Achronal generalized synchronization in mutually coupled semiconductor lasers

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Heil *et al.* [Phys. Rev. Lett. **86**, 795 (2001)] recently discovered achronal synchronization of chaos in mutually coupled semiconductor lasers. This paper offers an analytic interpretation of their experiment using a simple rate equation model. Local eigenvalue analysis shows that isochronal synchronization is unstable; achronal synchronization, on the other hand, is stable if a generalized synchronization function is introduced. Single- and multimode simulations have substantiated this rate equation interpretation. Finally, there is a brief examination of "chaos pass filtering."

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Pecora and Carrol's influential paper [1] set in motion work synchronizing chaotic electronic [2] and optical [3] systems. Optical systems are especially interesting as they can have infinite-dimensional [4,5] and spatiotemporal [5,6] chaos. Most optical systems use a feedback delay (τ_1) to drive the oscillator into a chaotic state and a coupling delay (τ_2) to couple the light into another oscillator [7]. If $\tau_1 \neq \tau_2$ then achronal synchronization, where the driven oscillator's dynamics lag or anticipate the driving oscillator's dynamics, occurs [8,9]. Mutually coupled lasers, where each laser's feedback is symmetrically replaced by the delayed electric field from the other, were not expected to have achronal synchronization since $\tau_1 = \tau_2$. However, Heil *et al.* [10] recently discovered achronal synchronization in mutually coupled lasers. Numerical models also possessed achronal synchronization [10,11] but did not explain the lack of isochronal synchronization—that is, both lasers having the same dynamics *at the same time*.

This paper offers an analytic interpretation of achronal synchronization in mutually coupled lasers. Isochronal synchronization can be described intuitively: each laser produces oscillations in its identical companion as if there were feedback with a time delay equal to the coupling delay. This solution is unstable. Achronal synchronization is stable, but it has a counter-intuitive construction: stable synchronization requires feedback with a delay time twice that of the coupling delay. This construction is an exact solution for only one laser, the other laser's oscillations are not described by this solution and a small error determined by the constructed solution will always exist. Boccaletti, Pecora, and Pelaez's framework for synchronization [12] classifies systems with these characteristics as generalized synchronization instead of the simpler identical synchronization.

The analysis begins with the standard single-mode rate equations for a semiconductor laser with delayed injection [13]

$$\frac{de_{1,2}(t)}{dt} = \Gamma a\{[n_{1,2}(t) - n_{tr}] - i\alpha n_{1,2}(t)\} \times e_{1,2}(t) - \alpha_{int} e_{1,2}(t) + \eta e_{2,1}(t-\tau),$$

$$\frac{dn_{1,2}(t)}{dt} = J - \frac{n_{1,2}(t)}{\tau_n} - a[n_{1,2}(t) - n_{tr}]|e_{1,2}(t)|^2. \quad (1)$$

Here $e_{1,2}(t)$ is the complex electric-field amplitude and $n_{1,2}(t)$ is the carrier density for either the first or second laser. The usual rate equation coefficients are used: Γ is the confinement factor, a is the linear differential gain, n_{tr} is the transparency carrier density, α is the linewidth enhancement factor, α_{int} is the internal loss (including the facet losses), J is the current pumping density, and τ_n is the effective carrier lifetime. The coupling term consists of an attenuation η and a delay τ.

First, isochronal synchronization is shown to be unstable. $e_1(t) = e_2(t) = e(t)$ exactly solves Eq. (1) if $e(t)$ is also the solution of an identical laser with feedback $e(t-\tau)$. This does not imply the existence of an external cavity but links, by analogy, the synchronized solution $e(t)$ to the Lang-Kobayashi solution [13]. Analogy between $e(t)$ and external cavity lasers permits the use of established results, associates a physical interpretation to a mathematical abstraction, and generally illuminates the analysis.

