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Quantum refrigeration cycles using spin-$\frac{1}{2}$ systems as the working substance

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The cycle model of a quantum refrigerator composed of two isothermal and two isomagnetic field processes is established. The working substance in the cycle consists of many noninteracting spin-$\frac{1}{2}$ systems. The performance of the cycle is investigated, based on the quantum master equation and semigroup approach. The general expressions of several important performance parameters, such as the coefficient of performance, cooling rate, and power input, are given. Especially, the case at high temperatures is analyzed in detail. The results obtained are further generalized and discussed, so that they may be directly used to describe the performance of the quantum refrigerator using spin-*J* systems as the working substance. Finally, the optimum characteristics of the quantum Carnot refrigerator are derived simply.

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I. INTRODUCTION

Quantum cycles are of much importance not only in theory but also in practice. The investigation relative to quantum cycles has attracted a good deal of attention. In recent years, the performance of quantum heat engines has been intensively studied [1–6] and the cycle models of quantum refrigerators have also been proposed [1,7–9]. Many conclusions have been obtained.

Similar to classical thermodynamic cycles, quantum cycles may have other typical cycle models, such as the Stirling cycle, Ericsson cycle, Brayton cycle, etc., besides the Carnot cycle. It is well known that the performance of the Carnot cycle is independent of the property of the working substance, while the performance of other cycles is, in general, dependent on the property of the working substance [10–12]. This conclusion is still true for quantum cycles. The working substance in a quantum cycle may be the spin systems, harmonic oscillator systems, ideal quantum gases, and so on. For different working substances, the performance of the cycle will be different. Thus, the property of the working substance must be analyzed when the performance of a quantum cycle is studied.

In the present paper, the property of a spin-$\frac{1}{2}$ system is given, based on the quantum master equation and semigroup approach. The performance of the quantum refrigeration cycle composed of two isothermal and two isomagnetic field processes is analyzed. The regenerative characteristics of the cycle are discussed. The important performance parameters such as the coefficient of performance, cooling rate, power input, and the temperatures of the working substance in two isothermal processes are optimized.

II. FIRST LAW OF THERMODYNAMICS IN A SPIN-$\frac{1}{2}$ SYSTEM

First of all, we consider a quantum system with a magnetic moment **M** placed in a magnetic field **B**. The direction of the magnetic field **B** is chosen constant and along the positive *z* axis. The magnitude of the magnetic field can change over time, but is not allowed to reach zero. The Hamiltonian of the interaction between the magnetic moment **M** in the quantum system and the magnetic field **B** is given by [13,14]

$$\hat{H}(t) = -\mathbf{M}\cdot\mathbf{B} = 2\mu_B \mathbf{S}\cdot\mathbf{B} = 2\mu_B B_z(t)\hat{S}_z, \qquad (1)$$

where μ_B is the Bohr magnetron, **S** is a spin angular momentum, $\hbar = h/(2\pi)$, and h is the Planck constant. Throughout this paper we adopt $\hbar = 1$ and define $\omega(t) = 2\mu_B B_z(t)$ for simplicity. ω is positive since the spin angular momentum and magnetic moment are in opposite directions. Thus, the Hamiltonian of an isolated single spin-$\frac{1}{2}$ system in the presence of the field $\omega(t)$ may be expressed as

$$\hat{H}(t) = \omega(t)\hat{S}_z. \qquad (2)$$

As described in Ref. [14], one can refer to ω rather than B_z as ''the field.'' The internal energy of the spin-$\frac{1}{2}$ system is of the expectation value of the Hamiltonian, i.e.,

$$E = \langle \hat{H} \rangle = \omega(t)\langle \hat{S}_z \rangle = \omega S. \qquad (3)$$

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Based on the statistical mechanics, the expectation value of the spin angular momentum S_z is expressed by the following relation

$$S=\langle\hat{S}_z\rangle=-\tfrac{1}{2}\tanh(\beta\omega/2), \tag{4}$$

where $-\frac{1}{2}<S<0$.

