Body versus surface forces in continuum mechanics: Is the Maxwell stress tensor a physically objective Cauchy stress?

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The Maxwell stress tensor (MST) $\mathbf{T}^M$ plays an important role in the dynamics of continua interacting with external fields, as in the commercially and scientifically important case of ''ferrofluids.'' As a conceptual entity in quasistatic systems, the MST derives from the definition $\mathbf{f}^M \overset{\text{def.}}{=} \boldsymbol{\nabla}\cdot\mathbf{T}^M$, where $\mathbf{f}^M(\mathbf{x})$ is a physically objective volumetric external body-force density field at a point $\mathbf{x}$ of a continuum, derived from the solution of the pertinent governing equations. Beginning with the fact that $\mathbf{T}^M$ is not uniquely defined via the preceding relationship from knowledge of $\mathbf{f}^M$, we point out in this paper that the interpretation of $\mathbf{T}^M$ as being a physical stress is not only conceptually incorrect, but that in commonly occuring situations this interpretation will result in incorrect predictions of the physical response of the system. In short, by elementary examples, this paper emphasizes the need to maintain the classical physical distinction between the notions of body forces $\mathbf{f}$ and stresses $\mathbf{T}$. These examples include calculations of the torque on bodies, the work required to deform a fluid continuum, and the rate of interchange of energy between mechanical and other modes.

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I. BACKGROUND

This paper focuses on the mechanical and dynamical distinction existing between body and surface forces in the physical description of continua. This important distinction is usually introduced in undergraduate courses in fluid mechanics and/or elasticity, where the material bodies being discussed are deformable. For a fortunate handful of students, the clear-cut conceptual differences existing between these two types of forces attains its ultimate degree of transparency in a formal course in continuum mechanics, which generally focuses on fundamentals rather than applications. The physical distinction between these two types of forces in classical continuum mechanics is essential in quantifying the behavior of continua. In the case of electromechanical body forces this fact has been abandoned by some in favor of the pragmatic mathematical advantages offered by the Maxwell stress tensor (MST) when calculating the force on a ponderable body [1]. Such schemes express the force $\mathbf{F}^M$ on the body as the surface integral, over a closed surface ∂V bounding the body, of a traction $\mathbf{T}^M_{\mathbf{n}}=\mathbf{n}\cdot\mathbf{T}^M$ arising from the ''Maxwell stress,'' $\mathbf{T}^M$, rather than as a volume integral, over the volume V of the body, of the physically objective volumetric body-force density $\mathbf{f}^M$. Though, mathematically, this surface integration invariably yields the correct force on the ponderable body, comparable use of the so-called ''stress'' field $\mathbf{T}^M$ as being a physical rather than a mathematical entity in other contexts may result in erroneous physical conclusions, as will be demonstrated.

Simply stated, and without referring to all the other physical arguments presented in what follows, the irreconcilable difference between representing the electromagnetic effect as a volumetric body-force density *versus* an electromagnetic stress hinges on the different ways of calculating the rate of working due to these two physically distinct quantities. Namely, the rate of working due to a volumetric body-force is given by the volume integral of the dot product of force and velocity, whereas that due to a stress is given by the surface integral of the corresponding traction dotted with the velocity of the moving material surface. That these two calculations are not equal for a ''Maxwellian'' force and its corresponding stress is shown below.

This paper is not intended as a criticism, *per se*, of the Maxwell stress tensor of classical electromagnetic theory, but rather as a caution that the *mathematical* body-force/surface-stress duality of electromagnetic field effects on ponderable matter is not physically acceptable in continuum mechanics. Only one of these two views is acceptable on physical grounds; either the electromagnetic effects enter as a body force or as a surface stress, but not both. Our adherence to the view that the electromagnetic effects enter the classical theory as long-range body forces is in accordance with the historical development of the field of electromagnetism. To our knowledge, this is still the commonly accepted notion.

A. General equations describing polarized continua

The standard equations governing momentum transport in fluid continua [6] are the respective Cauchy linear momentum and internal angular momentum equations, valid at each point $\mathbf{x}$ of the continuum [7,8],

$$\rho\frac{D\mathbf{v}}{Dt}=\boldsymbol{\nabla}\cdot\mathbf{T}+\mathbf{f}, \tag{1.1a}$$

$$\rho\frac{D\mathbf{a}}{Dt}=\boldsymbol{\nabla}\cdot\mathbf{C}+\mathbf{T}_{\times}+\mathbf{l}. \tag{1.1b}$$

It needs to be emphasized that though the symbols appearing in Eqs. (1.1) are completely arbitrary, their physical interpretation is not. For example, $\mathbf{T}$ could be replaced by another symbol, say $\boldsymbol{\tau}$, provided that the physical definition and, hence, interpretation of $\boldsymbol{\tau}$ remains the same as $\mathbf{T}$. As such, the symbols appearing in Eqs. (1.1) possess a precise physical meaning, whose significance is of signal importance in interpreting the behavior of the systems they describe.

The vector fields $\mathbf{v}$ and $\mathbf{a}$, respectively, represent the linear momentum density per unit mass (commonly identified as the mass-average velocity of the continuum) and the internal angular momentum density per unit mass. The internal angular momentum density is commonly assumed to possess the constitutive form $\kappa\Omega$, with the scalar κ the moment of inertia density, and the pseudovector Ω the intrinsic spin field [8]. The physical interpretation of the other terms in Eqs. (1.1) derives from the physical arguments underlying the analyses whereby these equations are obtained—namely, through macroscopic linear and angular momentum balances, as indicated in the following paragraphs.

1. Forces

Explicitly, the fields $\mathbf{T}$ and $\mathbf{f}$ appearing in these equations are introduced into continuum mechanics by the fundamental representation of the total external force $\mathbf{F}$ exerted on a control volume (closed with respect to mass) V bounded by a closed surface ∂V,

$$\mathbf{F}=\mathbf{F}_V+\mathbf{F}_S, \tag{1.2}$$

in which

$$\mathbf{F}_V=\int_V d\mathbf{F}_V, \tag{1.3a}$$

$$\mathbf{F}_S=\oint_{\partial V} d\mathbf{F}_S. \tag{1.3b}$$

This separation of the macroscopic force (and subsequently of other macroscopic quantities) into volumetric and surface contributions derives from classifying interactions between continuum elements into two types [9]: (1) interactions that diminish "slowly" with distance, and are therefore still effective over distances comparable to the macroscopic length scale, L, of the system; (2) interactions that are attenuated extremely "rapidly" over distances comparable to the characteristic linear dimension, l, of a continuum volume element, and whose effect does not penetrate the macroscopic continuum volume being considered. Thus, the distinction between volumetric and surface effects is linked to the continuum assumptions applicable in the asymptotic limit $l/L\to 0$.

In Eq. (1.3a), the extensive quantity $d\mathbf{F}_V$, termed a "body" or volume force, is a manifestation of so-called "action-at-a-distance" forces, resulting from the interaction between the material contents of a differential volume element dV (or corresponding mass element $dm=\rho dV$) centered at an interior point $\mathbf{x}$ of the body, and the "distant" surroundings, generally lying outside of the body, although not always so, as in the case of self-gravitating bodies. This serves to define the external volumetric density body-force field $\mathbf{f}(\mathbf{x})$, representing the proportionality factor in the linear relation $d\mathbf{F}_V=\mathbf{f}(\mathbf{x})dV$ existing between the body force $d\mathbf{F}_V$ and the volume element dV in the continuum limit.

In Eq. (1.3b), the extensive quantity $d\mathbf{F}_S$, which is a surface force, is a manifestation of the so-called "direct contact" or "push-pull-shear" forces exerted on a differential surface element dS centered at a surface point $\mathbf{x}$ of the body by the "contiguous" surroundings. This serves to define the stress vector $\mathbf{t_n}$ at the surface point as representing the proportionality factor in the linear relation $d\mathbf{F}_S=\mathbf{t_n}dS$ existing between the surface force $d\mathbf{F}_S$ and the surface element dS in the continuum limit. The stress vector is not itself a field quantity at a point $\mathbf{x}$ of the continuum since the force $d\mathbf{F}_S$ will generally depend upon the particular orientation $\mathbf{n}$ chosen for the surface element at $\mathbf{x}$. However, as shown by Cauchy, the stress vector $\mathbf{t_n}$ at a point $\mathbf{x}$ for a surface element possessing an orientation $\mathbf{n}$ can be expressed in terms of an orientation-independent tensor field $\mathbf{T}(\mathbf{x})$ possessing the property that $\mathbf{t_n}=\mathbf{n}\cdot\mathbf{T}(\mathbf{x})$, where $\mathbf{n}$ is drawn normal to the surface element. Cauchy's analysis, which usually proceeds by means of the so-called "tetrahedron" argument [10], amounts to a proof of the existence of the stress tensor field. Thus, upon defining a directed surface element $d\mathbf{S}\overset{\text{def.}}{=}\mathbf{n}dS$, and adopting the standard convention for the direction of $\mathbf{n}$ upon dS, one has that $d\mathbf{F}_S=d\mathbf{S}\cdot\mathbf{T}$, where $d\mathbf{F}_S$ is the force exerted by the material into which $\mathbf{n}$ is directed upon the material on the other side.

