PHYSICAL REVIEW E, VOLUME 65, 046234

Deriving exact energy solutions to the symmetrized *q*-difference Harper equation

E. Papp* and C. Micu
Physics Department, North University of Baia Mare, RO-4800 Baia Mare, Romania

(Received 26 October 2001; published 11 April 2002)

This paper deals with the application of the q calculus to second order q-difference equations, like the q symmetrized form of the Harper equation. One obtains three-term recurrence relations, for which a symmetrized version is written down. This opens the way to establish explicit energy results that are dependent on the commensurability parameter. The continuous realization of such energy results exhibits interesting patterns characterized by hierarchical structures.

DOI: 10.1103/PhysRevE.65.046234 PACS number(s): 05.45.−a, 52.38.Bv

I. INTRODUCTION

Starting from a certain gauge description, a symmetrized q-difference formulation like [1]

$$\mathcal{H}_q\psi(z)=i\left(\frac{1}{z}+qz\right)\psi(qz)-i\left(\frac{1}{z}+\frac{z}{q}\right)\psi(q^{-1}z)=E\psi(z), \tag{1}$$

of the celebrated Harper equation [2–5] has been proposed. Here q expresses a typical pure phase parameter

$$q=\exp\left(\frac{i}{2}\hbar^*\right), \tag{2}$$

whereas $\beta=\hbar^*/2\pi$ is the pertinent commensurability parameter expressing the number of magnetic flux quanta ($\Phi_0=h/e$) per unit cell. The independent dimensionless variable is denoted by z. Both equations serve to the description of Bloch electrons on a two-dimensional (2D) square lattice threaded by a magnetic field. We have to note, however, that in the former case one deals with a fixed point in the Brillouin zone. In addition, the corresponding wave function ansatz is rather different from the usual Bloch description (see Appendix A), such as applied in the derivation of the Harper equation (HE). The point is that the interest on the HE is even increasing nowadays. So remarkable advances have been made quite recently, such as duality between HE and the 2D d-wave superconductivity [6], the role of the energy spectrum of the HE in the description of the generalized Hall conductance [7,8], or the interconnection between the spectral determinant characterizing the HE and the partition function of the 2D Ising model [9]. This motivates us to study in some more detail the energies of Eq. (1), with the aim to achieve a deeper understanding of the HE itself. In the sequel we shall refer to second order q-difference equation (1) as the qSHE. It should be remarked that the HE and the qSHE are able to be related one to another [10,11], but they are not at all equivalent ones.

So far, the energy of the qSHE has been established in an implicit manner by resorting to a Bethe ansatz description [10–14]. However, Eq. (1) can also be solved by applying the q calculus [15,16], which results in three-term recurrence relations [17,18]. In both cases one considers rational values of the commensurability parameter, like

$$\beta=\frac{P}{Q},\quad P\in M_Q, \tag{3}$$

where by now P and Q are mutually prime integers. Fixing Q, one gets faced with selected P values like $P=P_s\in M_Q$, so that, e.g., $M_4=\{1,3\}$, but $M_7=\{1,2,3,4,5,6\}$. Further there is $q^{2Q}=1$, which amounts to consider Q-dimensional representations of the underlying quantum group $sl_q(2)$, for which q is a root of unity, as shown before [1]. Nevertheless, there are still some open points, such as the formulation of a quickly tractable symmetrized version of such recurrence relations. Further, it is of interest to display the underlying continuous $\hbar^*$ dependence of energies. This produces interesting graphs exhibiting hierarchical structures with complexity degrees increasing with Q. The graphs displayed in this manner can also be viewed, of course, as a source of nontrivial pattern formation.

This paper is organized as follows. Some preliminaries referring to the q calculus are presented in Sec. II. In Sec. III one deals with the derivation of symmetrized recurrence relations. General properties of energy solutions are discussed in Sec. IV. The $\hbar^*$ dependence of energies, which leads to quite interesting patterns, is displayed for some selected Q values like $Q=4-7$. The conclusions are presented in Sec. V. The derivation of the qSHE is discussed in Appendix A, whereas the $Q=6$ energy description is revisited in Appendix B.

II. PRELIMINARIES CONCERNING THE *q* CALCULUS

In order to handle Eq. (1), we shall resort to the symmetrized Jackson derivative

$$\partial_z^{(q)}f(z)=\frac{d_qf(z)}{d_qz}=\frac{f(qz)-f(q^{-1}z)}{z(q-q^{-1})}, \tag{4}$$

where q has now the general meaning of a (real or complex) deformation parameter. This reproduces, of course, the usual derivative as soon as $q\to1$. The Leibniz rule is given by

$$\partial_z^{(q)}[f(z)g(z)]=g(qz)\partial_z^{(q)}f(z)+f(q^{-1}z)\partial_z^{(q)}g(z), \tag{5}$$

---

*Email address: epapp@arad.ro

in which the f and g functions can also be interchanged. In particular, one has

$$\partial_z^{(q)} z^n = [n]_q z^{n-1}, \tag{6}$$

where the quantum number is

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{7}$$

We can say that the $\partial_z^{(q)}$ derivative is a $q \rightleftarrows q^{-1}$ symmetrization of the original Jackson derivative [15]

$$\frac{D_q f(z)}{D_q z} = \frac{f(qz) - f(z)}{z(q-1)}, \tag{8}$$

in which case

$$\frac{D_q z^n}{D_q z} = \llbracket n \rrbracket_q z^{n-1} \tag{9}$$

and

$$\llbracket n \rrbracket_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}. \tag{10}$$