It is assumed that the spin-$\frac{1}{2}$ system is not only coupled mechanically to the given "magnetic field" $\omega(t)$, but also coupled thermally to a heat reservoir at temperature T. Based on the semigroup formalism [14,15], the equation of motion of an operator in the Heisenberg picture is given by the quantum master equation [15–18], i.e.,

$$\frac{d\hat{X}}{dt}=i[\hat{H},\hat{X}]+\frac{\partial\hat{X}}{\partial t}+L_D(\hat{X}), \tag{5}$$

where

$$L_D(\hat{X})=\sum_a \gamma_a(\hat{V}_\alpha^+[\hat{X},\hat{V}_\alpha]+[\hat{V}_\alpha^+,\hat{X}]\hat{V}_\alpha) \tag{6}$$

is a dissipation term and originates from a thermal coupling of the spin to a heat reservoir, $\hat{V}_\alpha$ and $\hat{V}_\alpha^\dagger$ are operators in the Hilbert space of the system and are Hermitian conjugates, and γ_α are phenomenological positive coefficients. Substituting $\hat{X}$ in Eq. (5) by $\hat{H}$ and using Eq. (3), one can obtain the rate of change of the internal energy as

$$\frac{dE}{dt}=\frac{d}{dt}\langle\hat{H}\rangle=\left\langle\frac{\partial\hat{H}}{\partial t}\right\rangle+\langle L_D(\hat{H})\rangle=\frac{d\omega}{dt}S+\omega\frac{dS}{dt}. \tag{7}$$

Comparing Eq. (7) with the time derivative of the first law of thermodynamics

$$\frac{dE}{dt}=\frac{dW}{dt}+\frac{dQ}{dt}, \tag{8}$$

one can easily find that the instantaneous power is [19–21]

$$P=\frac{dW}{dt}=\left\langle\frac{\partial\hat{H}}{\partial t}\right\rangle=\frac{d\omega}{dt}S \tag{9}$$

and the instantaneous heat flow is

$$\frac{dQ}{dt}=\langle L_D(\hat{H})\rangle=\omega\frac{dS}{dt}. \tag{10}$$

It is thus clear that for a spin-$\frac{1}{2}$ system, Eq. (7) gives the time derivative of the first law of thermodynamics [16–20].

III. A QUANTUM REFRIGERATION CYCLE

Figure 1 shows a schematic diagram of a quantum refrigeration cycle, which is composed of two isothermal and two isomagnetic field processes. This cycle is a microscopic analog of the Ericsson refrigeration cycle [12,22]. For the convenience of writing, "temperature" will refer to β rather than T, where $\beta=1/T$ and T is the absolute temperature in energy units. In the isothermal processes, the working substance is coupled to the hot reservoir at constant "temperature" β_h and the cold reservoir at constant "temperature" β_c, respectively. Q_1 and Q_2 represent the amounts of heat exchange between the working substance and the heat reservoirs during the two isothermal processes, respectively. Due to finite-rate heat transfer between the working substance and the heat reservoirs, the "temperatures" of the working substance in two isothermal processes are different from those of the heat reservoirs. They are, respectively, given by β_1 and β_2 and there is a relation, $\beta_2\geqslant\beta_c>\beta_h\geqslant\beta_1$. In the isomagnetic field processes, a regenerator is often used to improve the performance of the cycle. Q_{bc} and Q_{da} represent the amounts of heat exchange between the working substance and the regenerator during the two isomagnetic field processes. ω_1 and ω_2 represent the high and low "magnetic field," respectively.

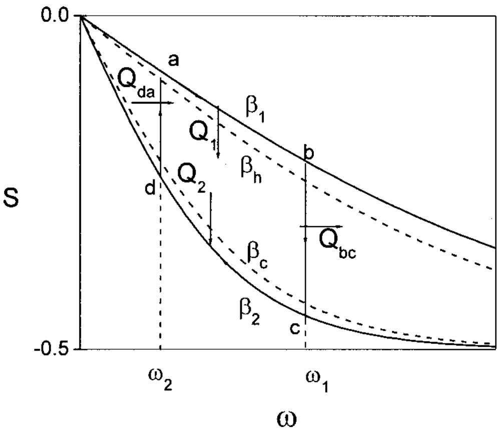

FIG. 1. The S-ω diagram of a spin-$\frac{1}{2}$ Ericsson refrigeration cycle, where the unit of ω is joules.