With the above pair of substitutions, Eqs. (1.3a) and (1.3b) adopt the respective forms:

$$\mathbf{F}_V=\int_V \mathbf{f}dV, \tag{1.4a}$$

$$\mathbf{F}_S=\oint_{\partial V} d\mathbf{S}\cdot\mathbf{T}. \tag{1.4b}$$

These arguments, based upon the clear-cut physical distinction existing between "distant" and "direct-contact" effects, serve to unequivocally establish the distinct physical significances attributed to the body-force volume density $\mathbf{f}$ and stress tensor $\mathbf{T}$. In circumstances where the stress field is continuous, one can invoke the divergence theorem to write Eq. (1.4b) in the form

$$\mathbf{F}_S=\int_V \boldsymbol{\nabla}\cdot\mathbf{T}dV.$$

While the latter expresses the surface force $\mathbf{F}_S$ as a volume integral, similar in appearance to Eq. (1.4a), it is apparent that the integrand $\boldsymbol{\nabla}\cdot\mathbf{T}\overset{\text{def.}}{=}\mathbf{f}'$, say, is physically not a body-force density field $\mathbf{f}'(\mathbf{x})$ since it does not arise from "distant" sources. Conversely, in circumstances where the constitutive equation for the body-force field $\mathbf{f}$ (assumed continuous) is such that it can be mathematically expressed as the divergence, $\boldsymbol{\nabla}\cdot\mathbf{T}'$, of a tensor, say $\mathbf{T}'(\mathbf{x})$, Eq. (1.4a) can be written in the alternative form

$$\mathbf{F}_V=\oint_{\partial V} d\mathbf{S}\cdot\mathbf{T}'. \tag{1.5}$$

While the latter expresses the volume contribution $\mathbf{F}_V$ as a surface integral, similar in appearance to Eq. (1.4b), it is apparent that $\mathbf{T}'$ is physically not a stress field since it does not arise from "direct-contact" forces. As such, while the divergence theorem allows one to define the *mathematical* fields $\mathbf{f}'(\mathbf{x})$ and $\mathbf{T}'(\mathbf{x})$ in the limited context of force calculations, this fact does not allow one to freely utilize these fields in *physical* contexts other than that of calculating forces in the absence of evidence of their physical legitimacy in these other situations. It is this simple argument, physics vs mathematics, that forms the basis for the discussion that follows.

The total force, Eq. (1.2), is given by the expression

$$\mathbf{F}=\int_V \mathbf{f}dV+\oint_{\partial V} d\mathbf{S}\cdot\mathbf{T}. \tag{1.6}$$

While the value obtained for this force is formally equivalent to that given by the "mathematically equivalent" expression

$$\mathbf{F}'=\int_V \mathbf{f}'dV+\oint_{\partial V} d\mathbf{S}\cdot\mathbf{T}'$$

(that is $\mathbf{F}=\mathbf{F}'$), the physical significance of the fields $(\mathbf{f}',\mathbf{T}')$ is vastly different from that of the comparable fields $(\mathbf{f},\mathbf{T})$ from which they derive. Accordingly, while Eq. (1.1a) could be formally rewritten as

$$\rho\frac{D\mathbf{v}}{Dt}=\boldsymbol{\nabla}\cdot\mathbf{T}'+\mathbf{f}',$$

it would be inappropriate to ascribe to $\mathbf{T}'$ the physical significance of a stress and to $\mathbf{f}'$ the physical significance of a body-force density, given their respective definitions. As will be shown, failure to appreciate this fact will generally lead to unequivocally incorrect physical results in situations where global properties of the continuum (other than the force on a body), functionally dependent upon the local physical body-force density and stress distributions, are sought. As such, the issue is not merely one of symbolism, but rather of the physical interpretation ascribed to these symbols.

Indeed, even apart from the physical issues involved, obvious mathematical issues signal the inability to uniquely convert volume forces to surface stresses and conversely. In this context, the amount of information embodied in each of these fields is pertinent. Thus, whereas a physical vector body-force density $\mathbf{f}$ entails but three independent scalar components, the derived "stress field" $\mathbf{T}'$, defined by the expression $\mathbf{f}\overset{\text{def.}}{=}\boldsymbol{\nabla}\cdot\mathbf{T}'$, generally requires nine independent scalar components for its unique specification. As such, at a minimum, the stress field $\mathbf{T}'$ derived from $\mathbf{f}$ in this manner lacks uniqueness. Conversely, the body-force field $\mathbf{f}'$, derived from the physical stress field $\mathbf{T}$ via the relation $\mathbf{f}'=\boldsymbol{\nabla}\cdot\mathbf{T}$ has but three independent scalar components, whereas the original field $\mathbf{T}$ from which it derives possesses nine independent scalar components. As such, relevant information embodying physically pertinent data embedded in $\mathbf{T}$ is obviously irrevocably "lost" in effecting the transition, $\mathbf{T}\rightarrow\mathbf{f}'$, from dyadic to vector field, whence the two fields cannot be physically equivalent in all of their consequences. While these facts may seem patently obvious, they are nevertheless widely ignored in the case of the Maxwell stress tensor, whose ubiquitous use permeates the field of electromechanics.

2. Torques

Analogous issues of symbolism and physicality apply to the terms appearing in the internal angular momentum equation (1.1b). Therein, $\mathbf{T}_\times$ is the pseudovector of the antisymmetric portion of the physical stress tensor $\mathbf{T}$, defined as $\mathbf{T}_\times\overset{\text{def.}}{=}-\varepsilon:\mathbf{T}$, with ε the unit pseudoisotropic triadic [equivalent to the permutation symbol ε_{ijk} in Cartesian tensor notation: $(T_\times)_i=-\varepsilon_{ijk}T_{kj}$]. The remaining terms in Eq. (1.1b), namely, $\mathbf{C}$ and $\mathbf{l}$, are introduced via a calculation of the total external torque $\mathbf{L}$ (about the arbitrary origin O from which the position vector $\mathbf{x}$ is drawn) exerted by the surroundings on the body, as follows:

$$\mathbf{L}=\mathbf{L}_V+\mathbf{L}_S, \tag{1.7}$$

in which

$$\mathbf{L}_V=\int_V d\mathbf{L}_V, \tag{1.8a}$$

$$\mathbf{L}_S=\oint_{\partial V} d\mathbf{L}_S. \tag{1.8b}$$

The extensive quantity $d\mathbf{L}_V$ appearing above is a manifestation of the so-called "action at a distance" torques about O, resulting from the interaction between the material contents of a differential volume dV (or a corresponding mass element $dm=\rho dV$) centered at an interior point $\mathbf{x}$ of the body, and the distant surroundings. This serves to define the external volumetric density body-torque pseudovector field, representing the proportionality factor in the linear relation $d\mathbf{L}_V=(\mathbf{x}\times\mathbf{f}+\mathbf{l})dV$ existing between the body torque $d\mathbf{L}_V$ and the volume element dV in the continuum limit. In this expression, with the position vector $\mathbf{x}$ measured with respect to an origin at O, the pseudovector $\mathbf{x}\times\mathbf{f}dV$ represents the torque about O arising from the macroscopic body-force density $\mathbf{f}$, whereas $\mathbf{l}dV$ is the intrinsic, origin-independent couple—the latter arising from the interaction of the polarized continuum (if, indeed, it is polarized) with a "distant" field. Thereby, Eq. (1.8a) yields

$$\mathbf{L}_V=\int_V(\mathbf{x}\times\mathbf{f}+\mathbf{l})dV. \tag{1.9}$$

Similarly, the extensive quantity $d\mathbf{L}_S$, which is a surface torque about O, is a manifestation of the so-called "direct contact" torques exerted on a differential surface element dS, centered at a surface point $\mathbf{x}$ of the body by the contiguous surroundings. Proceeding as in the comparable discussion of the surface force $d\mathbf{F}_S$ appearing in Eq. (1.3b), one eventually arrives at the expression $d\mathbf{L}_S=(\mathbf{x}\times\mathbf{t}_\mathbf{n}+\mathbf{c}_\mathbf{n})dS$ for the surface torque, where $\mathbf{c}_\mathbf{n}$ is the intrinsic, origin-independent couple-stress pseudovector, stemming from the direct contact torques arising from the polarized nature of the continuum. Analogous to the Cauchy stress tensor case, the latter serves to define the couple-stress pseudodyadic field $\mathbf{C}(\mathbf{x})$ in the relation $\mathbf{c_n}=\mathbf{n}\cdot\mathbf{C}$, whose existence is demonstrated via a comparable ‘‘tetrahedron argument.’’ Eventually, this leads to the fact that Eq. (1.8b) can be expressed as

$$\mathbf{L}_S=\oint_{\partial V}[\mathbf{x}\times(d\mathbf{S}\cdot\mathbf{T})+d\mathbf{S}\cdot\mathbf{C}]. \tag{1.10}$$

Availing ourselves of Eqs. (1.9) and (1.10), the total torque, Eq. (1.7), is thus given by the expression

$$\mathbf{L}=\int_V(\mathbf{x}\times\mathbf{f}+\mathbf{l})dV+\oint_{\partial V}[\mathbf{x}\times(d\mathbf{S}\cdot\mathbf{T})+d\mathbf{S}\cdot\mathbf{C}]. \tag{1.11}$$

As in the case of the comparable expression for the surface force contribution $\mathbf{F}_S$, in circumstances where the fields $\mathbf{T}$ and $\mathbf{C}$ are continuous, Gauss’ divergence theorem enables the surface torque contribution (1.10) to be written in the alternative volumetric form

$$\mathbf{L}_S=\int_V[\mathbf{x}\times(\boldsymbol{\nabla}\cdot\mathbf{T})+\mathbf{T}_\times+\boldsymbol{\nabla}\cdot\mathbf{C}]dV.$$

Comparison of the latter equation with Eq. (1.9) shows that, mathematically, one might be tempted to define, as before, a ‘‘body-force’’ field $\mathbf{f}'$ such that $\mathbf{f}'\overset{\text{def.}}{=}\boldsymbol{\nabla}\cdot\mathbf{T}$ and a ‘‘body-couple’’ field $\mathbf{l}'$ such that $\mathbf{l}'\overset{\text{def.}}{=}\mathbf{T}_\times+\boldsymbol{\nabla}\cdot\mathbf{C}$, giving the latter equation a symbolic appearance identical to Eq. (1.9), but with primed symbols replacing unprimed ones. Again, however, this would be physically inappropriate because of the very different origins of the ‘‘distant’’ and ‘‘direct-contact’’ contributions.