Both quantum numbers referred to above are interrelated as

$$[n]_q = \frac{1}{q^{n-1}} \llbracket n \rrbracket_{q^2}. \tag{11}$$

It is worthy of being mentioned that the original Jackson derivative is able to be produced by a radial reduction of the covariant derivative characterizing the quantum group of rotations on the noncommutative N-dimensional Euclidean space [19]. The q integral corresponding to Eq. (4) can also be easily established [16]. This amounts to consider explicit results like

$$\int_0^b z^n d_q z = \frac{b^{n+1}}{[n+1]_q}, \tag{12}$$

where b is an arbitrary number. Coming back to Eq. (2), one finds immediately the concrete realization

$$[n]_q = \frac{\sin(n\hbar^*/2)}{\sin(h^*/2)}, \tag{13}$$

which shows again that $[n]_q \to n$ as $q \to 1$. Using Eq. (4) and rescaling the energy

$$E = i\left(q - \frac{1}{q}\right) W, \tag{14}$$

it can be easily verified that Eq. (1) can be rewritten equivalently as

$$(\partial_z^{(q)} + z\partial_z^{(q)} z)\,\psi(z) = W\psi(z), \tag{15}$$

which can be solved in terms of three-term recurrence relations [17,18].

III. THE DERIVATION OF SYMMETRIZED RECURRENCE RELATIONS

Indeed, inserting the polynomial wave function

$$\psi(z) = \psi_q^{(Q)}(z) = \sum_{n=0}^{Q-1} C_n z^n, \tag{16}$$

into Eq. (15) yields the recurrence relation

$$[n+1]_q C_{n+1} + [n]_q C_{n-1} = W C_n, \tag{17}$$

where $C_0 = 1$. On the other hand one has $[Q]_q = 0$ by virtue of Eq. (3), in which case

$$C_{Q+1} = C_{Q+2} = \cdots = 0. \tag{18}$$

This shows that the energy levels corresponding to a fixed value of the Q parameter should be established via

$$C_Q = C_Q(q, W) = 0. \tag{19}$$

On the other hand there is

$$C_Q(q, W) = q^{N_Q} \frac{f_Q(q^2, W)}{\llbracket Q \rrbracket_{q^2}!}, \tag{20}$$

where $f_Q(q^2, W)$ denotes a polynomial of degree Q in W and where

$$N_Q = \frac{Q}{2}(Q-1). \tag{21}$$

Equation (20) comes from reasonable generalizations of explicit results. We have to realize, within the same context, that $f_Q(q^2, W) = f_Q(1/q^2, W)$, which means in turn that $C_Q(q, W) = C_Q(1/q, W)$ and $W(q^2) = W(1/q^2)$. Conversely, this latter equality implies the former one by virtue of Eq. (17), which means that the wave function itself is invariant under $q \to 1/q$, i.e., $\psi_q^{(Q)}(z) = \psi_{1/q}^{(Q)}(z)$. Accordingly, $E \equiv E_q \to E_{1/q} = -E_q$, which proceeds in accord with Eqs. (1) and (14).

Now what remains is to insert Eq. (20) into Eq. (17), which yields the symmetrized recurrence relations

$$f_Q(q^2, W) = W f_{Q-1}(q^2, W) - \Omega_{Q-2}^2 f_{Q-2}(q^2, W), \tag{22}$$

for $Q = 1,2,3,\ldots$, where

$$\Omega_{Q-2} = q^{2-Q} \llbracket Q-1 \rrbracket_{q^2}. \tag{23}$$

Using the combination

$$\Gamma_n = \Gamma_n(q) = \Gamma_n(1/q) = q^n + \frac{1}{q^n} = 2\cos\left(n\frac{\hbar^*}{2}\right), \tag{24}$$

which is invariant under $q \to 1/q$, it can be easily proved that

$$\Omega_{Q-2}(q) = \Omega_{Q-2}(1/q) = \begin{cases} \Gamma_1 + \Gamma_3 + \cdots + \Gamma_{Q-2}, & Q = \text{odd}, \\ 1 + \Gamma_2 + \cdots + \Gamma_{Q-2}, & Q = \text{even}. \end{cases} \tag{25}$$

Under such circumstances one obtains the eigenvalue equation

$$f_Q(q^2,W)=f_Q(1/q^2,W)=0, \tag{26}$$

by virtue of Eqs. (18) and (20), which produces precisely a number of Q real W roots, say

$$W=W_j^{(Q)}(q^2)=W_j^{(Q)}(1/q^2), \tag{27}$$

where $j=1,2,...,Q$. The q normalization of present wave function can also be readily done in terms of Eq. (12), as indicated before [18]. For this purpose we can choose $z\in[-1,1]$, but other normalization intervals like $z\in[0,b]$ can also be invoked.

The first six f_Q polynomials are given by

$$f_1(q^2,W)=W, \tag{28}$$

$$f_2(q^2,W)=W^2-1, \tag{29}$$

$$f_3(q^2,W)=W(W^2-3-\Gamma_2), \tag{30}$$

$$f_4(q^2,W)=W^4-6W^2+3+(2-3W^2)\Gamma_2+(1-W^2)\Gamma_4, \tag{31}$$

$$f_5(q^2,W)=W[W^4-10W^2+21+(17-6W^2)\Gamma_2 \\ +(11-3W^2)\Gamma_4+(5-W^2)\Gamma_6+\Gamma_8], \tag{32}$$

and

$$f_6(q^2,W)=W^6-15W^4+81W^2-37+(71W^2-10W^4-34) \\ \times\Gamma_2+(53W^2-6W^4-27)\Gamma_4 \\ +(33W^2-3W^4-18)\Gamma_6+(16W^2-W^4-10) \\ \times\Gamma_8+(5W^2-4)\Gamma_{10}+(W^2-1)\Gamma_{12}. \tag{33}$$

