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Decoherence as decay of the Loschmidt echo in a Lorentz gas

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Classical chaotic dynamics is characterized by exponential sensitivity to initial conditions. Quantum mechanics, however, does not show this feature. We consider instead the sensitivity of quantum evolution to perturbations in the Hamiltonian. This is observed as an attenuation of the Loschmidt echo $M(t)$, i.e., the amount of the original state (wave packet of width σ) which is recovered after a time reversed evolution, in the presence of a classically weak perturbation. By considering a Lorentz gas of size L, which for large L is a model for an *unbounded* classically chaotic system, we find numerical evidence that, if the perturbation is within a certain range, $M(t)$ decays exponentially with a rate $1/\tau_\phi$ determined by the Lyapunov exponent λ of the corresponding classical dynamics. This exponential decay extends much beyond the Eherenfest time t_E and saturates at a time $t_s \simeq \lambda^{-1}\ln[\tilde{N}]$, where $\tilde{N} \simeq (L/\sigma)^2$ is the effective dimensionality of the Hilbert space. Since τ_ϕ quantifies the increasing uncontrollability of the quantum phase (decoherence) its characterization and control has fundamental interest.

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Chaos justifies the observed macroscopic irreversibility within the reversible laws of classical mechanics. One of its characteristic features is the exponential divergence of trajectories corresponding to nearby initial conditions, which leads to deterministic unpredictability. However, quantum dynamics exhibits insensitivity to initial conditions [1] and hence prevents a dynamical definition of quantum chaos. Therefore, quantum signatures of chaos in systems with a chaotic classical equivalent are sought in their steady-state properties [2] such as spectral rigidity [3], wave function morphologies [4], and decaying parametric correlations of observables [5]. Early attempts [6] to address unitary quantum dynamics did not clarify the connections with the dynamical classical concepts of chaos, but the inclusion of interactions with a dissipative environment was expected [7] to produce an entropy growth controlled by chaos. In a purely Hamiltonian problem, quantum reversibility can be monitored through the amount of revival of a local density excitation, upon time reversal of its unitary quantum evolution, i.e., the Loschmidt echo [8]. By considering a Lorentz gas where the reversed evolution is disturbed by a static perturbation, we find a Loschmidt echo that attenuates exponentially with a rate associated with the chaos of the classical system. Under appropriate conditions, the dynamical instability of the classical system translates into quantum phase unpredictability (decoherence) and the classical Lyapunov exponent becomes the quantum irreversibility rate.

A hypersensitivity of time reversal to perturbations was observed in recent NMR experiments on many-body spin systems [9,10]. In essence, these experiments [11,12] involve the creation of a local density excitation represented by a state $|0\rangle$ which evolves under a many-spin Hamiltonian $\mathcal{H}$. The Loschmidt echo (LE) is the probability of returning to the initial state when a Hamiltonian evolution for a time t is followed by an identical period of imperfect reversal of that evolution, achieved by the transformation $\mathcal{H} \to -(\mathcal{H}+\Sigma)$, i.e.:

$$M(t) = |\langle 0|\exp[i(\mathcal{H}+\Sigma)t/\hbar]\exp[-i\mathcal{H}t/\hbar]|0\rangle|^2, \quad (1)$$

where Σ is a constant (or quasistatic) Hermitian perturbation representing the imperfection in the Hamiltonian reversal. Notice that in these experiments $-\ln[M(t)]$ is a measure [9] of the growing entropy. A striking finding [10] was that, for small Σ, the *decay rate* of $M(t)$ becomes independent of Σ, being proportional only to the dynamical scales of $\mathcal{H}$. The hints that chaotic systems may be unstable under fluctuating perturbations [7] and that many-body systems could be assumed to be intrinsically chaotic [13] suggested to us the hypothesis that the most relevant parameter of the Hamiltonian dynamics, the Lyapunov exponent λ, might control the decay of $M(t)$. Moreover, a semiclassical analysis assuming [14] a one-body classically chaotic Hamiltonian with a perturbative potential representing a quenched disorder predicted a LE decaying exponentially with a rate $1/\tau_\phi = \lambda$. These predictions, based on various approximations, encouraged us to perform exhaustive numerical experiments.