In any event, the internal angular momentum equation (1.1b) eventually derives from a total angular momentum balance using Eq. (1.11) for the total torque on the control volume, from which one subtracts the moment of the linear momentum equation (1.1a) [7].

3. Rate of working

In addition to their roles in quantifying the total force (1.6) and torque (1.11), the quantities that describe the rates of change of linear and angular momentum also play a pivotal role in providing the expression for ‘‘work’’ appearing in the principle of energy conservation, namely, the first law of thermodynamics. For a closed system contained in a region V, the first law is represented mathematically by the extensive equation

$$\frac{dE}{dt}=\dot{W}+\dot{Q}, \tag{1.12}$$

where E is the total energy contained within V, $\dot{W}$ is the rate of working of the surroundings on that region, and $\dot{Q}$ is the rate of heat transfer to the region V across its boundaries, ∂V.

The rate $\dot{W}$ of working per unit time performed by the surroundings on a closed, material fluid control volume V (i.e., one moving with the fluid), results from the translational and orientational motions of its generally polarized substructure. Calculation of this global work from knowledge of the continuum-dynamical and kinematical elements that enter into its formulation will serve to illustrate the signal physical significance that needs to be unequivocally assigned to the symbols appearing in Eq. (1.1). As in the preceding force and torque calculations, this rate of working is composed of both body and surface contributions

$$\dot{W}=\dot{W}_V+\dot{W}_S, \tag{1.13}$$

in which [11]

$$\dot{W}_V=\int_V d\dot{W}_V, \tag{1.14a}$$

$$\dot{W}_S=\oint_{\partial V} d\dot{W}_S. \tag{1.14b}$$

In Eq. (1.14a), the extensive quantity $d\dot{W}_V$ represents the rate of working by the ‘‘distant’’ surroundings on an interior volume element dV (or corresponding mass element $dm=\rho dV$) centered at an interior point $\mathbf{x}$ of the continuum domain, arising from the action of the long-range body forces and body couples. This serves to define the external volumetric rate of working density, representing the proportionality factor appearing in the linear relation $d\dot{W}_V=(\mathbf{f}\cdot\mathbf{v}+\mathbf{l}\cdot\Omega)dV$ existing between the extensive rate of working $d\dot{W}_V$ and the volume element dV in the continuum limit. Here, $\mathbf{v}$ is, as before, the mass-average velocity, and Ω is the internal spin-field of the structured continuum [8]. Equation (1.14a) can thus be expressed as the volume integral

$$\dot{W}_V=\int_V(\mathbf{f}\cdot\mathbf{v}+\mathbf{l}\cdot\Omega)dV. \tag{1.15}$$

This equation serves to focus attention upon the physical significance demanded of the symbols $\mathbf{f}$ and $\mathbf{l}$, appearing therein, through their energetic interpretations, since it will be demonstrated that replacement of these symbols by their ‘‘alternates,’’ namely, $\mathbf{f}'$ and $\mathbf{l}'$ as defined earlier in connection with our respective classification of forces and torques, will eventually lead to errors in the extensive rate of working $\dot{W}$, Eq. (1.13).

In Eq. (1.14b), the extensive quantity $d\dot{W}_S$ represents the rate of working performed by the ‘‘immediate’’ surroundings on the continuum domain V through a surface element dS centered at a surface point $\mathbf{x}$ of the body. This serves to define the rate of surface working per unit area as the proportionality factor in the linear relation $d\dot{W}_S=(\mathbf{t_n}\cdot\mathbf{v}+\mathbf{c_n}\cdot\Omega)dS$ existing between the rate of working $d\dot{W}_S$ and the surface element dS in the continuum. Upon introducing into the latter expression the prior definitions of the Cauchy stress field **T** and couple-stress dyadic **C**, equation (1.14b) can thereby be expressed as

$$\dot{W}_S = \oint_{\partial V} d\mathbf{S}\cdot(\mathbf{T}\cdot\mathbf{v}+\mathbf{C}\cdot\Omega). \tag{1.16}$$

As with Eq. (1.15), this latter equation serves to focus upon the physical significance to be attributed to the symbols **T** and **C** in the context of their respective usage in energetic calculations.

In combination, the expression for the total rate of working on the body V, given by Eq. (1.13), becomes [12]

$$\dot{W} = \int_V (\mathbf{f}\cdot\mathbf{v}+\mathbf{l}\cdot\Omega)\,dV + \oint_{\partial V} d\mathbf{S}\cdot(\mathbf{T}\cdot\mathbf{v}+\mathbf{C}\cdot\Omega). \tag{1.17}$$

The justification for Eq. (1.17) resides in the fact that it constitutes the most general application of the mechanical definition of work [13], based upon classical rigid-body mechanics principles applied to polarized systems. It is composed solely of force-times-displacement and couple-times-rotation terms. It is important to note that this equation implicitly assumes the subcontinuum structure to be composed of rigid-body elements. This does not represent a further restriction of the domain of validity of our analysis, as this assumption had already been implicitly invoked in adopting **v** and $\kappa\Omega$ as the respective linear and internal angular momentum densities per unit mass [8].

This expression for the rate of working on the material volume V plays a role in determining the rate of change of the total energy E associated with V through the first law of thermodynamics. This energy is frequently assumed to consist of the sum of several contributions, e.g., kinetic, potential, and internal energies. There is still debate over the form of other possible additional contributions, such as ''field energies,'' in the case of systems in electromagnetic fields [15–17]. Because of this lack of resolution, and to focus on the subject matter of this work (the physicality, or lack thereof, of the Maxwell stress tensor in continuum mechanics), we will merely assume that the total energy may be separated into a kinetic energy component K together with what we shall simply term ''other'' forms of energy, E_O. Of course, the ''other'' modes of energy may be further separated into subcategories, the nature of which is immaterial to the subsequent discussion. The first law, equation (1.12), thereby adopts the form

$$\frac{dK}{dt}+\frac{dE_O}{dt}=\dot{W}+\dot{Q}. \tag{1.18}$$

Next, we assume the following representation for the macroscopic kinetic energy relative to an inertial reference frame:

$$K_V=\int_V \rho\left(\frac{1}{2}v^2+\frac{1}{2}\kappa\Omega^2\right)dV, \tag{1.19}$$

consisting of translational and rotational contributions, where **v** and Ω are measured relative to the same reference frame as they are in Eq. (1.17).

A comment on potential energy is now in order. In writing Eq. (1.18) we have not included potential energy explicitly, choosing instead to include it as part of the work expression (1.17), as in Ref. [18]. This is further discussed in the Appendix.

An equation of change for the total kinetic energy is found by dot multiplying the mass-average velocity **v** and spin field Ω by the respective linear and internal angular momentum equations, (1.1a) and (1.1b), and adding the resulting expressions. When integrated over the region V this results in the following macroscopic equation:

$$\begin{aligned}\frac{d}{dt}\int_V \rho\left(\frac{1}{2}v^2+\frac{1}{2}\kappa\Omega^2\right)dV &= \oint_{\partial V} d\mathbf{S}\cdot(\mathbf{T}\cdot\mathbf{v}+\mathbf{C}\cdot\Omega)+\int_V(\mathbf{f}\cdot\mathbf{v}+\mathbf{l}\cdot\Omega\\ &\quad -\mathbf{T}^{\mathrm{T}}:\nabla\mathbf{v}-\mathbf{C}^{\mathrm{T}}:\nabla\Omega+\mathbf{T}_\times\cdot\Omega)\,dV,\end{aligned} \tag{1.20}$$

where the superscript T denotes the transposition operator.

Upon using Eq. (1.17) for the rate of working and subtracting Eq. (1.20) from Eq. (1.18), the following is obtained for the rate of change of the other forms of energy:

$$\frac{dE_O}{dt}=\Psi+\dot{Q}, \tag{1.21}$$

where, by definition,

$$\Psi \overset{\text{def.}}{=} \int_V\left[\mathbf{T}_S:(\nabla\mathbf{v})_S+\mathbf{C}^{\mathrm{T}}:\nabla\Omega+\mathbf{T}_\times\cdot\left(\frac{1}{2}\nabla\times\mathbf{v}-\Omega\right)\right]dV, \tag{1.22}$$

in which the subscript S refers to the symmetric component, $\mathbf{D}_S=\frac{1}{2}(\mathbf{D}+\mathbf{D}^{\mathrm{T}})$, of a dyadic **D**. Equation (1.22) is interpreted as representing the rate of transformation of mechanical energy into ''other,'' nonmechanical forms of energy contained within V.