The corresponding eigenvalue equations can be easily solved in terms of available formulas for quadratic and cubic equations, but further $Q>7$ cases remain to be solved numerically. It should be stressed, however, that explicit energy results established in this way are useful in order to probe several conjectures concerning the spectrum of the qSHE. So, we have to realize that f_Q can be represented as

$$f_Q(q^2,W)=W^{\alpha_Q}\sum_{n=0}^{\beta_Q} d_n^{(Q)}(W^2)\Gamma_{2n}, \tag{34}$$

in which $d_n^{(Q)}(W^2)$ are constituent polynomials in W^2. Furthermore,

$$\alpha_Q=0, \quad \beta_Q=\tfrac{1}{4}Q(Q-2) \tag{35}$$

for even Q values, whereas

$$\alpha_Q=1, \quad \beta_Q=\tfrac{1}{4}(Q-1)^2 \tag{36}$$

for odd Q values. So one finds

$$d_{\beta_Q}^{(Q)}(W^2)=(-1)^{Q/2}(1-W^2) \tag{37}$$

and

$$d_{\beta_Q}^{(Q)}(W^2)=(-1)^{(Q-1)/2}, \tag{38}$$

respectively. Concerning $d_0^{(Q)}(W^2)$, we can say that it is a polynomial of degree $(Q-\alpha_Q/2)$ in W^2, but the general description of remaining constituents is still an open problem.

IV. SPECTRAL PROPERTIES AND ENERGY PATTERNS

Accounting for Eqs. (14) and (27) yields the energy spectrum

$$S_Q[\hbar^*]=\{E_j^{(Q)}(\hbar^*)\}, \quad j=1,2,...,Q \tag{39}$$

for $Q=1,2,3,...$, where

$$E=E_j^{(Q)}(\hbar^*)=-2\sin\frac{\hbar^*}{2}W_j^{(Q)}(q^2), \tag{40}$$

and where we shall assume hereafter that $\hbar^*\in[0,2\pi]$. This serves for a better description of underlying symmetries. For the sake of discrimination we shall then put $x=\hbar^*$, thereby considering x as a continuous variable. Moreover, the x dependence of present energies has its own interest, as it serves as a source for nontrivial pattern formation. Having established the x dependence of energy levels, we are in a position to derive immediately the actual discrete spectrum of the qSHE by virtue of the intersection

$$\mathcal{E}_Q=S_Q[x]\cap M_Q. \tag{41}$$

This means that the actual spectrum is given by the set of crossing points between the x-dependent energy curves belonging to $S_Q[x]$ and vertical lines like $x=x_s$, where $x_s=2\pi P_s/Q$ and $P=P_s\in M_Q$.

A. Interconnections with usual results

Assuming that $Q\geqslant 3$, we have to realize that the ordered sequence of P_s points has an even number of constituents, say $2N_s(Q)$, so that $P_1^{(Q)}=1$ and $P_{2N_s-Pe+1}^{(Q)}=Q-1$. Accordingly, there are $N_s(Q)$ intervals centered around $P=Q/2$, which are included successively one into another:

$$I_1^{(Q)}\supset I_2^{(Q)}\supset\cdots\supset I_{N_s}^{(Q)}, \tag{42}$$

where

$$I_k^{(Q)}=[P_k^{(Q)},P_{2N_s-k+1}^{(Q)}], \tag{43}$$

$k=1,2,...,N_s(Q)$ and $P_k^{(Q)}+P_{2N_s-k+1}^{(Q)}=Q$. Both edges of the above interval produce the same energy [see also Eq. (51)]. One has, e.g., $N_s(Q)=3$ for $Q=7$, 9, 14, and 18, whereas $N_s(Q)=2$ for $Q=5$, 8, 10, and 12. This shows that the Q dependence characterizing $N_s(Q)$ is rather nontrivial.

Further interconnections with the usual Harper equation [see Eq. (A13)] are also worthy of being mentioned. Indeed, inserting $P=P_s\equiv P_k^{(Q)}$ into Eq. (26) yields a number of $N_s(Q)$ discrete polynomial realizations like $\tilde{P}_k^{(Q)}(E)=0$, where $\tilde{P}_k^{(Q)}(E)$ is a polynomial of degree Q in E. Such realizations have to be established in terms of a subsequent normalization, which proceeds by choosing the coefficient of E^Q to be unity. Accordingly

$$\tilde{P}_k^{(Q)}(E)=\left[i\left(q-\frac{1}{q}\right)\right]^Q f_Q(q^2,W)\Big|_{q=\exp(i\pi P_k^{(Q)}/Q)}, \quad (44)$$

which works in combination with Eq. (14). These polynomials are precisely the ones produced by applying either the secular equation method [20,21] or, equivalently, the transfer-matrix technique [4,22] to Eq. (A13). Indeed, in these latter cases one obtains eigenvalue equations like

$$\tilde{P}_k^{(Q)}(E)=\Lambda\equiv 2\cos(Q\theta_1)+2\cos(Q\theta_2), \quad (45)$$

