Self-organized critical system with no stationary attractor state

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A simple model economy with interacting producers and consumers is introduced. When driven by extremal dynamics, the model self-organizes *not* to an attractor state, but to an asymptote, on which the economy has a constant rate of deflation, is critical, and exhibits avalanches of activity with power-law distributed sizes. This example demonstrates that self-organized critical behavior occurs in a larger class of systems than so far considered: systems not driven to an attractive fixed point, but, e.g., an asymptote, may also display self-organized criticality.

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I. INTRODUCTION

It has been amply demonstrated by now that some driven extended dissipative systems will self-organize into a complex critical state in which events of all sizes occur. This phenomenon, called self-organized criticality (SOC) [1], has been invoked to explain phenomena such as the experimentally observed behavior of flux lines in high-T_c superconductors [2], solar flares, and earthquakes [3]. Several theoretical models that exhibit SOC have been constructed [1,4,5], for a recent review, see [6]. In all these models the SOC state is a (statistically) stationary state [7,8].

Here we demonstrate by example that one may observe SOC behavior also in systems that have no stationary attractor state. Thus, we extend the class of dynamical systems in which one may expect to find SOC. The example is a simple one-dimensional model economy driven by extremal dynamics. In this model, agents interact with each other through a fixed set of rules. As in standard economic theory [10], agents have utility functions that they try to maximize. But contrary to classical economic equilibrium theory, we have no "central agent," "market maker," or "auctioneer." Maximization of utility functions is left to individual agents.

Agents are rational and never change their strategies, only their prices and the quantities they buy and produce. By their transactions, agents make a profit, positive or negative. The agent who makes the most negative profit of all agents in the system then changes his price slightly in a manner that increases his profit. In the next time step, the agents do another round of optimized transactions, and the agent now having the most negative profit changes his price. This process is repeated *ad infinitum*.

After a transient period, the system arrives in a state with long-range spatial correlations (power laws) and deflation with constant rate. The distribution of profits displays a distinct threshold. Avalanches of causally connected price changes by agents with profits below this threshold are observed. The size distribution for avalanches follows a power law.

II. THE MODEL

Consider N agents numbered $n=1,2,\ldots,N$. Agent number n sells his product to agent number $n-1$ and buys the product produced by agent number $n+1$. We assume that individual agents do not consume their own production, so in order to consume they must trade, and in order to trade they must produce. Agent number n produces a quantity q_n of a good, which is sold at a price p_n per unit to his neighbor numbered $n-1$. He subsequently buys and consumes the quantity q_{n+1} of the good produced by his neighbor numbered $n+1$, who subsequently buys the good produced by *his* neighbor numbered $n+2$, etc. until all agents have made two transactions. This process is repeated, say once each day.

The goal of each agent is to maximize his utility function

$$u_n=-c(q_n)+d(q_{n+1}), \tag{1}$$

while satisfying the constraint

$$p_nq_n=p_{n+1}q_{n+1}. \tag{2}$$

The first term $-c$ in the utility function in Eq. (1) represents the agent's cost, or discomfort connected with the production of q_n units of the good he produces. This discomfort is an increasing function of q, and c is convex because, say, the individual agent grows tired. An example could be a pizza baker; by the end of a long working day the utility gained by taking in another customer is very small compared to the utility of going home to sleep. In other words, we are modeling individual agents and not economomies of scale.

The second term d is the utility of the good he buys from his neighbor. Its marginal utility is a decreasing function of quantity q, so d is an increasing, but concave, function. This choice of c and d is common in economics, see, e.g., [11].

The constraint Eq. (2) is the simplest possible. It expresses that the agents do not trust money, they accept money as currency, but do not want to possess any at the end of the day. Also, of course, the agents do not want to run out of money since that would prevent them from obtaining the utility they need. Agents have infinite credit in the sense that no strict limit to spending is enforced. However, they always try to balance their spending according to the constraint in Eq. (2).