Small perturbations (denoted by δ) may drive the system from synchronization. Stability is governed by

$$\lambda\begin{pmatrix}\delta e\\ \delta e^*\\ \delta n\end{pmatrix}=\begin{pmatrix}-\eta\frac{e(t-\tau)}{e(t)}(1+e^{-\lambda\tau}) & 0 & \Gamma a(1-i\alpha)\\ 0 & -\eta^*\frac{e^*(t-\tau)}{e^*(t)}(1+e^{-\lambda\tau}) & \Gamma a(1+i\alpha)\\ -a[n(t)-n_{tr}]|e(t)|^2 & -a[n(t)-n_{tr}]|e(t)|^2 & -\frac{1}{\tau_n}-a|e(t)|^2\end{pmatrix}\begin{pmatrix}\delta e\\ \delta e^*\\ \delta n\end{pmatrix}. \tag{2}$$

The uncoupled case $(\eta=0)$ has solutions

$$\lambda_0=0,$$

$$\lambda_\pm=\frac{-\left[\frac{1}{\tau_n}+a|e(t)|^2\right]\pm\sqrt{\left[\frac{1}{\tau_n}+a|e(t)|^2\right]^2-8\Gamma a^2[n(t)-n_{tr}]|e(t)|^2}}{2}. \tag{3}$$

$\lambda_\pm$ has a nonpositive real part providing that $n(t)-n_{tr}>0$. The time-averaged local eigenvalues determine stability:

Theorem 1 *Suppose $dy/dt=A(t)y$ and $A(t)$ is always diagonalizable. If every eigenvalue λ of $A(t)$ satisfies* $\mathrm{Re}(\lambda)\leqslant -f(t)$ *for some $f(t)>0$ then*

$$\left|\frac{y(t)}{y(0)}\right|\leqslant e^{-\int_0^t f(\zeta)d\zeta}. \tag{4}$$

This is not a rigorous stability criterion. Rigorous stability calculations are difficult, often tailored to a specific system (see [14] and references therein). Easier, nonrigorous methods can calculate stability without a loss of accuracy. One common method uses the Lyapunov exponents along a trajectory, with the system being stable if all Lyapunov exponents are negative [1], but it does not relate stability along the many possible trajectories in a chaotic attractor. However, the eigenvalues of the Jacobian determine the Lyapunov exponents along a trajectory, so showing that the Jacobian has no positive eigenvalues is equivalent to showing that the Lyapunov exponents are negative [15]. This connection establishes local eigenvalue analysis as a credible, nonrigorous tool for evaluating synchronization stability. Although Corron [16] found counterexamples to local eigenvalue stability analysis, he concluded that such stability analysis is still effective for most systems. All things considered, a system is unstable if a local eigenvalue is positive.

Small values of $\tau_r|\eta|$ (τ_r is the laser cavity round-trip time) perturb the eigenvalues from their $\eta=0$ values. This does not affect the stability of $\lambda_\pm$ but the marginally stable phase eigenvalue λ_0 is now either stable or unstable. Assuming the perturbation takes the form $\lambda=\lambda_0+\tau_r|\eta|\lambda_1+\tau_r^2|\eta|^2\lambda_2+\cdots$, where $\lambda_0=0$ and $\tau_r|\eta|\ll 1$, then to first order in $\tau_r|\eta|$

$$\begin{aligned}&\left[\lambda_1+2\frac{\eta^*}{|\eta|}\frac{e^*(t-\tau)}{e^*(t)}\right][\Gamma a^2(1-i\alpha)[n(t)-n_{tr}]|e(t)|^2]\\&\quad+\left[\lambda_1+2\frac{\eta}{|\eta|}\frac{e(t-\tau)}{e(t)}\right]\\&\quad\times[\Gamma a^2(1+i\alpha)[n(t)-n_{tr}]|e(t)|^2]\\&=0.\end{aligned} \tag{5}$$

Taking $\eta=|\eta|e^{i\omega_0\tau}$ and $e(t)=A(t)e^{i\phi(t)}$ gives

$$\begin{aligned}\lambda_1=&-2\frac{A(t-\tau)}{A(t)}\sqrt{1+\alpha^2}\\&\times\cos[\omega_0\tau+\arctan\alpha+\phi(t-\tau)-\phi(t)].\end{aligned} \tag{6}$$