IV. REGENERATIVE CHARACTERISTICS

Using Eqs. (4) and (10), we can calculate the amounts of heat exchange in the various processes as

$$\begin{aligned}Q_1&=\int_a^b\omega dS\\&=-\frac{1}{2}\omega_1\tanh(\beta_1\omega_1/2)+\frac{1}{2}\omega_2\tanh(\beta_1\omega_2/2)\\&\quad+\frac{1}{\beta_1}\ln\left[\frac{\cosh(\beta_1\omega_1/2)}{\cosh(\beta_1\omega_2/2)}\right],\end{aligned} \tag{11}$$

$$\begin{aligned}Q_2&=\int_c^d\omega dS\\&=-\frac{1}{2}\omega_2\tanh(\beta_2\omega_2/2)+\frac{1}{2}\omega_1\tanh(\beta_2\omega_1/2)\\&\quad+\frac{1}{\beta_2}\ln\left[\frac{\cosh(\beta_2\omega_2/2)}{\cosh(\beta_2\omega_1/2)}\right],\end{aligned} \tag{12}$$

$$Q_{bc}=\omega_1(S_c-S_b)$$
$$=\omega_1[-\tfrac{1}{2}\tanh(\beta_2\omega_1/2)+\tfrac{1}{2}\tanh(\beta_1\omega_1/2)], \quad (13)$$

and

$$Q_{da}=\omega_2(S_a-S_d)$$
$$=\omega_2[-\tfrac{1}{2}\tanh(\beta_1\omega_2/2)+\tfrac{1}{2}\tanh(\beta_2\omega_2/2)], \quad (14)$$

where S_a, S_b, S_c, and S_d are the mean values of the spin angular momentum in *a*, *b*, *c*, and *d* states, respectively. Using Eqs. (11)–(14), we obtain the work input per cycle as

$$W=|Q_1+Q_2+Q_{bc}+Q_{da}|$$
$$=\left|\frac{1}{\beta_1}\ln\frac{\cosh(\beta_1\omega_1/2)}{\cosh(\beta_1\omega_2/2)}-\frac{1}{\beta_2}\ln\frac{\cosh(\beta_2\omega_1/2)}{\cosh(\beta_2\omega_2/2)}\right|. \quad (15)$$

From Eqs. (13) and (14), one can find that the net amount of heat transfer between the working substance and the regenerator during the two isomagnetic field processes is determined by

$$\Delta Q=Q_{bc}+Q_{da}$$
$$=\frac{\omega_1}{2}[\tanh(\beta_1\omega_1/2)-\tanh(\beta_2\omega_1/2)]$$
$$+\frac{\omega_2}{2}[\tanh(\beta_2\omega_2/2)-\tanh(\beta_1\omega_2/2)]. \quad (16)$$

It is seen from Eq. (16) that there are three possible cases: (a) $\Delta Q>0$, (b) $\Delta Q=0$, and (c) $\Delta Q<0$. When $\Delta Q<0$, the amount of heat Q_{bc} flowing into the regenerator in one regenerative process is larger than that of Q_{da} flowing from the regenerator in the other regenerative process. The redundant heat in the regenerator per cycle must be released to the cold reservoir in a timely manner. This results in the reduction of the amount of refrigeration from Q_2 to Q_c. If not, the temperature of the regenerator would be changed such that the regenerator would not operate normally. Similarly, when $\Delta Q>0$, the amount of heat Q_{bc} flowing into the regenerator in one regenerative process is smaller than that of Q_{da} flowing from the regenerator in the other regenerative process. The inadequate heat in the regenerator per cycle must be compensated from the hot reservoir in a timely manner, while the amount of refrigeration Q_2 is unvarying. When $\Delta Q=0$, the parameters β and ω may not be chosen arbitrarily. They must satisfy a certain relation. In general, the quantum refrigeration cycle, which is composed of two isothermal and two isomagnetic field processes and whose working substance consists of noninteracting spin-$\frac{1}{2}$ systems, may not possess the condition of perfect regeneration.