An explicit utility function is chosen for illustration and analysis,

$$u_n=-\frac{1}{2}(q_n)^2+2\sqrt{q_{n+1}}. \tag{3}$$

An agent knows the prices of his two neighbors at all times but not how much they want to buy or sell. A similar, but locally driven, model with myopic agents was invoked in [12] in order to explain the dynamic origin of the value of money.

Based on his utility function and the prices he knows, each agent plans how much to produce and how much to purchase, assuming that everything he produces will be sold, and that all he wants to purchase will be available. The task is a simple optimization problem with solution

$$q_n^{(\mathrm{prod.})}=\left(\frac{p_n}{p_{n+1}}\right)^{1/3} \tag{4}$$

and

$$q_{n+1}^{(\mathrm{want})}=\left(\frac{p_n}{p_{n+1}}\right)^{4/3}. \tag{5}$$

We note in Eqs. (4) and (5) that the levels of production and intended consumption are independent of absolute prices, as they depend only on ratios. All prices may be multiplied by a common factor and leave quantities produced and consumed unchanged.

The process is initiated by choosing some initial, random values for the prices. Next, agent number n implements his plan by producing the quantity $q_n^{(\mathrm{prod.})}$, and setting it for sale at the price p_n. However, his customer, agent number $n-1$, has planned to buy the quantity $q_n^{(\mathrm{want})}$, and will do so if enough goods are available, i.e., if $q_n^{(\mathrm{want})}\leqslant q_n^{(\mathrm{prod.})}$. If $q_n^{(\mathrm{want})}>q_n^{(\mathrm{prod.})}$, agent $n-1$ buys the quantity available $q_n^{(\mathrm{prod.})}$. Thus, the traded amount is $q_n^{(\mathrm{trad.})}=\min(q_n^{(\mathrm{prod.})},q_n^{(\mathrm{want})})$.

At the end of the day, agent n has, unwillingly, made the profit

$$s_n=p_n q_n^{(\mathrm{trad.})}-p_{n+1}q_{n+1}^{(\mathrm{trad.})}. \tag{6}$$

After some time, agents in the economy who lose money, react by changing their prices, which is the only variable controlled by agents in this model. We assume that the agent losing most money per cycle is the first to react. With no loss of generality, we set the time that passes before this happens to a single cycle, say, one day. We also assume that after each price change, a new cycle and comparison of profits is made to find which agent now loses most money. As can be seen from Eqs. (4)–(6), an agent with negative profit increases his profit by lowering his price. This result does not depend on the specific choice in Eq. (3). Studying a more general version of the utility function $u_n=-aq_n^{\alpha}+bq_{n+1}^{\beta}$, we find with the specified set of rules that deflation will occur as long as $\alpha>\beta>0$.

We model the economic considerations of our agent by letting him lower his price by some small, random percentage η of his presently charged price. Agents that do not know calculus, have no memory, or in some other way have little economic insight can also be modeled. Such agents do not realize that they will improve their circumstances by lowering their price, instead they may attempt a random change in price—increasing or decreasing it by some random amount. If the direction of the price change is also random, the price performs a random walk. The number of price changes necessary for a single agent to ‘‘get it right,’’ i.e., to lower his price enough, is then distributed as the first-return time for an unbiased random walker. We choose to let our agents have the necessary insight to lower their price since the added randomness will only have the mentioned analytically predictable effect, hence it does not require computer simulation.

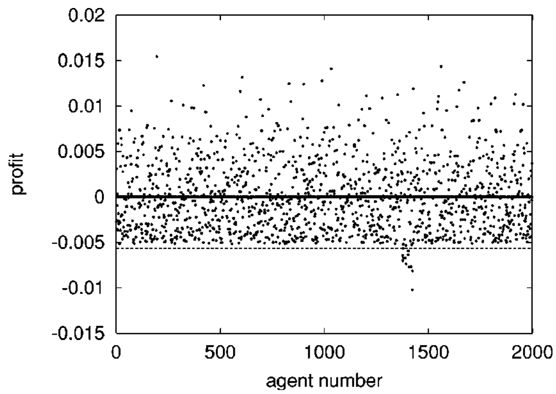

FIG. 1. Distribution of profits after 2×10^7 time steps, $\eta_{\max}=0.1\%$, rescaled by $\exp(2.5092\times10^{-7}t)$; notice the δ function at zero profit. The dashed line marks the threshold value $f_0=-0.0057\leqslant f_c$.