We consider a Lorentz gas, i.e., a particle in a square billiard of area L^2 where we fix an irregular array of N circular scatterers (impurities) of radius R. A particular realization of this system is represented in Fig. 1(a) where the length, in a minimal unit a, is $L=200a$. The scatterer concentration is $c = N\pi R^2/L^2 = 8\pi \times 10^{-2}$. The minimum distance allowed between two scatterer centers is $3R$. This requirement, together with the imposed periodic boundary conditions, prevents geometrical localization. The Lyapunov exponent of a Lorentz gas should be $\lambda = \beta v/\ell$, where ℓ is the collision mean free path, v the particle velocity, and $\beta \sim 1$ a geometrical factor [15] depending logarithmically on ℓ. In our system, we estimate $\ell \simeq L^2/(2NR) - \pi R/2$. Computation of the distribution of distances between collisions

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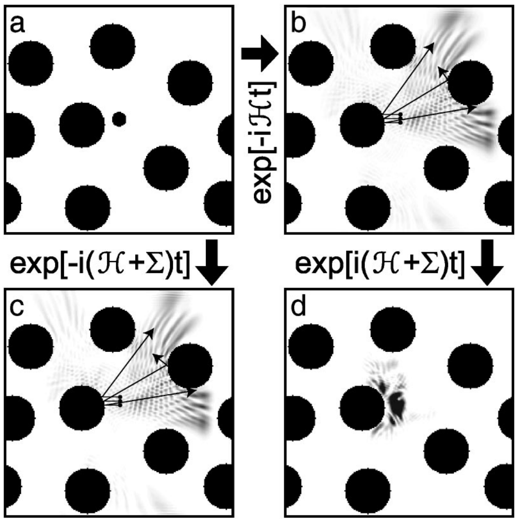

FIG. 1. Wave packet evolution in a system with $L/a=200$, $R/a=20$, and $c=8\pi\times10^{-2}$. (a) Initial state at $t=0$. The velocity points to the left. (b) State evolved with the unperturbed Hamiltonian for a time $t=30\hbar/V$. (c) State evolved with the perturbed Hamiltonian ($\alpha=0.04$) for a time $t=30\hbar/V$. (d) State evolved from that depicted in panel (b) with the perturbed Hamiltonian for a time $t=-30\hbar/V$. The square of the overlap (Loschmidt echo) between the states of panels (a) and (d) is $M(t)=0.09$, the same as that between the states of panels (b) and (c).

gives a shifted Poisson distribution whose mean value is ℓ. The Lyapunov exponent is obtained [16] from the average logarithm of the distance d_t, after a time t, between two classical trajectories initially separated by a small distance d_0. The condition of smallness for d_0 is that $d_t \ll R$. The longer the time the more precise the estimation of λ. By neglecting correlations in the position of the impurities, we obtain the estimation

$$\beta \simeq \int_R^\infty ds \frac{\exp(-s/\ell)}{\ell \exp(-R/\ell)} \int_0^1 dx \ln\left[1+\frac{s}{R\sqrt{1-x^2}}\right]. \quad (2)$$

The outer integral accounts for the distribution of free paths, s, and the internal one is the distribution of impact parameters x. Numerical $\lambda\ell/v$ verify Eq. (2). For the present case of $c=8\pi\times10^{-2}$, i.e., $\ell=96a$, we got $\beta=2.1\pm0.1$.