According to the regenerative characteristics mentioned above, the unified expression for the amount of refrigeration per cycle may be given by

$$Q_c=Q_2-\delta|\Delta Q|$$
$$=\delta\left[\frac{\omega_1}{2}\tanh(\beta_1\omega_1/2)-\frac{\omega_2}{2}\tanh(\beta_1\omega_2/2)\right]$$
$$+\frac{1}{\beta_2}\ln\frac{\cosh(\beta_2\omega_2/2)}{\cosh(\beta_2\omega_1/2)}+(1-\delta)\left[\frac{\omega_1}{2}\tanh(\beta_2\omega_1/2)\right.$$
$$\left.-\frac{\omega_2}{2}\tanh(\beta_2\omega_2/2)\right], \quad (17)$$

where $\delta=1$ when $\Delta Q<0$ and $\delta=0$ when $\Delta Q>0$.

V. TIME EVOLUTION OF THE SPIN ANGULAR MOMENTUM AND CYCLE PERIOD

In order to calculate the time of the heat-exchange processes, one must solve the equation of motion that determines the time evolution of the spin angular momentum. For a spin system, $\hat{V}_\alpha$ are chosen to be the spin creation and annihilation operators: $\hat{S}_+=\hat{S}_x+i\hat{S}_y$ and $\hat{S}_-=\hat{S}_x-i\hat{S}_y$, and $\hat{H}=\omega\hat{S}_z$. Substituting $\hat{S}_+$, $\hat{S}_-$, $\hat{H}$, and $\hat{X}=\hat{S}_z$ into Eq. (5), one can prove [14] that

$$\frac{dS}{dt}=-\alpha\exp(q\beta\omega)\{2[1+\exp(\beta\omega)]S+[\exp(\beta\omega)-1]\}, \quad (18)$$

where $a>0$ and $-1<q<0$. Solving Eq. (18), one can obtain the expression of time evolution as

$$t=-\frac{1}{a}\int_{s_i}^{s_f}\frac{dS}{\exp(q\beta\omega)\{2[1+\exp(\beta\omega)]S+\exp(\beta\omega)-1\}}, \quad (19)$$

where S_i and S_f are the initial and final values of S along a given path $S(\beta,\omega)$. Equation (19) is a general expression of time evolution for a spin-$\frac{1}{2}$ system coupling with the heat reservoir and the external magnetic field.

Substituting $S(\omega)=-\frac{1}{2}\tanh(\beta_1\omega/2)$, $\beta=\beta_h$, $S_i=S_i(\beta_1,\omega_2)$, and $S_f=S_f(\beta_1,\omega_1)$ into Eq. (19), one can obtain the time of the high-temperature isothermal process as

$$t_1=\frac{\beta_1}{2a}\int_{\omega_2}^{\omega_1}\{\exp(q\beta_h\omega)[\exp(\beta_h\omega)-\exp(\beta_1\omega)]$$
$$\times[1+\exp(-\beta_1\omega)]\}^{-1}d\omega. \quad (20)$$

Similarly, substituting $S(\omega)=-\frac{1}{2}\tanh(\beta_2\omega/2)$, $\beta=\beta_c$, $S_i=S_i(\beta_2,\omega_1)$, and $S_f=S_f(\beta_2,\omega_2)$ into Eq. (19), one can obtain the time of the low-temperature isothermal process as

$$t_2=\frac{\beta_2}{2a}\int_{\omega_1}^{\omega_2}\{\exp(q\beta_c\omega)[\exp(\beta_c\omega)-\exp(\beta_2\omega)]$$
$$\times[1+\exp(-\beta_2\omega)]\}^{-1}d\omega. \quad (21)$$

In two isomagnetic field processes, the "temperature" of the working substance changes from β_1 to β_2 or from β_2 to β_1, so they need a non-negligible time compared with the time of the isothermal processes. Substituting $S(\beta) = -\frac{1}{2}\tanh(\beta\omega_1/2)$, $\beta=\beta_r$, $S_i=S_i(\beta_1,\omega_1)$, and $S_f=S_f(\beta_2,\omega_1)$ into Eq. (19), one can obtain the time of one isomagnetic field ω_1 process as

$$t_3=\frac{\omega_1}{2a}\int_{\beta_1}^{\beta_2}\{\exp(q\beta_r\omega_1)[\exp(\beta_r\omega_1)-\exp(\beta\omega_1)] \times[1+\exp(-\beta\omega_1)]\}^{-1}d\beta, \tag{22}$$