which serves to establish Q bands via $-4\leqslant P_k^{(Q)}(E)\leqslant 4$. Equation (45) reproduces Eq. (26) via $Q\theta_l=(2n_l'+1)\pi/2$, where $l=1$, 2 and where n_1' and n_2' are integers. Conversely, Eq. (26) gets reproduced automatically for all odd Q values like $Q=2n'+1$ provided that $\theta_1=\theta_2=\pi/2$, where $n_1'=n_2'=n'$. Note that θ_1 and θ_2 are usual Brillouin phases characterizing Eq. (A13). One realizes, of course that the energy description characterizing the qSHE is produced by the HE one in so far as $\Lambda=0$. However, this would not work irrespective of Q, if one would resort, e.g., to $\theta_1=\theta_2$ fixings. In particular, one has

$$\tilde{P}_k^{(5)}(E)=\tilde{P}_\pm^{(5)}(E)=E(E^4-10E^2+t_\pm^{(5)}), \quad (46)$$

for $Q=5$, where $t_\pm^{(5)}=(7\pm\sqrt{5})/2$. However, such structures are rather relevant for large Q values (see, e.g., Fig. 1 in Ref. [4]), unions over Q's included, so that they will be disregarded in this paper. Explicit energy results have been presented before for $Q=1$–5 [17] and $Q=6$ [18]. So far one has $\Delta=1$, but $\Delta\neq 1$-generalizations of above polynomials, say $\tilde{P}_k^{(Q)}(E,\Delta)$, can be readily done using the transfer-matrix approach. Choosing again $Q=5$, then gives

$$\tilde{P}_k^{(5)}(E,\Delta)=\tilde{P}_\pm^{(5)}(E,\Delta)=E^5-5E^3(\Delta^2+1)+5E(\Delta^4+1)+\tfrac{5}{2}E\Delta^2(3\pm\sqrt{5}), \quad (47)$$

which works in combination with the generalized Λ form [22]

$$\Lambda(\Delta)=2\cos(Q\theta_1)+2\Delta^Q\cos(Q\theta_2), \quad (48)$$

such that $\Lambda(1)=\Lambda$ and $\tilde{P}_k(E,\Delta)=\Lambda(\Delta)$. It is understood that Δ denotes a gap parameter characterizing the energy dispersion law of the lattice [see Eqs. (A2) and (A13)].

B. Symmetry properties and characteristic patterns

The present energies are well ordered in the sense that following inequalities,

$$E_1^{(Q)}(x)\leqslant E_2^{(Q)}(x)\leqslant\cdots\leqslant E_j^{(Q)}(x), \quad (49)$$

are valid irrespective of $x\in[0,2\pi]$. This is synonymous with a noncrossing behavior, which means that one has just contact points $x=x_C^{(Q)}(E)$ corresponding to the equality signs in Eq. (49). One realizes that the x derivatives of energy eigenvalues are not continuous in such contact points. Next there is

$$E_j^{(Q)}(x)=-E_{Q-j+1}^{(Q)}(x), \quad (50)$$

which exhibits the so called energy reflection symmetry [23]. In addition

$$E_j^{(Q)}(0)=E_j^{(Q)}(2\pi)=0, \quad (51)$$

but, excepting the zero-energy solution, there is

$$E_j^{(Q)}(\pi)=\pm 2. \quad (52)$$

Correspondingly $x=\pi$ stands for a symmetry axis of the spectrum

$$E_j^{(Q)}(x)=E_j^{(Q)}(2\pi-x), \quad (53)$$

which holds for $0<x<2\pi$. This also means that $2\pi-x_C^{(Q)}(E)$ is a contact point as soon as $x_C^{(Q)}(E)$ does it. It is also clear that $x=0$ and $x=2\pi$ have the meaning of general multiple contact points, as indicated by Eq. (51).

So the x dependence of the five energy levels $E=E_j^{(5)}(x)$ corresponding to $Q=5$ is displayed in Fig. 1. One sees that both Eqs. (50) and (51) are fulfilled, whereas Eq. (52) works as

$$E_1^{(5)}(\pi)=E_2^{(5)}(\pi)=-2 \quad (54)$$

and

$$E_4^{(5)}(\pi)=E_5^{(5)}(\pi)=2. \quad (55)$$

Ruling out the contact points $x=0$ and $x=2\pi$, one sees that Eq. (49) proceeds as

$$E_1^{(5)}(x)<E_2^{(5)}(x)<0<E_4^{(5)}(x)<E_5^{(5)}(x), \quad (56)$$

where $E_3^{(5)}(x)=0$. A further interesting feature concerns mutual attraction (repulsion) effects exhibited by central energy curves like $E_{Q/2}^{(Q)}$ and $E_{(Q+2)/2}^{(Q)}$ ($E_{(Q-1)/2}^{(Q)}$ and $E_{(Q+3)/2}^{(Q)}$) for even (odd) Q values. This means that for $E=0$ and even $Q\geqslant 4$ values one has a relative larger number of zero-energy contact points, say $N_C^{(Q)}=N_C^{(Q)}(E=0)$, than for adjacent odd neighbors $Q-1$ and $Q+1$. Indeed, there is $N_C^{(2)}=2$, $N_C^{(4)}=4$, and $N_C^{(6)}=8$, whereas $N_C^{(3)}=2$, $N_C^{(5)}=2$, and $N_C^{(7)}=4$, which confirms clearly the above statement. Of course, the present energy patterns exhibit hierarchical structures with complexity degrees increasing with Q. Indeed the $Q=4$ ($Q=6$) patterns presented in Fig. 2 (Fig. 3) look less (more)