The delayed phase difference $\phi(t-\tau)-\phi(t)$ determines the isochronal synchronization stability. An upper bound on the delayed phase difference may be established by analogy between $e(t)$ and the equivalent Lang-Kobayashi solution. In the Lang-Kobayashi system, the delayed phase difference determines the external cavity fixed points [17] and their stability [18]. Specifically, near an equivalent unstable external cavity fixed point

$$\sqrt{1+\alpha}\cos[\omega_0\tau+\arctan\alpha+\phi(t-\tau)-\phi(t)]<-\frac{1}{\tau|\eta|}. \tag{7}$$

$e(t)$ approaches an unstable fixed point prior to an external cavity mode hop in the Lang-Kobayashi solution. As external cavity mode hops are a necessary condition for chaos in the Lang-Kobayashi system, Eq. (6) inevitably becomes positive and isochronal synchronization turns unstable. On startup there will be an initial period of isochronal synchronization lasting until immediately after the first equivalent external cavity mode hop in $e(t)$.

When isochronal synchronization loses stability a second solution may be chosen. Experiment and simulation suggest that achronal synchronization, consisting of a laggard solution $e_1(t)=[1+\delta e_1(t)]e(t)$ and a leader solution $e_2(t)=[1+\delta e_2(t+\tau)]e(t+\tau)$, is selected. Direct substitution of the synchronous solution $e(t)$ for $e_1(t)$ and $e_2(t)$ yields

$$\frac{de(t)}{dt}=\Gamma a\{[n(t)-n_{tr}]-i\alpha n(t)\}e(t)-\alpha_{int}e(t)+\eta\begin{cases} e(t) & (\text{subsystem } e_1)\\ e(t-2\tau) & (\text{subsystem } e_2)\end{cases}. \tag{8}$$

Two differences from isochronal synchronization are immediately obvious. First, a Lang-Kobayashi solution with feedback of $e(t-\tau)$ is a solution for neither subsystem. The equivalent Lang-Kobayashi solution requires a feedback term of $e(t-2\tau)$. Second, $e(t)$ only satisfies the equation of motion for the subsystem e_2; $e(t)$ is not an exact solution for the subsystem e_1. Such a situation, where the synchronized solution solves one subsystem exactly but not the other, is best classified as generalized synchronization, which ''associates the output of one system to a given function of the output of the other system'' [12]. The linearized subsystems are of two different types: δe_2 is homogeneous (which determines stability) and δe_1 is inhomogeneous (which determines the generalized synchronization function).

The homogeneous subsystem δe_2 is

$$\lambda\begin{pmatrix}\delta e_2\\ \delta e_2^*\\ \delta n_2\end{pmatrix}=\begin{pmatrix} -\eta\left[1+\dfrac{e(t-2\tau)}{e(t)}\right] & 0 & \Gamma a(1-i\alpha)\\ 0 & -\eta^*\left[1+\dfrac{e^*(t-2\tau)}{e^*(t)}\right] & \Gamma a(1+i\alpha)\\ -a[n(t)-n_{tr}]|e(t)|^2 & -a[n(t)-n_{tr}]|e(t)|^2 & -\dfrac{1}{\tau_n}-a|e(t)|^2\end{pmatrix}\begin{pmatrix}\delta e_2\\ \delta e_2^*\\ \delta n_2\end{pmatrix}. \tag{9}$$

As before, for $|\eta|=0$ Eq. (3) governs the stability. The marginally stable eigenvalue $\lambda_0=0$ may become unstable for nonzero values of $|\eta|$. Using the previous eigenvalue expansion and solving for λ_1 gives the stability condition

$$\lambda_1=-2\sqrt{1+\alpha^2}\Big[\cos[2\omega_0\tau+\arctan\alpha]+\frac{A(t-2\tau)}{A(t)}\cos[2\omega_0\tau+\arctan\alpha+\phi(t-2\tau)-\phi(t)]\Big]. \tag{10}$$

This expression is identical to Eq. (6) with the addition of a time-independent term $\cos[2\omega_0\tau+\arctan\alpha]$. The choice of $\arctan\alpha$'s branch guarantees that $\cos[2\omega_0\tau+\arctan\alpha]$ is greater than zero and furnishes a continually stabilizing force. As $\phi(t-2\tau)-\phi(t)$ is a chaotic variable with fluctuations greater than 2π, $\int_0^t\cos[\phi(\zeta-2\tau)-\phi(\zeta)]d\zeta$ goes to zero for large times. Hence, the time-independent term dominates the stability for long times and achronal synchronization is stable.