The perturbation parameter α controls the distortion of the diagonal components of the mass tensor, $m_{x,x}=m_0(1+\alpha)$ and $m_{y,y}=m_0/(1+\alpha)$, with m_0 the isotropic unperturbed mass. This perturbation is inspired by the effects of a slight rotation of the sample in the related problem of dipolar spin dynamics [17], which modifies the mass of the spin wave excitations. It is written as

$$\Sigma(\alpha) = -\frac{\alpha}{1+\alpha}\frac{p_x^2}{2m_0} + \alpha\frac{p_y^2}{2m_0}. \quad (3)$$

For a fixed initial position and velocity we find numerically that two evolutions with slightly different mass tensors lead to trajectories whose distance in phase space grows exponentially with the same Lyapunov exponent that amplifies initial distances, i.e., the classical dynamics of this system is equally sensitive to changes in the Hamiltonian as to changes in the initial conditions [18].

The Hamiltonian operator is obtained by a lattice discretization on a small scale a. The evolution is calculated using the Trotter-Suzuki algorithm [19]. Its basic idea is that, by choosing a short time unit τ, the evolution can be approximated as the finite product of evolution operators where each one is solved analytically. In the lowest order,

$$U(\tau) = \exp[i\mathcal{H}\tau/\hbar] \simeq \tilde{U}(\tau) = \prod_k^Q \exp[i\mathcal{H}_k\tau/\hbar], \quad (4)$$

where $\mathcal{H}=\Sigma_k^Q\mathcal{H}_k$. The conceptual virtue of this method is that $\tilde{U}(\tau)$ is always unitary. For an approximation of order n, the difference between $\tilde{U}(\tau)$ and $U(\tau)$ is of order τ^{n+1}. Our choice of $n=4$ and $\tau=0.1\hbar/V=0.2m_0a^2/\hbar$ allowed us to obtain high accuracy even for times an order of magnitude larger than needed.

Let us consider a typical system with $R/a=20$, $c=8\pi\times10^{-2}$, and size $L/a=200$. The initial state is a Gaussian wave packet of width $\sigma=3a$ and wave number $k_x=-0.7/a$, $k_y=0$ shown in Fig. 1(a). All further simulations will use $k_x=k_y=3\pi/8a$. Since $kR\gg1$, comparison with semiclassical calculations is justified. We took $2\pi/k\ll\ell=(96\pm2)a$, in order to minimize Anderson's localization effects. The density resulting from a typical evolution for a time $t=30\hbar/V$ [20] is shown in panel (b). An evolution with the perturbed Hamiltonian ($\alpha=0.04$) for the same time is shown in Fig. 1(c). In both panels the classical trajectories corresponding to three initial positions are shown for reference. While densities look similar to the eye, $M(t)$ is about 0.09 indicating the relevant role of the quantum phase in the attenuation of $M(t)$. Panel (d) shows the LE formed by the perturbed backward evolution of the state in panel (b). Analyzing $M(t)$ in different realizations we observe that, after the initial transient, $M(t)$ fluctuates around an exponential decay, with a characteristic time τ_ϕ. Notably this exponential decay persists much beyond the Ehrenfest time $t_E \simeq (1/\lambda)\ln[2R/\lambda_F] \simeq 40\hbar/V$, which is fixed by the local scale of the potential fluctuation [7]. In a Lorentz gas, in contrast with the usual case of chaotic cavities, t_E is independent of the system size [21]. Finally, $M(t)$ saturates on a time scale related to system size. A typical curve for $L/a=800$ is shown with a bold line in Fig. 2.

In principle, any given precision of τ_ϕ can then be obtained by studying a large enough system; however, this easily becomes computationally expensive. Alternatively, one can resort to the ensemble averaging of the observable $M(t)$, which reduces noise and defines τ_ϕ with the same precision in much smaller systems. This is exemplified in Fig. 2, where we present the ensemble averages $M(t)$ for billiards of three different sizes of $L/a=200$, 400, and 800 and fixed $\alpha=0.024$ and $c=8\pi\times10^{-2}$, which show the same exponential behavior with a progressively expanded range. Similar plots are obtained for the other parameters with an exponential $M(t)$, showing that the ensemble average does not introduce any spurious effect in the decay. Previous attempts to