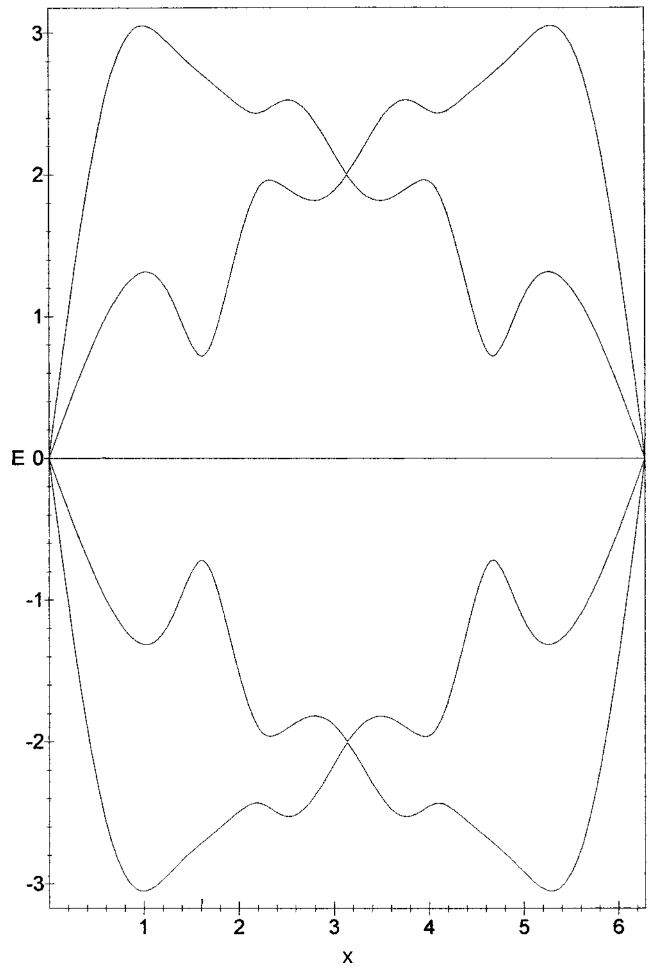

FIG. 1. The x dependence of the five energy levels $E = E_j^{(5)}(x)$, where $j=1,2,...,5$. Besides $x=0$ and $x=2\pi$, there are no $E=0$ contact points.

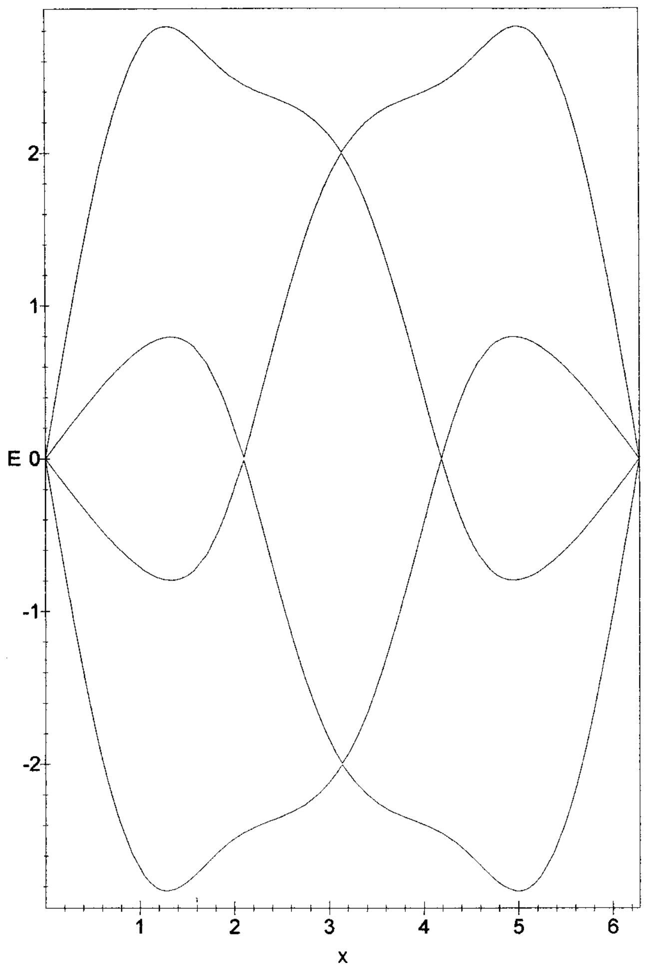

FIG. 2. The x dependence of the four energy levels $E = E_j^{(4)}(x)$, where $j=1$, 2, 3, and 4. One sees that $N_C^{(4)}(0)=2$, whereas $N_C^{(4)}(2)=N_C^{(4)}(-2)=1$. The corresponding contact points are $x_{C,1}^{(4)}(0)=2\pi/3$, $x_{c,2}^{(4)}(0)=4\pi/3$ and $x_{c,1}^{(4)}(\pm 2)=\pi$.

involved than the $Q=5$ ones. And we may wonder that the $Q=7$ patterns are actually even more sophisticated, as shown in Fig. 4. Comparing Figs. 3 and 4, one remarks a hierarchy of $0<x<\pi$ contact points at nonzero energy. This hierarchy, which starts with

$$x_C^{(6)}(E_0)=\frac{2\pi}{3}\cong 2.094, \tag{57}$$

at $E_0\cong 2.449$ (see Fig. 3), gets enhanced towards

$$x_C^{(7)}(E_1)\cong 1.244, \;\; x_C^{(7)}(E_2)\cong 2.513,$$

$$x_C^{(7)}(E_3)=x_C^{(7)}(0)\simeq 2.095, \tag{58}$$

if $Q=7$ (see Fig. 4), where $E_1\cong 1.360$, $E_2\cong 1.902$, and $E_3 \cong E_0$. In other words we found nontrivial patterns characterized by delightful symmetry realizations.