The inhomogeneous subsystem δe_1 defines a generalized synchronization function

$$\begin{aligned}\delta_D(t)&=e^{\int_{t_0}^{t}D(\zeta)d\zeta}\int_{t_0}^{t}e^{-\int_{t_0}^{\zeta}D(\xi)d\xi}Q^{-1}(\zeta)\times\begin{pmatrix}-\eta\left(1-\dfrac{e(\zeta-2\tau)}{e(\zeta)}\right)\\ -\eta^*\left(1-\dfrac{e^*(\zeta-2\tau)}{e^*(\zeta)}\right)\\ 0\end{pmatrix}d\zeta\\ &\quad+\delta_D(t_0)e^{\int_{t_0}^{t}D(\zeta)d\zeta}\\ &\equiv Q^{-1}(t)\Big[\theta[e(t)]+\delta e_1(t_0)e^{\int_{t_0}^{t}\lambda(\zeta)d\zeta}\Big].\end{aligned} \tag{11}$$

D is the diagonal matrix containing the local eigenvalues for the subsystem δe_1 [Eqs. (3) and (6) with $\tau\to 2\tau$], δ_D is the state vector of the subsystem δe_1 expressed in the basis of D, and Q is the change of coordinate matrix consisting of the local eigenvectors. $\theta[e(t)]$ is the generalized synchronization function in the original basis. Both D and Q are time-dependent matrices, making the exact calculation of the generalized synchronization function impractical (if not impracticable).

Numerical integration of Eq. (1) has verified the interpretation presented here. Both lasers have been started from the same state and noise has driven them apart. This initializa-

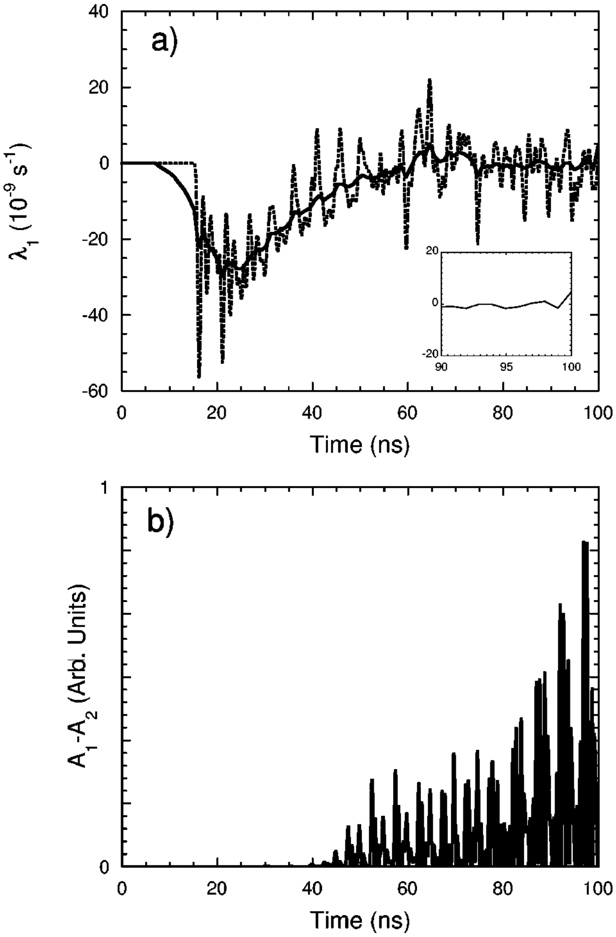

FIG. 1. Startup of the mutually coupled single-mode system. Both lasers have been started from the same initial conditions and noise has continuously driven the system away from an identical state. (a) Dashed line, eigenvalue λ_1 calculated for isochronal synchronization. Solid line, smoothed fit to λ_1. Inset, enlarged view of the smoothed fit to λ_1. (b) Error function $|A_1(t)-A_2(t)|$. Dimensionless parameters: $\Gamma=1.1$, $a=1$, $n_{tr}=1$, $\alpha=4$, $\alpha_{int}=0.27$, $J=4.7\times10^{-3}$, $\tau_n=333.3$, $\eta=0.21\times e^{i\pi/4}$, and $\tau=1515.15$ (5 ns).