where β_r is the "temperature" of the regenerator and $\beta_r > \beta$ because heat is transferred from the working substance to the regenerator in the isomagnetic field ω_1 process. Similarly, substituting $S(\beta)=-\frac{1}{2}\tanh(\beta\omega_2/2)$, $\beta=\beta_r'$, $S_i=S_i(\beta_2,\omega_2)$, and $S_f=S_f(\beta_1,\omega_2)$ into Eq. (19), one can obtain the time of another isomagnetic field ω_2 process as

$$t_4=\frac{\omega_2}{2a}\int_{\beta_2}^{\beta_1}\{\exp(q\beta_r'\omega_2)[\exp(\beta_r'\omega_2)-\exp(\beta\omega_2)] \times[1+\exp(-\beta\omega_2)]\}^{-1}d\beta, \tag{23}$$

where β_r' is the "temperature" of the regenerator and $\beta_r' < \beta$ because heat is transferred from the regenerator to the working substance in the isomagnetic field ω_2 process.

So far we have calculated the times of two isothermal and two isomagnetic field processes. Consequently, the cycle period is given by

$$t=t_1+t_2+t_3+t_4. \tag{24}$$

VI. OPTIMIZATION ON PERFORMANCE PARAMETERS

The coefficient of performance, cooling rate, and power input are three of the important performance parameters, which are often considered in the optimal design and theoretical analysis of refrigerators. Using Eqs. (15), (17), and (24), one can find that the coefficient of performance, cooling rate, and power input may be, respectively, expressed as

$$\varepsilon=\frac{Q_c}{W}=\left\{\delta\left[\frac{\omega_1}{2}\tanh(\beta_1\omega_1/2)-\frac{\omega_2}{2}\tanh(\beta_1\omega_2/2)\right]+\frac{1}{\beta_2}\ln\frac{\cosh(\beta_2\omega_2/2)}{\cosh(\beta_2\omega_1/2)}+(1-\delta)\left[\frac{\omega_1}{2}\tanh(\beta_2\omega_1/2)-\frac{\omega_2}{2}\tanh(\beta_2\omega_2/2)\right]\right\}\bigg/\left[\frac{1}{\beta_1}\ln\frac{\cosh(\beta_1\omega_2/2)}{\cosh(\beta_1\omega_1/2)}+\frac{1}{\beta_2}\ln\frac{\cosh(\beta_2\omega_1/2)}{\cosh(\beta_2\omega_2/2)}\right], \tag{25}$$

$$R=\frac{Q_c}{t}=\left\{\delta\left[\frac{\omega_1}{2}\tanh(\beta_1\omega_1/2)-\frac{\omega_2}{2}\tanh(\beta_1\omega_2/2)\right]+\frac{1}{\beta_2}\ln\frac{\cosh(\beta_2\omega_2/2)}{\cosh(\beta_2\omega_1/2)}+(1-\delta)\left[\frac{\omega_1}{2}\tanh(\beta_2\omega_1/2)-\frac{\omega_2}{2}\tanh(\beta_2\omega_2/2)\right]\right\}\bigg/(t_1+t_2+t_3+t_4), \tag{26}$$

and

$$P=\frac{W}{t}=\frac{\frac{1}{\beta_1}\ln\frac{\cosh(\beta_1\omega_2/2)}{\cosh(\beta_1\omega_1/2)}+\frac{1}{\beta_2}\ln\frac{\cosh(\beta_2\omega_1/2)}{\cosh(\beta_2\omega_2/2)}}{t_1+t_2+t_3+t_4}. \tag{27}$$

Using Eqs. (25)–(27), one can, in principle, optimize these important performance parameters of the quantum refrigeration cycle.

At very low temperatures, $\beta_2\to\infty$ so that $\tanh(\beta_2\omega_2/2)\to 1$ and $\tanh(\beta_2\omega_1/2)\to 1$. In such a case, the amount of refrigeration per cycle $Q_2=0$ and the refrigerator has lost its role.

