langle U_{f,i} \rangle}{\partial x_j}. \quad (92)$$

The form of this equation strongly depends on the expression that is chosen for $\mathbf{A}_s$ in the Langevin equation for $\mathbf{U}_s$ and therefore the form given above should not be considered as the last word.

VII. CONNECTIONS WITH PREVIOUS WORK

In this work, a probabilistic description of polydispersed turbulent two-phase flows has been presented in the form of a two-point PDF (one fluid-particle point and one discrete-particle point). The trajectories of the pairs of particles are given by diffusion processes and thus the Fokker-Planck equation verified by the PDF is known. In other words, a closed mesoscopic approach is provided. It is worth bringing out some important features of the present formalism.

(i) The correspondence between a PDF equation and

mean field equations has been noticed, for the description of the discrete phase, by several authors. However, in these approaches, there is no systematic path from the PDF description to the field description, i.e., the correspondence between Eulerian and Lagrangian quantities is not clearly made. In the present formalism, there is a natural path from a Lagrangian PDF equation to an Eulerian PDF equation, and therefore to mean field equations.

(ii) By contraction of the present PDF formalism, the different approaches encountered in the literature can be obtained. There is a hierarchy (different levels of information) between the models that is clearly identified.

(ii.1) By contraction over all discrete particle properties, the PDF equation verified by the marginal p_f^L (the one-point fluid PDF) is retrieved. This corresponds to the classical PDF approach to turbulent single-phase flows [8,9].

(ii.2) By contraction over all fluid particle properties, the PDF equation verified by the marginal p_p^L (the one-point particle PDF) is retrieved. This corresponds to the classical Lagrangian models [2]. These methods are rather easy to implement but this very easiness may hide consistency issues and a lack of theoretical analysis can lead to the creation of specific problems [12]. In the present approach a mathematical framework is provided and such problems (spurious drifts, correspondence with a PDF,...) are easily avoided (see [12] for detailed explanations).

(ii.3) The PDF approach to polydispersed turbulent two-phase flows is often encountered in the literature in the form of a kineticlike equation for the discrete phase, e.g., [18,19]. In these work, one secks the Fokker-Planck equation verified by the marginal p_p^L or more precisely by one of its marginals. Indeed, the fluid velocity seen is often considered as an external variable and one has to resort to functional calculus to provide a closed form of the Fokker-Planck equation. In such derivations, Gaussianity has to be assumed [15] and in nonhomogeneous turbulence (when the velocity of the fluid seen is bound to deviate from Gaussianity), this approximation might be too strong. It has been shown [12], by Gaussian integration by parts, that the Fokker-Planck equation obtained by most authors is in fact a contraction of Eq. (36).

(iii) In the present formalism, from a *closed* mesoscopic description, it is demonstrated how a closed two-field model can be derived, provided that some additional hypotheses are made. The path that is proposed in this work is rigorous and not model dependent. Once the models for the trajectories of the pairs of particles have been chosen, the derivation of the mean field equations is straightforward and the classical problem of finding closure laws at the macroscopic level can then be avoided.

(iv) Many two-field models are often derived by time or/and volume averaging the local instantaneous field equations and by introducing closure laws at the macroscopic level. Other two-field models combine ensemble averaging operators (for the fluid) and probabilistic tools (kinetic equations for the particles). In the latter models, there is, once again, no clear correspondence between Eulerian and Lagrangian quantities and closure, for the fluid, is performed at the macroscopic level. In the former models, apart from the problem-dependent macroscopic closures, it is often intricate to establish the link between the computed quantities and what can be measured by experiment. This is not the case in our approach where all operators are in fact, in discrete form, mass-weighted averages.

VIII. CONCLUSION

In the present paper, a probabilistic description of polydispersed turbulent two-phase flows has been presented in the form of a two-point PDF (one fluid-particle point and one discrete-particle point) where the models are given in terms of the trajectories of the associated diffusion process. By doing so, a closed mesoscopic model is provided and it is shown that there is a clear equivalence between Langevin equations and the Fokker-Planck equation verified by the associated PDF.

By giving the relations between the field of distribution functions (Eulerian PDF) and the two-point Lagrangian PDF, it is demonstrated how mean field equations can be derived in a consistent manner without having recourse directly to macroscopic closures. It is then emphasized that, in order to derive a two-field model that can be used in practical computations, supplementary assumptions have to be made that greatly limit the types of flows that can be considered. In flows where, at least, one of these hypotheses cannot be made, it is shown that the natural alternative is the mean field PDF approach where one uses the mean field approach for the fluid but where one keeps the PDF approach for the discrete phase. It is important to stress that these two approaches are often compared in a misleading way. As clearly seen in the present work, they cannot be compared directly since they correspond to a different level of information for the discrete phase. In the case of the mean field approach, only the two first moments are available (with all supplementary assumptions and limitations that are needed) whereas in the mean field-PDF approach any expectation for a function of the variables attached to a discrete particle can be evaluated. As a matter of fact, when two methods are compared, one should not only judge the computational effort that is needed (an error that is often made) but the optimal ratio between the level of information that can be obtained and the simulation time.

Finally, it should be pointed out that the use of probabilistic tools in the present form is not the ultimate answer to our problem of modeling polydispersed turbulent two-phase flows. In other words, the probabilistic tools that are employed cannot be declared as fundamental but as a practical way to model complex systems within the frame of well-established mathematical theories such as the theory of diffusion processes.

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