V. CONCLUSIONS

Our main emphasis in this paper was on the application of the q calculus to the derivation of explicit and exact energy results for the qSHE. This amounts to solve three-term recurrence relations, for which a symmetrized and quickly tractable version has been formulated, as shown by Eq. (22). Explicit formulas for characteristic $f_Q(q^2,W)$ polynomials have been written down for $Q=1-6$, but further cases can be treated in a similar manner. We are led to the formulation of relevant properties of energy formulas just by solving eigenvalue equations for a few number of Q values. Concerning the efficiency, there are reasons to say that the present method of three-term recurrence relations is comparable with the secular equation method [20,21] as well as with the method of transfer matrices [4,22]. The Bethe ansatz solution to the qSHE [1] looks somewhat more intricate, but it is, of course, of an actual interest from a general theoretical point of view. Having obtained $f_Q(q^2,W)$ opens the way to establish P_s derivatives of energy eigenvalues via

$$\frac{\partial}{\partial P_s}=i\,\frac{\pi}{Q}\,q\,\frac{\partial}{\partial q}\bigg|_{x=x_s}, \tag{59}$$

where q stands, of course, for exp($ix/2$) and where Q is fixed.

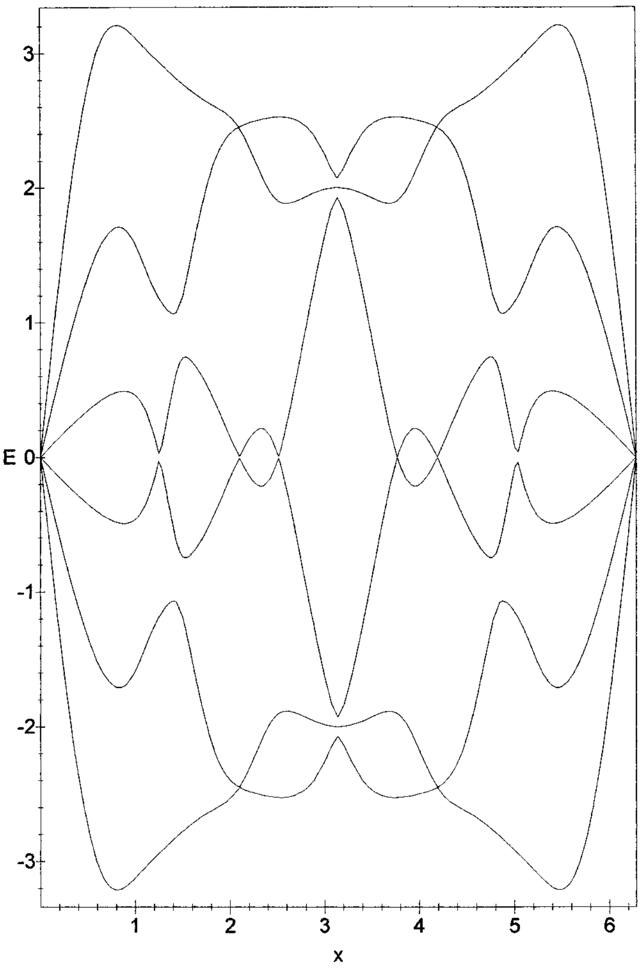

FIG. 3. The x dependence of the six energy levels $E=E_j^{(6)}(x)$, where $j=1,2,...,6$. Now $N_C^{(6)}(0)=6$, whereas both $(\pi, -2)$ and $(\pi, 2)$ are triple contact points. In addition, $N_C^{(6)}(E_0)=N_C^{(6)}(-E_0)=2$, where $E_0 \cong 2.449$.

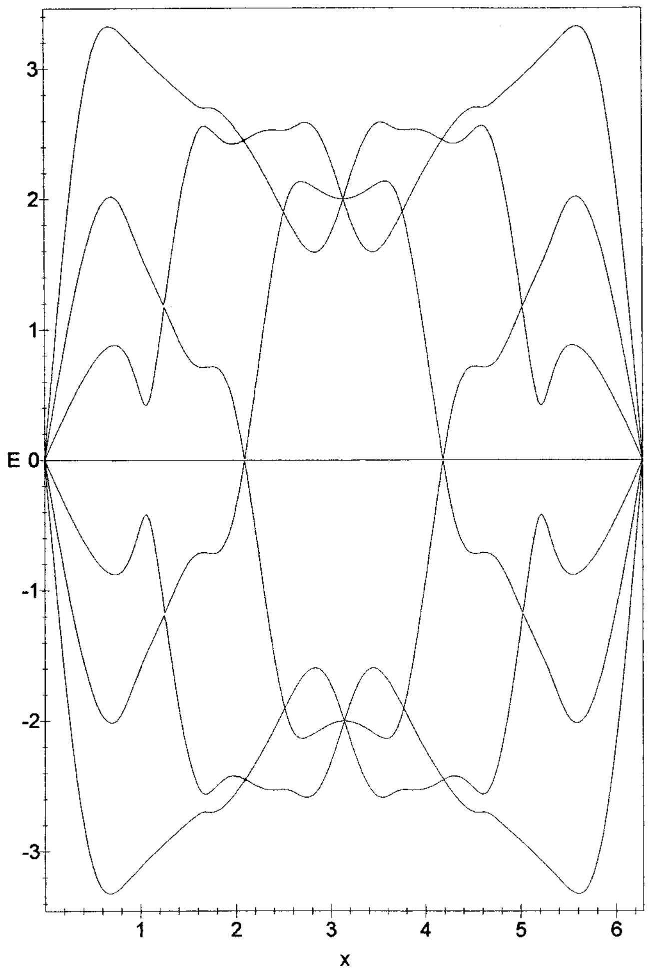

FIG. 4. The x dependence of the seven energy levels $E=E_j^{(7)}(x)$, where $j=1,2,...,7$. One has $N_C^{(7)}(0)=2$ and $N_C^{(7)}(2)=N_C^{(7)}(-2)=1$, but $N_C^{(7)}(E_i)=N_C^{(7)}(-E_i)=2$ for $E_1 \cong 1.360$, $E_2 \cong 1.902$, and $E_3 \cong 2.449$.