At high temperatures, $\beta\omega \ll 1$. The results obtained above may be simplified. For example, Eqs. (11)–(16), (20)–(23), and (25) may be, respectively, simplified as,

$$Q_1=\beta_1(\omega_2^2-\omega_1^2)/8, \tag{28}$$

$$Q_2=\beta_2(\omega_1^2-\omega_2^2)/8, \tag{29}$$

$$Q_{bc}=\omega_1^2(\beta_1-\beta_2)/4, \tag{30}$$

$$Q_{da}=\omega_2^2(\beta_2-\beta_1)/4, \tag{31}$$

$$W=(\beta_2-\beta_1)(\omega_1^2-\omega_2^2)/8, \tag{32}$$

$$\Delta Q=(\omega_1^2-\omega_2^2)(\beta_1-\beta_2)/4<0, \tag{33}$$

$$t_1=\frac{\beta_1}{4a(\beta_h-\beta_1)}\ln\left(\frac{\omega_1}{\omega_2}\right), \tag{34}$$

$$t_2=\frac{\beta_2}{4a(\beta_2-\beta_c)}\ln\left(\frac{\omega_1}{\omega_2}\right), \tag{35}$$

$$t_3=\frac{1}{4a}\int_{\beta_1}^{\beta_2}\frac{d\beta}{\beta_r-\beta}, \tag{36}$$

$$t_4=\frac{1}{4a}\int_{\beta_2}^{\beta_1}\frac{d\beta}{\beta_r'-\beta}, \tag{37}$$

and

$$\varepsilon=\frac{2\beta_1-\beta_2}{\beta_2-\beta_1}. \tag{38}$$

It should be noted that the "temperatures" β_r and β_r' of the regenerator in two isomagnetic field processes are not constant and vary with time. If there is not any additional assumption, Eqs. (36) and (37) cannot be calculated further. One of the simplest assumptions is that both $\beta_r-\beta$ and $\beta_r'-\beta$ are kept constant. Then, the times of two isomagnetic field processes may be expressed as

$$t_3+t_4=\gamma(\beta_2-\beta_1), \tag{39}$$

where γ is a proportional constant, which is independent of temperature. It will be seen from another assumption given below that this simple assumption is reasonable.

In general, the larger the temperature difference of the working substance in the two isothermal processes is, the larger the amount of regeneration and the longer the time of the regenerative processes will be. When the time of the regenerative processes is assumed to be directly proportional to the amount of regeneration, the time of two isomagnetic field processes may be written as

$$t_3+t_4=\varsigma(|Q_{bc}|+Q_{da})=\varsigma\frac{\omega_1^2+\omega_2^2}{2}(\beta_2-\beta_1)=\gamma(\beta_2-\beta_1), \tag{40}$$

where ς is also a proportional constant, which is independent of temperature. Equations (39) and (40) just show that the two assumptions mentioned above are equivalent to each other.

Using Eqs. (34), (35), and (39), one obtains the cycle period as

$$t=\frac{\beta_1}{4a(\beta_h-\beta_1)}\ln\left(\frac{\omega_1}{\omega_2}\right)+\frac{\beta_2}{4a(\beta_2-\beta_c)}\ln\left(\frac{\omega_1}{\omega_2}\right)+\gamma(\beta_2-\beta_1). \tag{41}$$

Substituting Eq. (41) into Eqs. (26) and (27) gives

$$R=\frac{b(2\beta_1-\beta_2)}{d[\beta_1/(\beta_h-\beta_1)+\beta_2/(\beta_2-\beta_c)]+\gamma(\beta_2-\beta_1)} \tag{42}$$

and

$$P=\frac{b(\beta_1-\beta_2)}{d[\beta_1/(\beta_h-\beta_1)+\beta_2/(\beta_2-\beta_c)]+\gamma(\beta_2-\beta_1)}, \tag{43}$$

where $b=(\omega_1^2-\omega_2^2)/8$ and $d=\ln(\omega_1/\omega_2)/4a$.