When agent n lowers his price, his estimate of how much he should optimally produce and consume also drops. Conversely, his customer, agent $n-1$, raises his estimate of how much he should optimally buy. Agent n's supplier, agent $n+1$, does not change *his* estimates of how much he should produce and consume, and consequently he risks producing more than he can sell. In this way an agent with negative profit increases his profit by lowering his price, while potentially ‘‘passing on’’ the problem of negative profit to his supplier.

III. COMPUTER SIMULATION

In a simulation of the model, N agents are initially given random prices drawn from a uniform distribution on the interval [1,2]. The interval [1,2] is an arbitrary choice. The only demand on the initial prices is that they are positive. Relative price changes η are drawn from a uniform distribution on the interval $[0,\eta_{\max}]$.

The updating scheme is (i) the levels of production and intended consumption are found from Eqs. (4) and (5), (ii) the profit of each agent is determined from Eq. (6), (iii) the agent with the lowest (most negative) profit is found, and given a new, lower price $p\rightarrow p(1-\eta)$; (iv) go to (i).

We employed periodic boundary conditions with $N+1=1$ and $N=0$ and studied systems of various sizes, ranging from 200 to 20 000 agents, on time scales from some hundreds to 10^8 updates, and with $\eta_{\max}$ ranging from 0.1% to 10%. Results turned out to be insensitive to the particular value used for $\eta_{\max}$. After an initial transient period, the system organized itself into a state where the spatial distribution of profits exhibit a clear threshold $f_c(t)$, see Fig. 1.

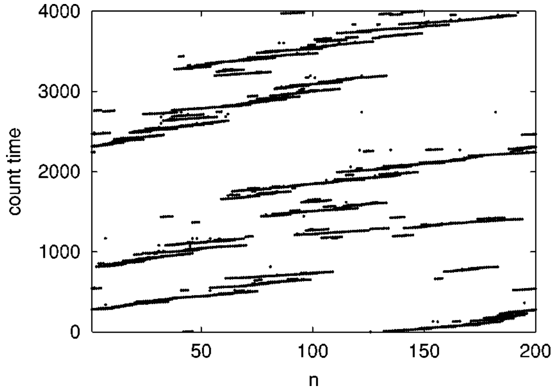

FIG. 2. Spatiotemporal distribution of the losing agents in an economy with $N=200$ agents, after the initial transient. Abscissa shows loser's coordinate. Ordinate shows time.

Few or no agents are found to have profits below this threshold, and those found tend to be spatially located near the "loser."

Figure 2 shows how the loser's role moves through the system. There is a clear drift in one direction, because of the left-right asymmetry of the utility function.

The spatial correlations of the loser positions were examined by measuring the distribution of distances between successive losers. If the spatial jump *x*, between two successive losers was more than half the system size to the right, it was counted as a jump to the left. The distribution of distances between successive losers follows a power-law distribution asymptotically at large distances, i.e., the system is critical. We fitted [13] the distribution of distances between successive losers to the expression

$$P(x)=Ax^{-\pi^{(\mathrm{right})}}+B(N-x)^{-\pi^{(\mathrm{left})}}+C, \tag{7}$$

see Fig. 3, and found the exponent values $\pi^{(\mathrm{right})}=1.844 \pm 0.002$ and $\pi^{(\mathrm{left})}=2.021\pm 0.002$. Here C is a constant that takes into account the approximately flat distribution of "avalanche starters." An avalanche starter is an agent with profit above the threshold, who is chosen as the loser by our algorithm on the rare occasions when no losers with subthreshold profits are left in the systems.