In the classical description of Newtonian fluids the rate of mechanical energy exchange with other modes is written as [18,19]

$$\Psi^N=\int_V(-p\nabla\cdot\mathbf{v}+\boldsymbol{\tau}:\nabla\mathbf{v})\,dV, \tag{1.23}$$

with p the thermodynamic pressure and $\boldsymbol{\tau}$ the deviatoric viscous stress. For these systems the first term in the integrand of Eq. (1.23) is identified as the ''reversible'' rate of change of the internal energy of the system due to fluid compressibility (in such fluid systems E_O is solely composed of the internal energy), whereas the second term in Eq. (1.23) is identified as the irreversible rate of internal energy increase, the so-called viscous dissipation rate [18,19].

Various reasons exist for not effecting similar steps and subsequent interpretative identifications with Eq. (1.21) at this stage. First, we wish to keep our analysis general with respect to the constitutive forms chosen for the dynamical quantities $\mathbf{f}$, $\mathbf{l}$, $\mathbf{T}$, and $\mathbf{C}$. Second, we do not wish to enter the dispute [17,20–23] over the exact form of the nonmechanical energy term when, for example, electromagnetic fields are present. Finally, in order to be able to ascribe physical significance to the terms resulting from Eq. (1.22), one requires knowledge of the reversible thermodynamics of these systems (involving second law considerations). In the case of Newtonian fluids it is assumed *a priori* that the reversible work contributing to the internal energy consists solely of a $-p\,dV$ term, whence this term is identified with the $-p\boldsymbol{\nabla}\cdot\mathbf{v}$ term appearing in Eq. (1.23). We stress that this does not imply that the $-p\boldsymbol{\nabla}\cdot\mathbf{v}$ term of Eq. (1.23) is to be regarded as a volumetric "work density" in the irreversible formulation. Two reasons exist for this: First, in violation of the mechanical definition of work, it cannot be expressed as the product of a vector force and a vector displacement rate without combining it with other terms. Second, if in fact

$$-\int_V p\boldsymbol{\nabla}\cdot\mathbf{v}\,dV$$

were indeed a work term, the expression for the total work (1.17) would have been incomplete, and the preceding analysis given for Newtonian fluids would have been incorrect. Historically, the classical thermodynamic interpretation of the $-p\boldsymbol{\nabla}\cdot\mathbf{v}$ term may be traced to the interpretation of Joule's original experiments, which laid the foundation for the subject, wherein the term $-p\boldsymbol{\nabla}\cdot\mathbf{v}$ was identified with the "disappearance" of "external" mechanical energy.

Mathematically, Eq. (1.22) does not add any new physics to the total force, torque, and work trio of Eqs. (1.6), (1.11), and (1.17), but serves merely in an interpretative role. It should be clear from its derivation that any uncertainty in any of the latter three entities will manifest itself as a comparable uncertainty in Eq. (1.22). The purpose of considering this equation, in addition to this trio, resides in the conceptual and physical interpretation of the consequences of any such uncertainties in the total force, torque, and work calculations. Equation (1.22) simply identifies that portion of the work and mechanical energy that is transformed into nonmechanical forms of energy within the system, information that would be relevant in problems where mechanical dissipation effects were important, or where changes in the thermodynamic state of a system were relevant.

B. "Maxwellian" forces and the Maxwell stress tensor

A volumetric external force-density field, $\mathbf{f}^M \equiv \mathbf{f}^M(\mathbf{x})$, will be said to be "Maxwellian" if it can be written as the divergence of a dyadic field $\mathbf{T}^M \equiv \mathbf{T}^M(\mathbf{x})$,

$$\mathbf{f}^M = \boldsymbol{\nabla}\cdot\mathbf{T}^M. \tag{1.24}$$

For a prescribed constitutive equation governing $\mathbf{f}^M$ this represents the constitutive definition, albeit necessarily nonunique, of $\mathbf{T}^M$. In the electromagnetic theory of charges and currents in vacuum, the dyadic $\mathbf{T}^M$ is referred to as the Maxwell stress tensor (MST). A particular example of its use occurs in the area of magnetic fluids (so-called "ferrofluids" [24]), where $\mathbf{f}^M$ obeys the constitutive relation [25]

$$\mathbf{f}^M = \mu_0 \mathbf{M}\cdot\boldsymbol{\nabla}\mathbf{H}, \tag{1.25}$$

applicable to incompressible media, with $\mathbf{M}$ the magnetization, $\mathbf{H}$ the local magnetic field, and μ_0 the permeability of free space. In the case of a force density described by this constitutive equation it can easily be shown, using the pertinent equations describing the magnetostatic field in the absence of free currents [26], namely, $\boldsymbol{\nabla}\cdot\mathbf{B}=0$ and $\boldsymbol{\nabla}\times\mathbf{H}=\mathbf{0}$, together with some elementary vector-dyadic identities, that the corresponding Maxwell stress tensor is

$$\mathbf{T}^M = \mathbf{B}\mathbf{H} - \frac{1}{2}\mu_0 H^2 \mathbf{I}, \tag{1.26}$$

with $\mathbf{B}=\mu_0(\mathbf{M}+\mathbf{H})$, and $\mathbf{I}$ the unit tensor [27]. This tensor, though symmetric for linear magnetic media (where $\mathbf{M}$ is collinear with $\mathbf{H}$) may, in general, be asymmetric, as in the case of ferrofluid flows.

The Maxwell stress tensor is introduced in several different ways in standard textbooks on electromagnetism and its applications. For example, in Stratton's treatise on electromagnetism [26] it is shown to be a mathematical consequence of Maxwell's equations that the body-force density within a system of charges and currents in vacuum may be written as the divergence of a tensor field added to the local time derivative of $c^{-2}\mathbf{E}\times\mathbf{H}$ (the "electromagnetic momentum" of the field at a point $\mathbf{x}$), with $\mathbf{E}$ the electric field and c the speed of light in vacuum. In quasistatic fields, this latter term is neglected, whence the force is represented by the divergence of the MST. This tensorial representation of the dynamical electrostatic and magnetostatic state of a body is analogous to that of an elastic body, owing to the existence of the Cauchy stress tensor for such elastic bodies. Historically, this analogy was consistent with existing aether theories of electromagnetism towards the end of the 19th century. However, as Stratton ultimately states: "all that can be said ... is that mutual forces between elements of charge can be correctly calculated on the assumption that there exists a *fictitious* state of stress" (emphasis ours).

De Groot and Mazur [28] introduce the MST through a statement of conservation of total (i.e., mechanical and electromagnetic) momentum. Their continuum formulation is a consequence of Newton's momentum conservation principle for discrete point mass systems when applied to charges and currents in electromagnetic fields, owing to the formal representation of the forces acting upon such systems as consisting of the sum of the divergence of the MST and the time derivative of the electromagnetic momentum. Superficial discussion is devoted by these authors to the issue of the reality of the MST as a "physical stress," although they point out that the Maxwell stress tensor is not uniquely defined by Eq. (1.24), but, rather, is arbitrary to within an additive divergenceless tensor, as we have remarked above.

In Melcher's book [29] on continuum electromechanics, on the other hand, the MST is introduced merely as a mathematical artifice for the purpose of calculating the quasistatic

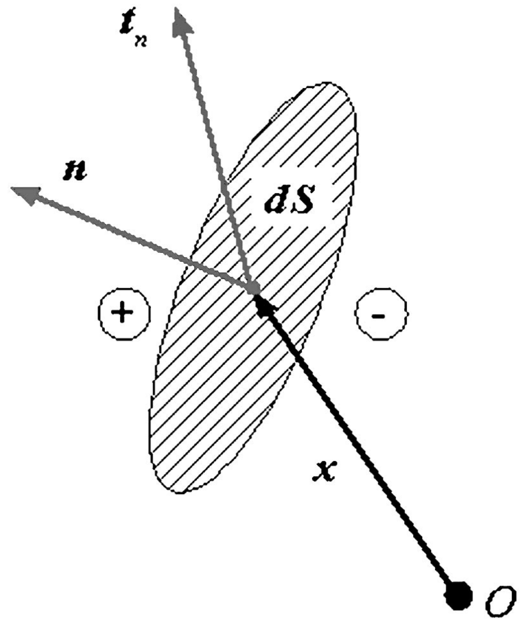

FIG. 1. Classical definition of the traction $\mathbf{t_n}$ as the force per unit area exerted by the fluid on the $(+)$ side of a directed surface element $d\mathbf{S}=\mathbf{n}dS$ centered at the point $\mathbf{x}$, on the fluid in the $(-)$ side.

electromagnetic force on a ponderable body. Melcher also observes that the conceptual replacement of $\mathbf{f}^M$ by an ''equivalent'' stress field $\mathbf{T}^M$ furnishes the correct torque on a body only for a *symmetric* MST. By ''conceptual replacement'' is meant regarding the effects of the Maxwellian force $\mathbf{f}^M$ as arising from a stress distribution $\mathbf{T}^M$. Melcher's book [29] provides an exhaustive compendium of the varied forms adopted by the MST for different electrostatic and magnetostatic force densities.

Stratton's and Melcher's books each emphasize that the ''traction'' due to the MST is devoid of physical significance unless it is integrated over a *closed* surface, a consequence of the application of Gauss' theorem upon converting a volume integral to a surface integral, such as was done with Eq. (1.5). Indeed, this definition of the ''electromagnetic traction'' is reversed from the classical definition in continuum mechanics. A traction $\mathbf{t_n}$ is classically defined, referring to Fig. 1, as the force exerted by the fluid present on the $(+)$ side of a surface element dS, centered at a point $\mathbf{x}$ of the fluid continuum, upon the fluid lying on the $(-)$ side of the surface. Using Cauchy's tetrahedron argument, this traction may be shown to equal the vector dot product of the unit normal $\mathbf{n}$ to the surface element dS and a dyadic $\mathbf{T}$, called the stress. Thus, the traction $\mathbf{t_n}$ has a distinct physical meaning as a vector force per unit area. Upon integration over an open surface S, the traction $\mathbf{t_n}$ correctly gives the force exerted by the fluid on one side of the surface upon the fluid immediately proximate on the opposite side.