tion assured that the system began in the isochronal synchronization state. After the first external cavity mode hop, isochronal synchronization became unstable. When the instability occurs has been estimated from the error function $|A_1(t)-A_2(t)|$. For isochronal synchronization, the perturbation eigenvalue λ_1 and the error function are plotted in Fig. 1. Both have been averaged with a 2τ full width at half maximum (FWHM) Gaussian filter, which (1) simulates a finite detector response, (2) averages the inhomogeneous contribution from $\theta[e(t)]$, and (3) satisfies the conditions for local eigenvalue stability analysis. For a delay of 5 ns, the effective bandwidth of the simulated detector is 100 MHz. Chaotic oscillations had set in by 40 ns, and the system was showing isochronal synchronization (as measured by the error function) until 80 ns. Prior to 80 ns, λ_1 had some positive spikes but still had an overall negative value. At 80 ns λ_1 oscillates about 0, clearly violating the conditions for stability, and $e(t)$ has had its first external cavity mode hop (Fig.

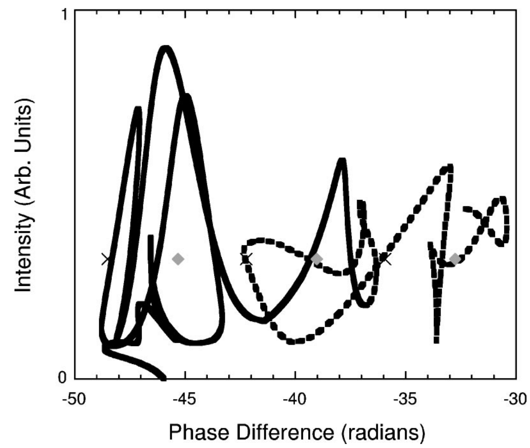

FIG. 2. Phase-space trajectory for the single-mode system. Equivalent stable (unstable) external-cavity modes have been denoted as diamonds (crosses). Up to 80 ns (solid line) the trajectory had remained near the first stable external cavity mode. After 80 ns (dashed line) the trajectory has jumped to the next stable external cavity mode and isochronal synchronization has become unstable.

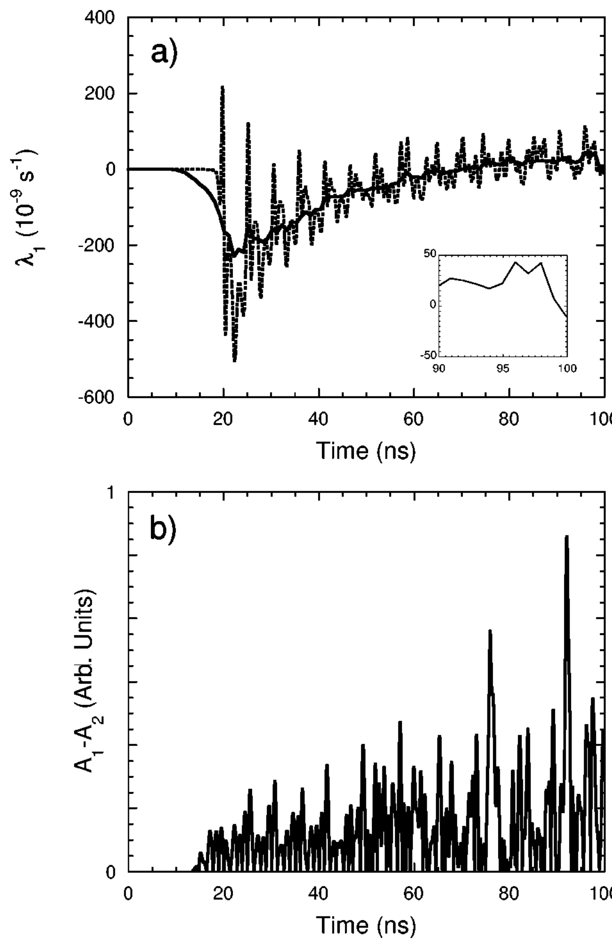

FIG. 3. As in Fig. 1 but for the multimode system. (a) Dashed line, eigenvalue λ_1 calculated for isochronal synchronization. Solid line, smoothed fit to λ_1. Inset, enlarged view of the smoothed fit to λ_1. (b): Error function $|A_1(t)-A_2(t)|$. A many-body model [11] with the same parameters as the single-mode model has been used.