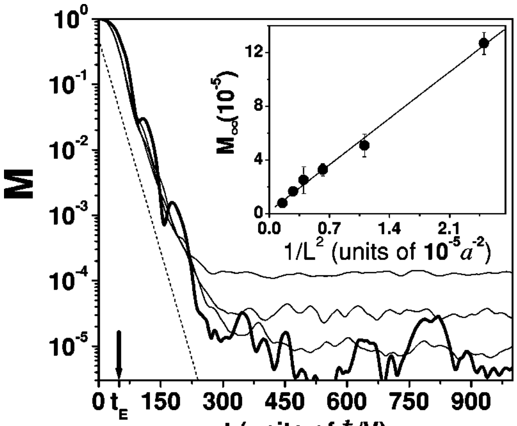

FIG. 2. Bold line, $M(t)$ for an individual system with $L/a=800$ and $\alpha=0.024$. The Ehrenfest time is shown with an arrow. Dashed line, shown for reference: exponential decay with the calculated Lyapunov exponent of the classical equivalent. Solid lines: average $M(t)$ for different system sizes $L/a=200$, 400, and 800 and the same perturbation. In the inset M_∞ is shown for $L/a=200$, 300, 400, 500, 600, and 800 as a function of $(a/L)^2$. The straight line is the best fit, with $M_\infty=(5\pm0.2)(a/L)^2$.

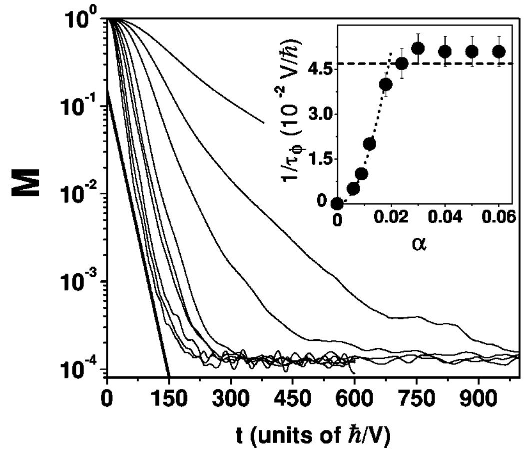

FIG. 3. Average $M(t)$ for a system with $L=200a$, $R=20a$, which makes $\ell=(96\pm2)a$. The values of α are, from top to bottom, 0.006, 0.009, 0.012, 0.018, 0.024, 0.03, 0.04, 0.05, and 0.06. The thick line corresponds to an exponential with the calculated numerical Lyapunov exponent of the system. For long times $M(t)$ saturates to a finite value M_∞. In the inset the inverse characteristic decay times are shown, evidencing the regime where the decay of M is given by the intrinsic properties of the system. The dotted line is the fit, $\tau_\phi^{-1}=(130\pm6)\alpha^2V/\hbar$, while the dashed line is the classical Lyapunov exponent.

characterize quantum chaos [6] considered the limiting value M_∞ for the Loschmidt echo at asymptotically large times. $M_\infty\propto1/\tilde{N}$ was proposed where $\tilde{N}$ is the number of energy levels appreciably represented in the initial state. In our case $\tilde{N}\simeq(L/\sigma)^2$. The numerical results verify a complete independence of M_∞ on the perturbation. The calculated M_∞ for various sizes are shown in the inset of Fig. 2 for a fixed $\alpha=0.024$. The predicted relationship is shown with a straight line, obtaining $M_\infty=(5.0\pm0.1)(a/L)^2$. For all the sizes we kept c (and hence ℓ, λ, and t_E) fixed and we verified that τ_ϕ does not depend on L/a. These results imply that the Lyapunov exponential behavior persists up to a time $t_s\simeq\lambda^{-1}\ln[\tilde{N}]$.