The definition $\mathbf{t}_\mathbf{n}^M \overset{\text{def.}}{=} \mathbf{n}\cdot\mathbf{T}^M$, of the electromagnetic traction $\mathbf{t}_\mathbf{n}^M$, proceeds in the opposite direction [30]. First, the MST is defined via Eq. (1.24), following which it is dot multiplied with the unit normal vector $\mathbf{n}$ of the test surface dS centered at $\mathbf{x}$. Though this is a mathematically acceptable operation, it is the physical interpretation ascribed to it, namely application of the term ''traction'' to the dot product $\mathbf{n}\cdot\mathbf{T}^M$, that is the issue at dispute. The electromagnetic traction, upon integration over an open surface S, will *not* correctly furnish the force exerted by the fluid lying on one side of the surface upon the contiguous fluid immediately opposite.

As pointed out by Stratton [26], introduction of the MST into electromagnetism occurred during the prerelativity era, at which time the so-called aether was regarded as an elastic material entity, pervading all of space, and capable of sustaining stresses in response to electromagnetic forces. In this context, Maxwell stresses were regarded as possessing definitive physical existence within the material aether continuum. This view of the aether was set aside by the work of Einstein [31]. Despite this, the concept of the Maxwell stress as an objective physical entity was not abandoned concurrently with the discredited aether theory. Indeed, its ubiquitous use in applications has persisted to this day, leading, for example, to inadequacies in calculations of energy transfer occurring during magnetohydrodynamic flows [35], and conceptually spurious proofs of the inapplicability of the mechanical definition of stress [23] (both discussed in Sec. II), as well as conceptual inadequacies in formulations of key equations describing magnetic fluid phenomena [24,32–34] (discussed in Sec. III).

Moreover, during the prerelativity era, the concept of a separate Cauchy moment-of-momentum equation appears to have been unknown, leading to the erroneous conclusion that all stresses (both electromagnetic and otherwise) were symmetric [36,37]. This contributed further to the continued view of the MST as being a physical entity; for under these conditions, use of the stress field $\mathbf{T}^M$ as being equipollent in its consequences to the body-force field $\mathbf{f}^M$ that spawned it, yields not only the correct force on a body but equally the correct torque, as is demonstrated below. However, this is not the case for higher-order moments of the force distribution $\mathbf{f}^M$, namely, those beyond the zeroth and first, nor for other physical quantities, such as: (i) the work done by the distant and local surroundings upon a body via the action of forces and couples; and (ii) the accompanying rate of mechanical energy conversion into other forms, as discussed in the following section.

II. CONSEQUENCES OF REPLACING THE MAXWELLIAN FORCE $\mathbf{f}^M$ BY A MAXWELL STRESS $\mathbf{T}^M$

Despite the discredited physical validity of the Maxwell stress tensor as a physical state of stress existing within a hypothetical elastic continuum, namely, the ''aether,'' contemporary advocates of its continued use in electromagnetic applications appear willing to set aside this fact in favor of its pragmatic utility when calculating the forces on bodies. However, this success in furnishing the correct force on a body (and even the correct torque under certain well-defined circumstances) obscures the fact that the very idea of replacing a body-force density field $\mathbf{f}^M$ by an ''equivalent'' stress field $\mathbf{T}^M$ is conceptually flawed on physical grounds and, as such, may lead to invalid conclusions when the MST concept is indiscriminately used in related physical contexts arising in mechanics and electrodynamics. Such errors include incorrect calculations of the torque on bodies in circumstances where the MST is asymmetric, as well as incorrect predictions of the work done by the surroundings on an electromagnetic continuum and, hence, of the rate of kinetic energy exchange with other forms of energy. Even apart from the specifics of these examples is the physical inappropriateness of linearly adding the "stress" $\mathbf{T}^M$ stemming from a Maxwellian *body* force $\mathbf{f}^M$ to the Cauchy stress $\mathbf{T}$ existing within elastic or fluid-mechanical systems so as to obtain a so-called "total" stress tensor as, for example, is often done in the case of ferrofluids [24,32–34]. In any given physical problem, for example, in fluid mechanics, there exists but a *single* stress, namely, the Cauchy stress resulting solely from the existence of contiguous matter in all its attributes, thereby rendering the concept of a "total" stress an oxymoron.

To establish the circumstances under which the quantity $\mathbf{T}^M$ defined in Eq. (1.24) might be inappropriately identified as a stress, add and subtract the vector field $\mathbf{f}^M$ to Eq. (1.1a), and use Eq. (1.24) to write the linear momentum equation in the form

$$\rho\frac{D\mathbf{v}}{Dt}=\nabla\cdot\mathbf{T}^{\dagger}+\mathbf{f}^{\dagger}, \tag{2.1}$$

where, by definition,

$$\mathbf{T}^{\dagger}\overset{\text{def.}}{=}\mathbf{T}+\mathbf{T}^{M}, \tag{2.2}$$

is the so-called "total" stress [24], and

$$\mathbf{f}^{\dagger}\overset{\text{def.}}{=}\mathbf{f}-\mathbf{f}^{M} \tag{2.3}$$

is a "modified" body-force density (representing, in the case of magnetic fluids, the effect of other forces besides the magnetic).

While this total stress is frequently envisioned [3,5,24,32–34] as a physical stress, rather than as a purely mathematical artifice, this view is physically without merit. Were this view correct, the vector invariant appearing in Eq. (1.1b) would have to be $\mathbf{T}_{\times}^{\dagger}=\mathbf{T}_{\times}+\mathbf{T}_{\times}^{M}$, rather than $\mathbf{T}_{\times}$, in order to be consistent with the interpretation of $\mathbf{T}^{\dagger}$ in Eq. (2.1) as the "stress."

Were one to calculate the force $\mathbf{F}^{\dagger}$, torque $\mathbf{L}^{\dagger}$, rate of working $\dot{W}^{\dagger}$, and rate of mechanical energy exchange with other forms $\Psi^{\dagger}$ in terms of the fields $\mathbf{T}^{\dagger}$ and $\mathbf{f}^{\dagger}$ appearing in the linear momentum equation (2.1), the expressions for these four entities would respectively be, by analogy with Eqs. (1.6), (1.11), (1.17), and (1.22):

$$\mathbf{F}^{\dagger}=\int_V \mathbf{f}^{\dagger}dV+\oint_{\partial V}d\mathbf{S}\cdot\mathbf{T}^{\dagger}, \tag{2.4a}$$

$$\mathbf{L}^{\dagger}=\int_V \mathbf{x}\times\mathbf{f}^{\dagger}dV+\oint_{\partial V}\mathbf{x}\times(d\mathbf{S}\cdot\mathbf{T}^{\dagger})+\int_V \mathbf{l}dV+\oint_{\partial V}d\mathbf{S}\cdot\mathbf{C}, \tag{2.4b}$$

$$\dot{W}^{\dagger}=\int_V \mathbf{f}^{\dagger}\cdot\mathbf{v}dV+\oint_{\partial V}d\mathbf{S}\cdot\mathbf{T}^{\dagger}\cdot\mathbf{v}+\int_V \mathbf{l}\cdot\Omega dV+\oint_{\partial V}d\mathbf{S}\cdot\mathbf{C}\cdot\Omega, \tag{2.4c}$$

$$\Psi^{\dagger}=\int_V \mathbf{T}_S^{\dagger}:(\nabla\mathbf{v})_S dV+\int_V \mathbf{C}^{\mathrm{T}}:\nabla\Omega dV+\int_V \mathbf{T}_{\times}^{\dagger}\cdot\left(\frac{1}{2}\nabla\times\mathbf{v}-\Omega\right)dV. \tag{2.4d}$$

Subtract Eqs. (2.4) from their respective counterparts in Eqs. (1.6), (1.11), (1.17), and (1.22), denote the differences as $\Delta\mathbf{F}^{\dagger}=\mathbf{F}^{\dagger}-\mathbf{F}$, etc., and use Eq. (1.24) together with some elementary vector-dyadic identities to obtain

$$\Delta\mathbf{F}^{\dagger}=\mathbf{0}, \tag{2.5a}$$

$$\Delta\mathbf{L}^{\dagger}=-\int_V \mathbf{T}_{\times}^{M}dV, \tag{2.5b}$$

$$\Delta\dot{W}^{\dagger}=\int_V \mathbf{T}_S^{M}:(\nabla\mathbf{v})_S dV+\int_V \mathbf{T}_{\times}^{M}\cdot\frac{1}{2}\nabla\times\mathbf{v}dV, \tag{2.5c}$$

$$\Delta\Psi^{\dagger}=\int_V (\mathbf{T}_S^{M})^{\mathrm{T}}:(\nabla\mathbf{v})_S dV+\int_V \mathbf{T}_{\times}^{M}\cdot\left(\frac{1}{2}\nabla\times\mathbf{v}-\Omega\right)dV. \tag{2.5d}$$

Were the Maxwellian stress $\mathbf{T}^M$ to be physically equivalent *in all respects* to the Maxwellian body-force density $\mathbf{f}^M$ from which it derives, the quantities $\Delta\mathbf{F}^{\dagger}$, $\Delta\mathbf{L}^{\dagger}$, $\Delta\dot{W}^{\dagger}$, and $\Delta\Psi^{\dagger}$ would then each be identically zero. That, with the exception of the force, they are not zero confirms that there can be one, and only one, correct stress tensor in a given physical situation. Moreover, the disparity existing in the rate of working and mechanical energy dissipation, namely, Eqs. (2.5c) and (2.5d), respectively, reveals that the fundamental issue arising from use of the Maxwell stress does not disappear even when the stress tensor is symmetric ($\mathbf{T}_{\times}^{M}=\mathbf{0}$). Indeed, the only general case for which Eq. (2.5c) and Eq. (2.5d) are both identically zero occurs for a rigid-body motion, $\mathbf{v}=\mathbf{A}(t)+\mathbf{x}\times\mathbf{B}(t)$, together with a symmetric Maxwell stress, $\mathbf{T}_{\times}^{M}=\mathbf{0}$, where $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are, respectively, a position-independent vector and pseudovector.