2). After 80 ns achronal synchronization is stable and its associated eigenvalue is $\lambda_1 = -2\ \mathrm{s}^{-1}$. This eigenvalue is larger than the isochronal eigenvalue ($\sim 10^{-8}\ \mathrm{s}^{-1}$), is time independent, and is suitable for standard linear stability analysis. So, in a single-mode simulation, the external cavity modes have destabilized isochronal synchronization while achronal synchronization has not been affected.

Mutually coupled lasers are not likely to run single mode, even if the solitary laser runs nominally single mode. Feedback [19] and external injection [20] excite multimode dynamics in single-mode lasers and multimode dynamics can change the synchronization stability [5]. Luckily, in many instances multimode systems conserve the single-mode structure, but such requires verification before affixing a single-mode interpretation to a potentially multimode system. Previous work established that achronal synchronization is stable for the multimode case [11]. The perturbation eigenvalue λ_1 and the error function $|A_1(t) - A_2(t)|$ have also been calculated for the multimode system and are plotted in Fig. 3. Initially isochronal synchronization predominated but had lost stability within the first 100 ns, switching to achronal synchronization. As in the single-mode case the eigenvalue λ_1 had initially been negative, turning positive when isochronal synchronization lost stability. Unlike the single-mode situation, following the onset of chaos λ_1 had brief periods where it became positive, hampering isochronal synchronization. This relates to the increased difficulty of synchronizing infinite-dimensional spatiotemporal chaos, which is present in the multimode system [5]. Still the multimode laser has yielded to achronal synchronization by 80 ns and λ_1 has become positive (the onset of the first power dropout drives λ_1 negative at 100 ns). Thus the single-mode interpretation applies to multimode systems as well.

Finally, the interpretation developed here can shed light on a surprising result from [10]. In [10], periodic perturbations made to the laggard laser altered the leader laser's response while periodic perturbations made to the leader laser did not disturb the laggard. The effect was labeled "chaos pass filtering" and from this the researchers concluded that achronal synchronization acted as a unidirectional coupled system: the leader laser was the driving subsystem and the laggard laser was the driven subsystem.

In achronal synchronization, the difference between the two lasers is

$$\begin{aligned}\delta e(t) &= e_1(t) - e_2(t-\tau)\\ &= e^{-\int_{t_0}^{t}\lambda(\zeta)d\zeta}\delta e_1(t_0)\\ &\quad - e^{-\lambda_1 t}e^{-\int_{t_0}^{t}\lambda(\zeta)d\zeta}\delta e_2(t_0) - \theta[e(t)]\\ &= [\delta e_1(t_0) - e^{-\lambda_1 t}\delta e_2(t_0)]e^{-\int_{t_0}^{t}\lambda(\zeta)d\zeta} - \theta[e(t)].\end{aligned} \tag{12}$$

If the leader system is perturbed [$\delta e_1(t_0) = 0$ and $\delta e_2(t_0) \neq 0$], $\delta e(t)$ decays to $\theta[e(t)]$, achronal generalized synchronization is unaffected, and "chaos pass filtering" is observed. If the laggard system is perturbed [$\delta e_1(t_0) \neq 0$ and $\delta e_2(t_0) = 0$], achronal synchronization is affected and the system may be driven to a different solution such as observed in [10].

This paper has shown that, because of a phase instability, achronal synchronization is preferred over isochronal synchronization in mutually coupled lasers. Achronal synchronization requires a construction that results in the two lasers having different dynamics; viewed as such it is the first example of generalized synchronization in optical systems. Single- and multimode simulations explicitly show the phase instability's onset. Finally, "chaos pass filtering" is understood as a natural consequence of achronal generalized synchronization.

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