Such results are useful for the derivation of lateral derivatives in contact points as well as for the study of related magnetic properties via $\partial/\partial P_s=(2\pi\hbar/ea^2)\partial/\partial B$. It is then clear that a further point of interest is the derivation of $\Delta \neq 1$-generalizations of present $f_Q(q^2,W)$ polynomials, which serve as continuous counterparts of polynomials like $\widetilde{P}_k^{(Q)}(E,\Delta)$ already mentioned before. For this purpose we have to resort to the $\Delta \neq 1$ version of Eq. (1), so that the same concerns Eq. (A8). This is still an open point and related investigations remain to be done.

On the other hand, having obtained explicit energy results enables us to obtain useful information. Indeed, having established $\widetilde{P}_k^{(Q)}(E)$ polynomials enables us to establish the density of states for the Harper equation in terms of elliptic functions, as shown previously [21]. Further informations concerning the density of states is able to be derived by using the difference of lateral derivatives in contact points [24]. Nontrivial patterns have been displayed in Figs. 1–4. For this purpose relatively small Q values have been invoked, so that present patterns can be viewed as being complementary to the Hofstadter butterfly [4]. It is a nice exercise to derive numerically energy patterns produced by larger Q values, the actual spectrum of the qSHE included. Nonpolynomial solutions to the qSHE can be derived by using again recurrence relations, but so far this was done for the zero energy only [25].

ACKNOWLEDGMENT

We are indebted to CNCSIS/Bucharest (Project 307) for financial support.

APPENDIX A: DERIVATION OF qSHE

In order to make our paper self-contained, let us present briefly the derivation of Eq. (1). One starts, as usual, from the energy dispersion law for a 2D lattice

$$E_{\mathrm{disp}}(\mathbf{k})=\varepsilon_1 \cos\theta_1+\varepsilon_2 \cos\theta_2, \tag{A1}$$

which is quite familiar in solid state physics, where $\theta_l = k_l a_l$ $(l=1,2)$, $\mathbf{k}=(k_1,k_2)$ is the wave vector, whereas the lattice spacings are denoted by a_l. The gauge is chosen as

$$A_l=(-1)^l\frac{B}{2}(x_1+x_2+\alpha_l a), \tag{A2}$$

where $a_1=a_2=a$, which serves to the implementation of the Peierls substitution

$$k_l\rightarrow -i\frac{\partial}{\partial x_l}+\frac{e}{\hbar}A_l. \tag{A3}$$

So far α_l's are arbitrary parameters. Combining Eqs. (A1)–(A3) yields an energy band Hamiltonian, say $\mathcal{H}_{\mathrm{disp}}$, which serves to the formulation of the related eigenvalue equation as

$$\mathcal{H}_{\mathrm{disp}}\Psi(\mathbf{x})=E_d\Psi(\mathbf{x}), \tag{A4}$$

where E_d denotes the pertinent eigenvalue.

Within the next step one considers the Bloch-like wave function ansatz

$$\Psi(\mathbf{x})=\exp(i\mathbf{k}\cdot\mathbf{x})\varphi_B(\mathbf{x}), \tag{A5}$$

where $\mathbf{k}\cdot\mathbf{x}=k_1x_1+k_2x_2$, but this time

$$\varphi_B(\mathbf{x})=\varphi(x_1+x_2), \tag{A6}$$

which represents a rather special choice. Inserting Eq. (A5) into Eq. (A4) and accounting for the Baker-Campbell-Hausdorff formula

$$e^Ae^B=e^{A+B}e^{(1/2)[A,B]}, \tag{A7}$$

yields the discrete equation

$$i\left(\frac{1}{q^n}+q^{n+1}\right)\varphi_{n+1}-i\left(q^{n-1}+\frac{1}{q^n}\right)\varphi_{n-1}=E\varphi_n, \tag{A8}$$

for $\theta_1=\theta_2=\pi/2$, $\alpha_2=-\alpha_1=1/2$, $\varepsilon_2=\Delta\varepsilon_1$, and $\Delta=1$, where $E=2E_d/\varepsilon_1$ and $\varphi_n=\varphi(na)$. The space discretization is performed via $x_l=n_la$, so that $n=n_1+n_2$ is an integer. Moreover, the $\hbar^*$ parameter gets introduced as

$$\hbar^*=2\pi\frac{\Phi}{\Phi_0}, \tag{A9}$$

where $\Phi=Ba^2$ is the magnetic flux through the unit cell.

After having arrived at this stage, we are ready to convert Eq. (A8) into Eq. (1) by resorting to the wave function

$$\psi(z)=\sum_{n=\infty}^{\infty}\varphi_nz^n. \tag{A10}$$

Alternatively, we can make the identifications $q^n=z$, $q^{n+1}=qz$ and $q^{n-1}=z/q$, in which case Eq. (A8) could produce Eq. (1) via

$$\psi(z)=\psi(q^n)=\varphi_n. \tag{A11}$$

However, for this purpose we have to consider the z variable as it stands in Eq. (1) as being independent of q, which looks somewhat questionable from a strict mathematical point of view.