Using a refrigerator, one always wants to obtain a cooling rate as large as possible for a given power input. For this purpose, we introduce the Lagrangian

$$L=R+\lambda P=\frac{b[(2-y)+\lambda(y-1)]}{d[1/(\beta_h-\beta_1)+y/(y\beta_1-\beta_c)]+\gamma(y-1)}, \tag{44}$$

where λ is the Lagrange multiplier and $y=\beta_2/\beta_1$. Using Eq. (44) and the external condition $\partial L/\partial\beta_1=0$, we can obtain an optimal relation

$$\beta_1=\frac{\beta_c+y\beta_h}{2y}. \tag{45}$$

Substituting Eq. (45) into Eqs. (38), (42), and (43), we find that the fundamental optimal relations between some important parameters and the coefficient of performance are, respectively, given by

$$\beta_1=\beta_h\left[1-\frac{\varepsilon_r-\varepsilon}{2\varepsilon_c(\varepsilon+2)}\right], \tag{46}$$

$$\beta_2=\beta_c\left[1+\frac{\varepsilon_r-\varepsilon}{2(\varepsilon+1)(\varepsilon_c+1)}\right], \tag{47}$$

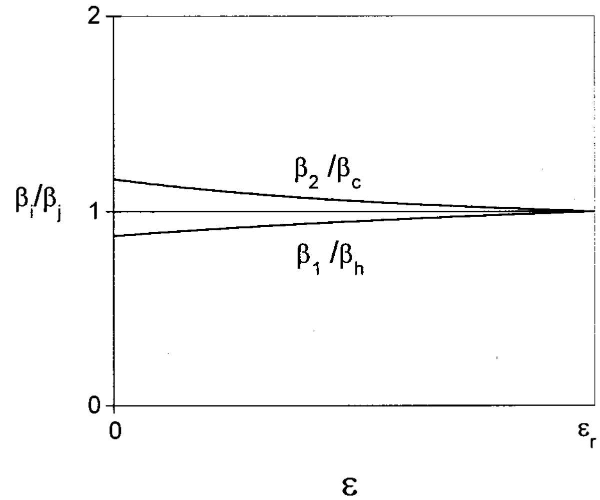

FIG. 2. The β_1/β_h-ε and β_2/β_c-ε characteristic curves.

$$R=\frac{B\beta_h\varepsilon(\varepsilon_r-\varepsilon)}{\varepsilon_c(\varepsilon+1)(\varepsilon+2)+D\beta_h(\varepsilon_r-\varepsilon)}, \tag{48}$$

and

$$P=\frac{B\beta_h(\varepsilon_r-\varepsilon)}{\varepsilon_c(\varepsilon+1)(\varepsilon+2)+D\beta_h(\varepsilon_r-\varepsilon)}, \tag{49}$$

where $B=b/4d$, $D=\gamma/4d$, $\varepsilon_c=\beta_h/(\beta_c-\beta_h)$ is the coefficient of performance of a Carnot refrigerator, and $\varepsilon_r=(2\beta_h-\beta_c)/(\beta_c-\beta_h)$ is the coefficient of performance of a magnetic Ericsson refrigerator [10,22].

Using Eqs. (46)–(49), we can plot the β_i/β_j-ε ($i=1,2$ and $j=h,c$), R^*-ε, P^*-ε, and R^*-P^* characteristic curves, as shown in Figs. 2–5, where $R^*=R/(B\beta_h)$ and $P^*=P/(B\beta_h)$ are the dimensionless cooling rate and power input, respectively. It is seen from Fig. 3 or 5 that there exists a maximum cooling rate. Starting from Eq. (48), one can prove that when the coefficient of performance

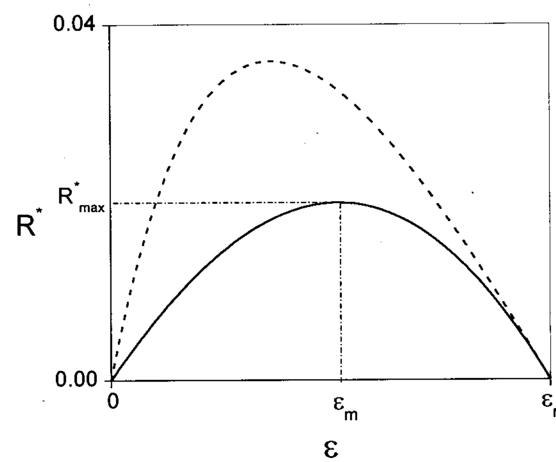

FIG. 3. The dimensionless cooling rate $R^*(=R/B\beta_h)$ versus coefficient of performance ε. Dashed ($D\beta_h=0$) and solid ($D\beta_h=10$) curves are presented for $\beta_c/\beta_h=1.5$.