Since agents keep lowering their prices, the threshold in profit distributions decreases to zero, exponentially in time, $f_c(t)\propto \exp(-kt)$, where $k=\langle\eta\rangle/[N(1-\langle\eta\rangle)]$ to leading order in $\langle\eta\rangle$, and $\langle\cdot\rangle$ denotes an ensemble or time average. The relation for k is only a first approximation since the loser often has to change his price more than once in order to increase his profit enough. A better approximation including this effect is used in the actual rescaling. When all profits are rescaled by $\exp(kt)$, we obtain stationariness in the threshold $f_c=f_c(t)$.

We next consider the activity below a threshold $f_0\leqslant f_c$ and define an avalanche as the duration of causally connected activity below this threshold. We refer to this duration as the avalanche size S. When all agents are above the threshold there is no active avalanche by our definition of avalanches.

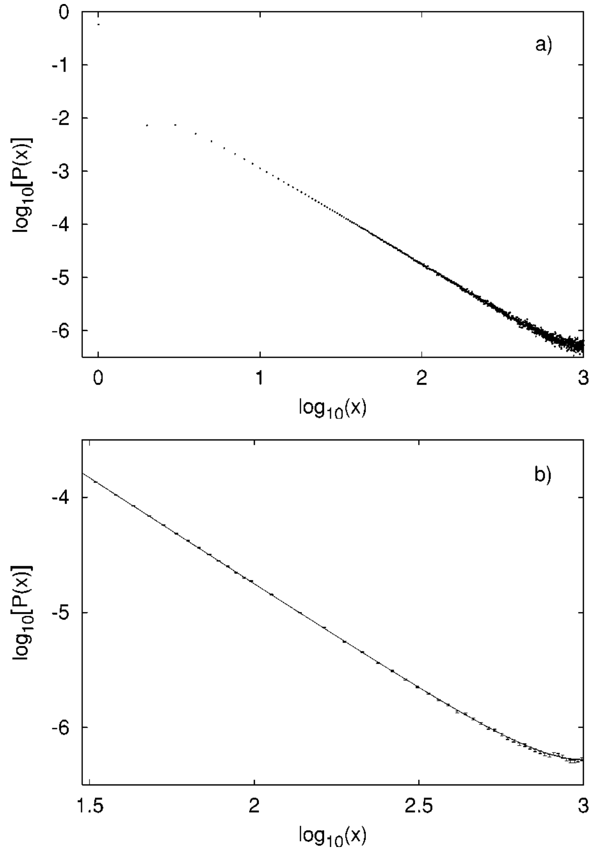

FIG. 3. Distribution of the spatial separation between successive losers. Economy with 2000 agents, $\eta_{\max}=0.1\%$, 10^8 time steps sampled after the initial transient phase. (a) Jumps to the right. (Jumps to the left have a similar looking distribution). (b) Same data as in (a), but binned. Plot shows mean and root mean square deviation of data in each bin, as well as a χ^2 fit of Eq. (7) to the data shown in (a) and similar data for jumps to the left. Exponents $\pi^{(\mathrm{right})}=1.844\pm 0.002$ and $\pi^{(\mathrm{left})}=2.021\pm 0.002$. The backing of the fit is $P_{n'}(>\chi^2)=70\%$.

Figure 4 shows a log-log plot of the avalanche size distribution for a system of 2000 agents. Measurements were made during 10^7 time steps, after discarding the first 10^6 time steps. We clearly see a power law $P(S)\propto S^{-1.48\pm 0.03}$. The value of the exponent $\tau=1.48\pm 0.03$ is indistinguishable from 3/2, the latter being the exponent of the distribution of first-return times for an unbiased random walker.