The physical interpretation of the results embodied in Eqs. (2.5) is as follows: The force $\mathbf{F}^{\dagger}$ on the fluid domain V is correctly calculated upon replacing the body force by an "equivalent" Maxwell stress tensor, but the torque $\mathbf{L}^{\dagger}$ is not (compared with the corresponding quantity without the † superscript). As a consequence, the rate of spin-kinetic energy change is incorrectly calculated. When combined with the incorrect estimate for the rate of working $\dot{W}^{\dagger}$, this leads to a false estimate for the rate of mechanical energy exchange with other modes, $\Psi^{\dagger}$, and concomitantly to an incorrect estimate of the thermodynamic state of the fluid region V (this being related to the ''other'' forms of energy by an equation of state).

The conclusion emanating from Eq. (2.5b) agrees with Melcher's [29] comment regarding the required symmetry of the Maxwell stress tensor, although this caution is widely ignored in the literature. It is also important to note that the MST must be symmetric throughout the *entire* domain V in order for the correct torque to result [i.e., for Eq. (2.5b) to be identically zero]. This contrasts with statements in the literature [38–47] to the effect that, when calculating the torque on a system using the MST, it suffices for the latter to be symmetric *only over the domain of integration*, namely, the bounding surface ∂V. In fact, a survey of the current literature bearing on use of the Maxwell stress concept to calculate torques reveals basically three types of comments on the issue: (1) those in which no mention is made of the required symmetry of the MST for torque calculations [38–44]; (2) those that claim that the MST need only be symmetric on the surface ∂V of the body, thus ignoring the possibility of effects arising from the existence of stress asymmetry within the body itself [45,46]; (3) those that actually use the MST to calculate the torque on a body in circumstances wherein the MST is asymmetric [47]. Judging from Eq. (2.5b), these views all lack validity, and Melcher's comment is correct. However, certain subtleties exist in magnetostatic and electrostatic cases that must be further addressed before rendering an unequivocal judgement regarding the accuracy of torque calculations based on the Maxwell stress concept. These are discussed in the next section.

A specific example drawn from the recent literature, wherein the MST is inappropriately used to calculate work — leading thereby to an incorrect result for the work — is found in Ref. [35]. These authors analyze instabilities in solar convective flow in the context of an analysis of energy conversion mechanisms. In calculating the conversion of ''magnetic energy'' into kinetic energy, the MST is used as a physical stress in performing a surface integration, identical in spirit to that used to calculate the work arising from the classical Reynolds fluid-mechanical stress. The ''magnetic energy'' used in Ref. [35] would be part of what is here classified as ''other'' energies. The conclusion derived from Eq. (2.5d) suggests that this approach is without merit.

Another example involving use of the Maxwell stress in calculations of work is found in Ref. [23], where the authors examine the thermodynamic consistency of the continuum-mechanical definition of stress. The authors analyze the change in Helmholtz free energy for two processes possessing the same initial and final states, from which they determine that one of the components of their ''mechanical'' stress is object-shape dependent, a contradiction of their definition of stress. However, in their analysis they treat electromagnetic interactions through the corresponding Maxwell stress tensor, thereby invalidating their analysis on the basis of the arguments alluded to above. We refrain from presenting a more detailed analysis of the specifics of their problem, as surfaces of discontinuity exist in some of their material properties, thereby raising questions as to the correct form of the corresponding surface-excess forces [48].

As noted above, the MST, defined by Eq. (1.24), is arbitrary to within an additive divergenceless tensor [28]. However, the relations outlined above necessarily apply for any MST, no specific form of the MST [such as the constitutive form (1.26)] having been assumed *a priori*. This precludes the possibility of adding a solenoidal ''correction tensor'' to any form of the MST in order to nullify the errors displayed in Eq. (2.5).

III. THE "MAXWELL STRESS" IN FERROFLUIDS

A subtle exception to Eq. (2.5b), one explicitly involving replacement of a Maxwellian body-force density by a Maxwell stress tensor, and the concomitant lack of objectivity alluded to above, occurs in the case of magnetic fluids (''ferrofluids''). These materials provide commercially and scientifically important examples of fluids characterized rheologically by asymmetric states of stress. The description of the rheological state of ferrofluids in terms of a ''total'' stress, as defined above, was, in fact, the initial motivation behind the present work.

For such fluids, the volumetric body-force field $\mathbf{f}^M$, Eq. (1.25), arising from magnetic forces acting on the ferrofluid, is sometimes replaced by the magnetic Maxwell stress tensor $\mathbf{T}^M$ [24,32–34], Eq. (1.26), following which the linear and internal angular momentum equations are subsequently reformulated using the ''total'' stress in place of the true Cauchy stress. For the reasons outlined above, this ''equivalent'' substitution is, in general, physically invalid, in the sense that it may lead to discrepancies in those applications involving calculations other than that of determining the force (and possibly the torque) on a body. However, a seemingly fortuitous relation existing between the body-couple density field $\mathbf{l}$ and the vector invariant $\mathbf{T}^M_\times$ of the Maxwell stress confounds the issue [49]. In the ferrofluid case, the body-couple density field, say $\mathbf{l}^M$, due to magnetic forces is given constitutively as [24]

$$\mathbf{l}^M=\mu_0\mathbf{M}\times\mathbf{H}. \tag{3.1}$$

It is then a simple matter to verify, using Eq. (1.26), that for ferrofluids the following relation holds:

$$\mathbf{T}^M_\times=\mathbf{l}^M. \tag{3.2}$$

This makes it possible to rewrite the linear and angular momentum equations (1.1a) and (1.1b) in the respective forms

$$\rho\frac{D\mathbf{v}}{Dt}=\boldsymbol{\nabla}\cdot\mathbf{T}^\ddagger+\mathbf{f}^\ddagger, \tag{3.3a}$$

$$\rho\frac{D\mathbf{a}}{Dt}=\boldsymbol{\nabla}\cdot\mathbf{C}+\mathbf{T}^\ddagger_\times+\mathbf{l}^\ddagger. \tag{3.3b}$$

Here, as employed in the ferrofluids literature [32–34], the new symbol $\mathbf{T}^\ddagger$ corresponds to the so-called ''total stress'' at a point $\mathbf{x}$ of the ferrofluid, whereas $\mathbf{f}^\ddagger$ and $\mathbf{l}^\ddagger$, respectively, correspond to force and couple densities arising from ''distant'' sources (not including magnetic forces and couples).

These symbols are related to the comparable quantities appearing in Eqs. (1.1) and (1.24) by the respective expressions [50]

$$\mathbf{T}^{\ddagger} \overset{\text{def.}}{=} \mathbf{T} + \mathbf{T}^{M}, \tag{3.4}$$

$$\mathbf{f}^{\ddagger} \overset{\text{def.}}{=} \mathbf{f} - \mathbf{f}^{M}, \tag{3.5}$$

$$\mathbf{l}^{\ddagger} \overset{\text{def.}}{=} \mathbf{l} - \mathbf{l}^{M} \equiv \mathbf{l} - \mathbf{T}_{\times}^{M}. \tag{3.6}$$

In a manner analogous to that of the previous section, the apparent force $\mathbf{F}^{\ddagger}$, torque $\mathbf{L}^{\ddagger}$, rate $\dot{W}^{\ddagger}$ of working, and rate $\Psi^{\ddagger}$ of mechanical energy interchange with other modes experienced by a body, corresponding to the physical interpretation assigned to the symbols appearing in Eqs. (3.3), are respectively given by the expressions

$$\mathbf{F}^{\ddagger} = \int_V \mathbf{f}^{\ddagger} dV + \oint_{\partial V} d\mathbf{S} \cdot \mathbf{T}^{\ddagger}, \tag{3.7a}$$

$$\mathbf{L}^{\ddagger} = \int_V \mathbf{x} \times \mathbf{f}^{\ddagger} dV + \oint_{\partial V} \mathbf{x} \times (d\mathbf{S} \cdot \mathbf{T}^{\ddagger}) + \int_V \mathbf{l}^{\ddagger} dV + \oint_{\partial V} d\mathbf{S} \cdot \mathbf{C}, \tag{3.7b}$$

$$\dot{W}^{\ddagger} = \int_V \mathbf{f}^{\ddagger} \cdot \mathbf{v} dV + \oint_{\partial V} d\mathbf{S} \cdot \mathbf{T}^{\ddagger} \cdot \mathbf{v} + \int_V \mathbf{l}^{\ddagger} \cdot \Omega dV + \oint_{\partial V} d\mathbf{S} \cdot \mathbf{C} \cdot \Omega, \tag{3.7c}$$

$$\Psi^{\ddagger} = \int_V \mathbf{T}_S^{\ddagger} : (\nabla \mathbf{v})_S dV + \int_V \mathbf{C}^{\mathrm{T}} : \nabla \Omega dV + \int_V \mathbf{T}_{\times}^{\ddagger} \cdot \left(\frac{1}{2} \nabla \times \mathbf{v} - \Omega \right) dV. \tag{3.7d}$$