The above procedure differs from the derivation of the HE for which one considers the Landau gauge $\mathbf{A}=(0,Bx_1,0)$ and the usual wave function ansatz

$$\Psi(\mathbf{x})=e^{ik_2x_2}\varphi(x_1), \tag{A12}$$

instead of Eq. (A5). This leads to the HE

$$\varphi_{n+1}+\varphi_{n-1}+2\Delta\cos(n\hbar^*+\theta_2)\varphi_n=E\varphi_n, \tag{A13}$$

which incorporates the Brillouin parameter θ_2, where $x_1=na$ and where, as before, $\varepsilon_2=\Delta\varepsilon_1$. Thus the HE is a well defined band energy equation with the periodic boundary condition

$$\varphi_{n+Q}=\varphi_n, \tag{A14}$$

which differs from the rather special qSHE.

APPENDIX B: REVISITING $Q=6$ ENERGY FORMULAS

Solving Eq. (33) yields the six ordered W roots

$$W_1^{(6)}=-W_6^{(6)}=(F_+^{1/3}+F_-^{1/3}+L)^{1/2}, \tag{B1}$$

$$W_2^{(6)}=-W_5^{(6)}=(\varepsilon_-F_+^{1/3}+\varepsilon_+F_-^{1/3}+L)^{1/2}, \tag{B2}$$

and

$$W_3^{(6)}=-W_4^{(6)}=(\varepsilon_+F_+^{1/3}+\varepsilon_-F_-^{1/3}+L)^{1/2}, \tag{B3}$$

where

$$W_1^{(6)}\geqslant W_2^{(6)}\geqslant W_3^{(6)} \tag{B4}$$

and

$$L=\tfrac{1}{3}(\Gamma_8+3\Gamma_6+6\Gamma_4+10\Gamma_2+15). \tag{B5}$$

Inserting $Q=6$, one realizes that Eq. (42) gets fulfilled in terms of Eqs. (40) and (B1)–(B4). Accordingly [26]

$$\varepsilon_{\pm}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}, \tag{B6}$$

$$F_{\pm}=-\frac{S}{2}\pm\sqrt{D}, \tag{B7}$$

such that

$$S=-\sum_{n=0}^{12} b_n\Gamma_{2n} \tag{B8}$$

and

$$D=-\frac{1}{108}\sum_{n=0}^{20} c_n\Gamma_{2n}, \tag{B9}$$

where the c_n coefficients are, of course, different from the ones in Eq. (16). The values of b_n and c_n are presented in Table 1 in Ref. [18].

[1] P. B. Wiegmann and V. A. Zabrodin, Phys. Rev. Lett. **72**, 1890 (1994).
[2] P. G. Harper, Proc. Phys. Soc., London, Sect. A **68**, 874 (1955).
[3] A. Rauh, Phys. Status Solidi B **69**, K9 (1975).
[4] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[5] M. Wilkinson, J. Phys. A **20**, 4337 (1987).
[6] Y. Morita and Y. Hatsugai, Phys. Rev. Lett. **86**, 151 (2001).
[7] C. Albrecht, J. H. Smet, K. von Klitzing, D. Weiss, V. Umansky, and H. Schweizer, Phys. Rev. Lett. **86**, 147 (2001).
[8] M. Koshino, H. Aoki, K. Kuroki, S. Kagoshima, and T. Osada, Phys. Rev. Lett. **86**, 1062 (2001).
[9] I. V. Krasovsky, Phys. Rev. Lett. **85**, 4920 (2000).
[10] P. B. Wiegmann and V. A. Zabrodin, Nucl. Phys. B **422**, 495 (1994).
[11] I. V. Krasovsky, Phys. Rev. B **59**, 322 (1999).
[12] L. D. Faddeev and R. M. Kashaev, Commun. Math. Phys. **169**, 181 (1995).
[13] Y. Hatsugai, M. Kohmoto, and Y. S. Wu, Phys. Rev. B **53**, 9697 (1996).
[14] A. G. Abanov, J. C. Talstra, and P. B. Wiegmann, Nucl. Phys. B **525**, 571 (1998).
[15] F. H. Jackson, Mess. of Math. **38**, 57 (1909).
[16] G. Gasper and M. Rahman, *Basic Hypergeometric Series* (Cambridge University Press, Cambridge, 1990). The q derivative (q integral) is presented, e.g., in pp. 22–23 (Chap. 1.11).
[17] C. Micu and E. Papp, Mod. Phys. Lett. B **11**, 773 (1997).
[18] E. Papp and C. Micu, J. Phys. A **33**, 6615 (2000). In this paper some corrections are in order, as indicated in Appendix B.
[19] E. Papp, *Derivation of q Analogs for the Radial Schrödinger Equation in N Space Dimensions* (Nova Science, New York, 1997), Chap. 3.3.
[20] M. Shifman and A. Turbiner, Phys. Rev. A **59**, 1791 (1999).
[21] G. H. Wannier, G. M. Obermair, and R. Ray, Phys. Status Solidi B **93**, 337 (1979).
[22] Y. Hatsugai and M. Kohmoto, Phys. Rev. B **42**, 8282 (1990).
[23] M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. **50**, 1870 (1983).
[24] Y. Y. Wang, B. Pannetier, and R. Rammal, J. Phys. (Paris) **48**, 2067 (1987).
[25] C. Micu and E. Papp, J. Phys. A **31**, 2881 (1998).
[26] The present Eqs. (B1)–(B3) have to be considered instead of Eqs. (15)–(17) in Ref. [18], since in this latter case there are some regrettable transcription errors.