When studying the avalanche size distribution for varying positions of the threshold f_0 [15], a distribution function of a form well-known from percolation theory [16] suggests itself $P(S)=S^{-\tau}g(S(f_c-f_0)^{1/\sigma})$, where the Fisher exponent τ, now plays the role of the avalanche size distribution coefficient. $g(x)$ is a scaling function, with the properties $g(x) \to 0$ for $x\to\infty$, $g(x)\to g(0)$ for $x\to 0$, and σ is the avalanche cutoff exponent [17]. Hence, the system cannot be adequately described in terms of a simple unbiased random walker.

The exponent value 3/2 is also characteristic of mean-field theory, and was, e.g., obtained in the mean-field treatment of the Bak-Sneppen model [4], which has some similarity with the model treated here. However, as shown in [18] for the

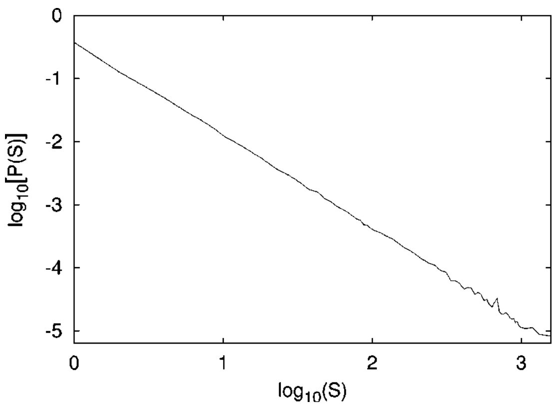

FIG. 4. Distribution of avalanche sizes in the critical state. The size of an avalanche is the number of subsequent system updates with (rescaled) profits less than $f_0 = -0.0057$.

''simplest possible SOC system'' [19], the exponent 3/2 can also occur in a system with fluctuations. Finally the exponent 3/2 is common to all critical branching processes [20]. So one cannot, from the value of our exponent, conclude that mean-field theory is exact for our model in one dimension.

IV. DISCUSSION AND CONCLUSION

We have shown that our model economy evolves to a critical state when driven by extremal dynamics. This occurs without fine tuning of parameters, i.e., the system is self-organized. We emphasize that the extremal driving of our model makes it nonlocal. Attempts at making extremally driven SOC models local by assigning an update probability to *all* sites have so far only led to the loss of criticality and speculations that some temperaturelike parameter has to be set to zero in order for criticality to appear [21].

We studied the spatial correlations in the system by measuring the distribution of spatial separations of consecutive activity in the system. Two power laws (left and right) were found with exponents $\pi^{(\mathrm{left})} = 2.02$ and $\pi^{(\mathrm{right})} = 1.84$, and a 70% backing of the fit.

The system's dynamics does *not* have an attractive fixed point, but only an attractive asymptote, and contrary to other SOC systems *never* reaches a stationary state. We then rescaled the system to a (statistically) stationary state where the definition of avalanches was possible. After this rescaling, we found a power law for the distribution of avalanche sizes with exponent $\tau = 1.5$.

Thus, we have demonstrated by example that many more driven dynamical systems than hitherto realized may be found to be self-organized critical. The class of systems that may exhibit SOC behavior now also include systems that do *not* have an attractive fixed point for their dynamics.

The deflation in our model economy is not caused by a simultaneous change of prices by all agents in the system, nor by a random scatter of such changes. Rather, price changes propagate in avalanches. So although the effect is gradual at the macroscale of the whole economy, the world is quite ''turbulent'' on the microscale experienced by the individual agent.

The system studied here is brutally minimalistic. There is room for several amendments towards improved realism, with little loss in simplicity. In one dimension, a richer, but also more complicated, dynamics can be achieved simply by letting agents with zero profit increase their prices. This allows for an economy with alternating periods of inflation and deflation. On a two-dimensional square lattice, each agent can have two suppliers and two customers, allowing for competition, hence a marketlike scenario. In this sense, networks with higher coordination numbers are more realistic. We expect criticality also in these cases, but with different exponents.

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