As before, sequentially subtract these equations from their respective counterparts, Eqs. (1.6), (1.11), (1.17) and (1.22), denote the differences by $\Delta \mathbf{F}^{\ddagger} = \mathbf{F}^{\ddagger} - \mathbf{F}$, etc., and use Eqs. (1.24) and (3.2) together with some elementary vector-dyadic identities to eventually obtain the expressions

$$\Delta \mathbf{F}^{\ddagger} = \mathbf{0}, \tag{3.8a}$$

$$\Delta \mathbf{L}^{\ddagger} = \mathbf{0}, \tag{3.8b}$$

$$\Delta \dot{W}^{\ddagger} = \int_V \mathbf{T}_S^{M} : (\nabla \mathbf{v})_S dV + \int_V \mathbf{T}_{\times}^{M} \cdot \left(\frac{1}{2} \nabla \times \mathbf{v} - \Omega \right) dV, \tag{3.8c}$$

$$\Delta \Psi^{\ddagger} = \int_V \mathbf{T}_S^{M} : (\nabla \mathbf{v})_S dV + \int_V \mathbf{T}_{\times}^{M} \cdot \left(\frac{1}{2} \nabla \times \mathbf{v} - \Omega \right) dV. \tag{3.8d}$$

Again, the disparities Δ represented by Eqs. (3.8) must be identically zero if the body-force/body-couple density and Maxwell stress descriptions of magnetic effects are to be physically equivalent. The fact that the differences $\Delta \dot{W}^{\ddagger}$ and $\Delta \Psi^{\ddagger}$ are not identically zero in general further supports the contention that there exists but a single physically meaningful stress tensor in magnetic fluids, namely, the classical Cauchy stress tensor $\mathbf{T}$.

The physical interpretation of the results embodied in Eqs. (3.8) is as follows: conceptual replacement of a body-force and body-couple field by their corresponding Maxwell stress counterparts will give correct estimates for the total force $\mathbf{F}^{\ddagger}$ and torque $\mathbf{L}^{\ddagger}$ acting on a fluid domain V; hence, the rate of kinetic energy change will be correctly calculated using the MST, through an analog of Eq. (1.20). However, the rate of working $\dot{W}^{\ddagger}$ on the fluid domain and the concomitant rate of mechanical energy exchange with other forms, $\Psi^{\ddagger}$, will be incorrect (as compared with the quantities without the superscript ‡), with the additional work going directly into the "other" energies through Eq. (1.21).

As discussed in the previous sections of this paper, and as shown by Eqs. (3.8), the analysis of magnetic fluid motion presented in Refs. [24,32–34], together with the corresponding physical description of the state of stress in ferrofluids, is questionable in the sense that it replaces the Maxwellian magnetic force density (1.25) with the corresponding Maxwell stress tensor, Eq. (1.26). This is not to say that the fundamental equations used in ferrohydrodynamics are *mathematically* wrong, but rather that their physical interpretation is based on an invalid conceptual framework. Failure to recognize this fact may result in incorrect physical predictions.

The analysis of this section can be trivially extended to more general circumstances, involving magnetoquasistatic and electroquasistatic systems described by Kelvin-type force and couple densities [29], where it can be shown that a condition equivalent to Eq. (3.2) applies. In the light of these results, specifically that for the torque difference (3.8b), we concur with Melcher's [29] assertion that the MST will give the correct torque on a body only for a symmetric MST, albeit subject to the following caveat: When a relation exists between the body-couple density $\mathbf{l}$ and vector invariant $\mathbf{T}_{\times}^{M}$ of the Maxwell stress, such as is embodied in Eq. (3.2), replacement of the body-force density $\mathbf{f}^{M}$ *and* body-couple density $\mathbf{l}^{M}$ by the equivalent Maxwell stress tensor $\mathbf{T}^{M}$ will yield the correct torque on a body. Failure to do so would result in double counting the magnetic couple effect in torque calculations. Referring to the three classes of commentary regarding the symmetry issue alluded to in the previous section, we find that the conclusion of the first group is correct. Explicitly, it is unnecessary for the constitutive response of the material to be linear along the path of integration (or indeed anywhere in the body) in order for the MST to give the correct torque in the *special case* of materials described by Kelvin-type forces and couples.

IV. EXAMPLE—FLOW OF A FERROFLUID IN A CYLINDRICAL CONTAINER SUBJECTED TO A UNIFORM ROTATING MAGNETIC FIELD

Consider a ferrofluid contained in an infinitely long circular cylinder of radius R whose walls are held stationary [51].

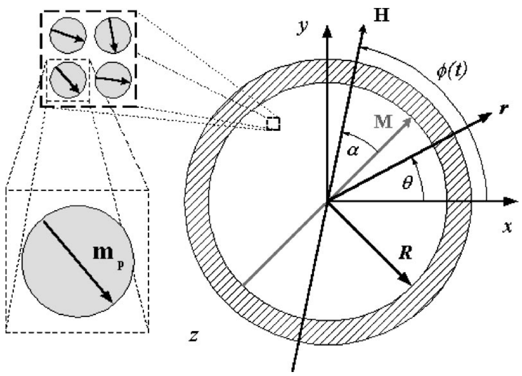

FIG. 2. Ferrofluid contained in a fixed cylinder and subjected to a rotating magnetic field **H**. The field rotates at an angular rotation rate $\dot{\phi}$, relative to the fixed container walls. The ferrofluid consists of a subcontinuum suspension of permanently magnetized particles with particle magnetization $\mathbf{m_p}$. The resulting suspension-scale magnetization field is uniform and lags the magnetic field by an angle $\alpha < \pi/2$.

The ferrofluid is subjected to a magnetic field that is uniform throughout the fluid volume and which rotates steadily (relative to the cylinder walls) at a specified angular velocity ω. This can be achieved by placing the cylindrical ferrofluid container in the gap of a two-pole magnetic induction machine. The solution for the flow equations used here is that presented in Ref. [24], and is consistent with the linear and internal angular momentum equations for an incompressible ferrofluid whose rheological behavior is described by a Newtonian constitutive relation for the symmetric portion of the stress together with an antisymmetric stress [52]. The couple-stress dyadic is given a Newtonian-like constitutive equation [52], whereas the body-force and body-couple densities are given by Eqs. (1.25) and (3.1), respectively, with the corresponding Maxwell stress given by Eq. (1.26). The resulting solution is also consistent with the magnetization equation for ferrofluids [24], provided that the spin time scale, Ω^{-1}, is much larger than the ferrofluid relaxation time τ $(\Omega\tau \ll 1)$.

Because the magnetic field is uniform throughout the ferrofluid, no body forces influence the motion, whereas the body couple density is given by

$$\mathbf{l}(\mathbf{x}) = l_z(r)\mathbf{i}_z, \quad l_z(r) = \mu_0 MH \sin\alpha, \tag{4.1}$$

with α the lag angle between the magnetic field and the magnetization vector, as in Fig. 2. The magnitude of the magnetization vector and its lag angle with respect to the magnetic field are related to the magnetic field rotation rate ω and the Brownian relaxation time constant τ of the ferrofluid through the magnetization equation [24]. Explicitly, these dependencies are

$$M = M_0[1 + (\omega\tau)^2]^{-1/2}, \tag{4.2}$$

$$\alpha = \arctan(\omega\tau), \tag{4.3}$$

where M_0 is the magnitude of the magnetization when in equilibrium with the magnetic field **H**.

In circular cylindrical coordinates (r,θ,z), the fluid streamlines lie in concentric circles about the axis of rotation, and are given by

$$\mathbf{v}(\mathbf{x}) = v_\theta(r)\mathbf{i}_\theta, \quad v_\theta(r) = v_0\left[\frac{r}{R} - \frac{I_1(\kappa r)}{I_1(\kappa R)}\right], \tag{4.4}$$

whereas the spin-velocity field Ω is axially directed:

$$\Omega(\mathbf{x}) = \Omega_z(r)\mathbf{i}_z, \quad \Omega_z(r) = \frac{\eta_e}{\eta_R}\left(\frac{\mu_0 MH}{4\zeta}\sin\alpha\right)\left[1 - \frac{I_0(\kappa r)}{I_0(\kappa R)}\right]. \tag{4.5}$$

Here, $I_n(x)$ is the modified Bessel function of the first kind of order n, and $(\mathbf{i}_r, \mathbf{i}_\theta, \mathbf{i}_z)$ are unit vectors in the indicated directions. The parameters appearing in these equations are related to the physical and geometrical properties characterizing the problem as follows:

$$v_0 = \frac{1}{2\kappa\eta_R}(\mu_0 MH\sin\alpha)\frac{I_1(\kappa R)}{I_0(\kappa R)}, \tag{4.6}$$

$$\eta_R = \eta + \zeta\left[1 - \frac{2I_1(\kappa R)}{\kappa R I_0(\kappa R)}\right], \tag{4.7}$$

$$\kappa^2 = \frac{4\eta\zeta}{\eta_e\eta'}. \tag{4.8}$$

In these equations, η, ζ, η', and $\eta_e = \eta + \zeta$ are, respectively, the shear, vortex, spin-shear, and effective viscosities of the ferrofluid.