A representative dependence of τ_ϕ on the parameter α is obtained by considering the smallest sample size compatible with a good observation of the exponential, in this case $L/a=200$. Figure 3 shows $M(t)$ averaged over 100 realizations that contain at least one scatterer in the classical trajectory of the wave packet. For $\alpha>\alpha_c\simeq0.02$, all the exponential decays coincide, within the numerical error, with the Lyapunov decay associated with the classical system, shown with a thick line for comparison. The initial perturbative parabolic decay, $M(t)\simeq1-b\ (\alpha t)^2$ with $b\simeq0.37(V/\hbar)^2$, has a characteristic scale which *does* depend quadratically on the perturbation strength and prevents the curves from being superimposed. We show in the inset of Fig. 3 the numerical values of τ_ϕ^{-1} *extracted from the exponential part*. It shows an initial quadratic dependence on α and a crossover at $\alpha_c\simeq0.02$ to saturation at a value close to the classical Lyapunov exponent.

The semiclassical analysis predicts a universal behavior provided that the perturbation is strong enough to be quantically significant, but weak enough to be classically irrelevant. The first condition implies that the length $\tilde{\ell}=v\tilde{\tau}$ required for an important change in the phase as a consequence of the perturbation must be shorter than the distance associated with the Lyapunov exponent v/λ. A rough estimation for our perturbation $\delta m_{x,x}=\alpha m_0$ is

$$\tilde{\tau}\simeq\frac{\pi\hbar}{\delta E_k}=\frac{\pi m_0a^2}{\alpha\hbar(1-\cos[ka])}\lesssim1/\lambda, \qquad (5)$$

which would predict a critical value $\alpha_c\simeq0.13$. However, we see the universal exponential even for perturbations $\alpha\geqslant0.02$ implying a critical value an order of magnitude smaller than Eq. (5). This suggests that either this estimation or the condition required in Ref. [14] is too strong [22].

The condition for a classically weak perturbation means that it must not modify the system's global properties. Otherwise, even in the absence of chaos (no collisions) the perturbation would spread out the classical trajectories. In our case, this $M(t)$ fits a Gaussian decay. If at the time $1/\lambda$ required for the chaos to set in the overlap has already decayed because of the perturbation, the exponential decay will not be observed. This sets an upper bound for the perturbation from $M(t=1/\lambda)=\exp[-b(\alpha/\lambda)^2]\geqslant M_\infty$. Replacing with the parameters of the system shown in Fig. 3 one gets the condition $\alpha\leqslant0.3$, consistent with our results.

Therefore, we have shown that in a wide range of parameters $M(t)$ in any large enough individual system decays exponentially until an asymptotic value is reached. The characteristic decay time does not depend on the perturbation, but rather on the intrinsic properties of the Hamiltonian. The range of perturbation parameters with an exponential decay is much broader than hinted by the previous semiclassical analysis. From our numerical results one can infer that infinite systems not showing Anderson's localization should present an unbounded exponential decay. The Lorentz gas represents a broad class of chaotic systems, where the same overall behavior is expected. It has the additional advantage over other chaotic cavities that here the saturation time ($t_s \simeq (2/\lambda)\ln[L/\sigma]$) can be made arbitrarily longer than the Ehrenfest time. This manifests the quantum nature of the observed effect. The simplicity of our perturbation evidences a very remarkable property: *there is no need for chaoticity or stochasticity in the "environment" to introduce irreversibility*. Figure 1 manifests that the irreversibility produced by small perturbations is caused by their effect on the wave function's phase. In this sense, the definition of classical chaos in terms of hypersensitivity to initial conditions and perturbations translates, in the quantum world, into a sensitivity of phases to perturbations in the Hamiltonian, i.e., decoherence. Since we restricted ourselves to a Hamiltonian problem, its solution clarifies how chaos limits our control of dynamics even within the reversible framework of quantum mechanics of closed systems. This concept has deep consequences for the diverse fields where decoherence must be minimized, such as classical wave propagation [23], mesoscopic and molecular electronics [24], and quantum information [25]. In view of the entropic meaning of $-\ln[M(t)]$ the interest for the foundations of statistical physics [26] is clear.

In summary, we have presented numerical evidence supporting the fact that decoherence, as measured by the exponential decay of the Loschmidt echo, is controlled by the same parameters that govern classical chaos. This renders $M(t)$, defined in Eq. (1), a very practical tool to study dynamical quantum chaos.

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