To calculate the rate of working on the ferrofluid volume V, we apply Eq. (1.17), to obtain

$$\dot{W} = (\mu_0 MH\sin\alpha)^2\left(\frac{\eta_e}{4\eta_R\zeta}\right)\left[1 - 2\frac{I_1(\kappa R)}{\kappa R I_0(\kappa R)}\right]V, \tag{4.9}$$

in which all of the work is effected via the action of the volumetric body-couple and spin-velocity terms. This is a consequence of the assumed no-slip boundary condition [24] applied to both the translational and spin velocities at the cylinder wall.

Had we considered the effect of magnetic interactions as resulting from an electromagnetic stress, we would have replaced the effects of both the magnetic force and couple densities by the corresponding Maxwell stress, given by Eq. (1.26), and used Eq. (3.7c) to obtain the rate of working on the cylinder contents. By replacing the body force and couple densities by their corresponding Maxwell stress tensor, we find that the first two terms on the right-hand side of Eq. (3.7c) are now zero because $\mathbf{f}^\ddagger = \mathbf{0}$ and $\mathbf{l}^\ddagger = \mathbf{0}$, whereas the last two terms are zero owing to the no-slip conditions $\mathbf{v} = \mathbf{0}$ and $\mathbf{\Omega} = \mathbf{0}$ prevailing at the cylinder walls. Equation (3.7c) then yields

$$\dot{W}^\ddagger = 0, \tag{4.10}$$

according to which no work is done on the ferrofluid.

Note that the difference $\Delta\dot{W}^{\ddagger}$ between Eqs. (4.9) and (4.10) could have been obtained directly from Eq. (3.8c).

As stated earlier, there are no magnetic body forces in this example. The effects arise solely through magnetic body couples acting on the magnetically polarized suspension. However, if we consider the same physical situation, but allow the fields to vary in the radial and azimuthal directions (such as would occur if the cylinder of ferrofluid were placed in the gap of a four-pole magnetic induction machine), azimuthal body forces would arise. In that case it is clear that these body forces would do additional work on the flowing ferrofluid, above and beyond the work done by the action of the body couples, as in Eq. (4.9). However, should these body forces and couples be replaced by their corresponding Maxwell stress representations, the same arguments would apply as were used in obtaining Eq. (4.10), whence the Maxwell stress viewpoint would still indicate that no work was being done on the flowing ferrofluid.

V. CONCLUDING REMARKS

While the Maxwell stress tensor possesses pragmatic computational utility in simplifying the algebraic manipulations required to calculate the force exerted on bodies in electromagnetic and related fields, its ubiquitous use in this single physical context assigns to it an apparent role as a stress that belies the fact that this tensor is not equivalent in all of its physical consequences to the body-force density field from which it derives. Lacking such equivalent objectivity, its unquestioned use as a true Cauchy stress in contexts other than that of calculating the force on a body, can lead to both conceptual errors and consequent incorrect physical predictions. As such, use of the apparent body-force/Maxwell-stress duality of the electromagnetic effect, so fruitful in classical electromagnetic theory, cannot be unequivocally accepted when such effects are considered in a continuum-mechanical context involving the interaction of electromagnetic fields with matter.

Closely related to this observation is the fact that the concept of a "total stress" as consisting constitutively of a sum of separate stresses, each existing in the absence of the other, is an oxymoron. There is only one stress. In contrast, one could, without ambiguity, refer to a "total body-force density," consisting constitutively of separate body-force densities, when the distant sources giving rise to the contributing forces do not physically interact with one another, and hence possess separate and distinct existences (as, for example, with forces arising from classical gravitational and electromagnetic interactions). Should the distant sources of these forces interact in any significant way, the separateness of these forces would be lost (whence the magnetic and electric forces are not constitutively "separate" in this sense, as the electric and magnetic fields interact through Maxwell's equations, even in classical systems).

Most importantly, we have illustrated through physical arguments, mathematical manipulations, and simple examples the fundamental distinction existing between surface and body forces in continuum mechanics, and have shown that their unambiguous *physical* meaning should not be confounded through *purely mathematical* manipulations.

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APPENDIX: A COMMENT ON POTENTIAL ENERGY

The potential energy Φ assigned to a material volume V is typically said to consist of that energy which is a consequence of its position, shape, or state (including gravitational energy, electrical energy, nuclear energy, and chemical energy). Here we restrict our comments to potential energies associated with action-at-a-distance interactions (such as the gravitational and electrical potential energies), i.e., to potential energy as a consequence of the position of the material volume V in a body-force field $\mathbf{f}$.

When the body-force per unit mass, $\hat{\mathbf{f}} \overset{\text{def.}}{=} \rho^{-1}\mathbf{f}$, is expressible in terms of the gradient of a scalar function $\hat{\phi}(\mathbf{x},t)$, as in

$$\hat{\mathbf{f}} = -\nabla\hat{\phi}, \tag{A1}$$

the volumetric work associated with this force can eventually be manipulated into the form

$$\int_V \rho\hat{\mathbf{f}}\cdot\mathbf{v}\,dV = \int_V \rho\left(\frac{\partial\hat{\phi}}{\partial t} - \frac{D\hat{\phi}}{Dt}\right)dV, \tag{A2}$$

with $D\hat{\phi}/Dt$ the material derivative of the scalar $\hat{\phi}$. Combining this expression with Eqs. (1.13) and (1.18) yields

$$\frac{dE}{dt} = \int_V\left(\rho\frac{\partial\hat{\phi}}{\partial t} - \rho\frac{D\hat{\phi}}{Dt} + \mathbf{l}\cdot\Omega\right)dV + \oint_{\partial V} d\mathbf{S}\cdot(\mathbf{T}\cdot\mathbf{v} + \mathbf{C}\cdot\Omega) + \dot{Q}. \tag{A3}$$

If the scalar function $\hat{\phi}$ is restricted to be a function of position only, i.e., independent of time, then the term $\partial\hat{\phi}/\partial t$ vanishes, whereas the $D\hat{\phi}/Dt$ term may be moved to the left-hand side, and the Reynolds transport theorem [18] applied, to obtain

$$\frac{d}{dt}(E+\Phi) = \int_V \mathbf{l}\cdot\Omega\,dV + \oint_{\partial V} d\mathbf{S}\cdot(\mathbf{T}\cdot\mathbf{v} + \mathbf{C}\cdot\Omega) + \dot{Q}, \tag{A4}$$

where the potential energy Φ assigned to the material volume V is defined as

$$\Phi(t) = \int_V \rho(\mathbf{x},t)\,\hat{\phi}(\mathbf{x})\,dV, \tag{A5}$$

with the time-independent scalar function $\hat{\phi}$ interpreted as the potential energy density per unit mass.

Equation (A4) represents an equation of change for the quantity $E^{\dagger} \stackrel{\text{def.}}{=} E+\Phi$, which may be called the ''total energy'' of the system. However, we note that this identification and physical interpretation is clear only when the body forces considered are conservative body-force fields, *independent of time* [and hence derivable from a scalar function $\hat{\phi}(\mathbf{x})$]. Had this not been the case, a term of the form $\rho\partial\hat{\phi}/\partial t$ would remain in the volume integral of the right-hand side of Eq. (A4). Such a term lacks physical interpretation as a ''work term,'' in contrast with the other terms on the right-hand side of Eq. (A4), because it does not conform to the mechanical definition of work as the scalar product of a vector force and a vector displacement.

These arguments and mathematical manipulations show how potential energies derivable from conservative, time independent, body-force fields may be included unambiguously in the work term of Eq. (1.17). Examples of such potential energies are those arising in time independent external gravitational fields. Furthermore, we have illustrated how these ''potential energies'' cease to possess their usual physical interpretations when the scalar potential fields $\hat{\phi}$ from which they derive are time dependent (as will in general be the case in electromechanical systems, where mechanical displacements and deformations of the system under study will affect the *sources* of the electromagnetic field and hence lead to time-dependent electric and magnetic fields).

In effect, potential energy is a *shared* property of a system and its ''surroundings'' as, for example, in the case of the gravitational potential energy shared by a system interacting with Earth (the latter constituting the ''primary'' field source). In particular, the potential energy does not belong to the *system* alone, but rather to the Earth-system combination. Accordingly, it is only when the Earth undergoes no changes in state as a result of the changes of state occurring in the system that the change $d\Phi$ in potential energy can be assigned to the system alone. It is this fact which demands that $\partial\hat{\phi}/\partial t$ vanish in order for the work term arising from the ''distant'' body-force source (Earth) to be identified with a change in potential energy of the system alone. Were Earth to undergo a change in state simultaneous with that of the system as a result of their mutual interactions, the rate of working on the system, namely, $\int_V \rho\hat{\mathbf{f}}\cdot\mathbf{v}dV$, with $\hat{\mathbf{f}}=-\boldsymbol{\nabla}\hat{\phi}(\mathbf{x},t)$, would continue to give the correct work rate, but the integral would no longer be interpretable as a rate of change, $d\Phi/dt$, of the energy of the system.

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relations

$$\nabla\cdot\mathbf{g}=-4\pi G\rho,$$

$$\nabla\times\mathbf{g}=0,$$

where $\rho(\mathbf{x})$ is the mass density and G the universal gravitational constant. The volumetric force density $\mathbf{f}^M(\equiv\rho\mathbf{g})$ due to the gravitational field can be represented in the form of Eq. (1.24), with the "gravitational stress field" $\mathbf{T}^M$ given by the expression

$$\mathbf{T}^M=-\frac{1}{4\pi G}\left(\mathbf{gg}-\mathbf{I}\frac{1}{2}g^2\right).$$

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