$$\varepsilon_m=\frac{-(2\varepsilon_c+D\beta_h\varepsilon_r)+[(2\varepsilon_c+D\beta_h\varepsilon_r)^2+(\varepsilon_c\varepsilon_r+3\varepsilon_c-D\beta_h)(2\varepsilon_c\varepsilon_r+D\varepsilon_r^2\beta_h)]^{1/2}}{3\varepsilon_c+\varepsilon_c\varepsilon_r-D\beta_h}, \tag{50}$$

the cooling rate attains its maximum, i.e.,

$$R_{\max}=\frac{B\beta_h\varepsilon_m(\varepsilon_r-\varepsilon_m)}{\varepsilon_c(\varepsilon_m+1)(\varepsilon_m+2)+D\beta_h(\varepsilon_r-\varepsilon_m)}. \tag{51}$$

In such a case, the power input and the "temperatures" of the working substance in two isothermal processes are given by

$$P_m=\frac{B\beta_h(\varepsilon_r-\varepsilon_m)}{\varepsilon_c(\varepsilon_m+1)(\varepsilon_m+2)+D\beta_h(\varepsilon_r-\varepsilon_m)}, \tag{52}$$

$$\beta_{1m}=\beta_h\left[1-\frac{\varepsilon_r-\varepsilon_m}{2\varepsilon_c(\varepsilon_m+2)}\right], \tag{53}$$

and

$$\beta_{2m}=\beta_c\left[1+\frac{\varepsilon_r-\varepsilon_m}{2(\varepsilon_m+1)(\varepsilon_c+1)}\right]. \tag{54}$$

It is also seen from Figs. 2–5 that when $\beta_1=\beta_h$ and $\beta_2=\beta_c$, $\varepsilon=\varepsilon_r$, $R=0$, and $P=0$. When $R<R_{\max}$, there are two coefficients of performance for a given R, where one is smaller than ε_m and the other is larger than ε_m. When $\varepsilon<\varepsilon_m$, the cooling rate decreases as the coefficient of performance decreases. Obviously, the region of $\varepsilon<\varepsilon_m$ is not optimal for a quantum refrigerator. The optimal region of the coefficient of performance should be

$$\varepsilon_m\leqslant\varepsilon<\varepsilon_r. \tag{55}$$

When a quantum refrigerator is operated in this region, the cooling rate will increase as the coefficient of performance decreases, and vice versa. It is thus clear that $R_{\max}$ and ε_m are two important parameters. $R_{\max}$ determines the upper bound of the cooling rate, while ε_m determines the allowable value of the lower bound of the optimal coefficient of performance.

It is of interest to compare the results obtained here with those derived from a magnetic Ericsson refrigeration cycle. When the magnetic refrigerant in the Ericsson refrigeration cycle is described by the Curie law and heat transfer between the working substance and the heat reservoirs obeys the linear heat-transfer law in irreversible thermodynamics [23–26]. The fundamental optimum relations of the magnetic Ericsson refrigeration cycle are, respectively, given by [27]

$$T_1^{-1}=T_h^{-1}\left[1-\frac{\varepsilon_r-\varepsilon}{(1+u)\varepsilon_c(\varepsilon+2)}\right], \tag{56}$$

$$T_2^{-1}=T_c^{-1}\left[1+\left(\frac{u}{1+u}\right)\frac{\varepsilon_r-\varepsilon}{(\varepsilon_c+1)(\varepsilon+1)}\right], \tag{57}$$

$$R=\frac{k_1T_h^{-1}\varepsilon(\varepsilon_r-\varepsilon)}{(1+u)^2\varepsilon_c(\varepsilon+1)(\varepsilon+2)+k_1\nu T_h^{-1}(\varepsilon_c-\varepsilon)}, \tag{58}$$

and

$$P=\frac{k_1T_h^{-1}(\varepsilon_r-\varepsilon)}{(1+u)^2\varepsilon_c(\varepsilon+2)(\varepsilon+1)+k_1\nu T_h^{-1}(\varepsilon_r-\varepsilon)}, \tag{59}$$

where ν is a parameter that is dependant on the magnetic fields in two isomagnetic field processes but independent of temperature, $u=\sqrt{k_1/k_2}$, and k_1 and k_2 are the thermal conductances between the working substance and the heat reservoirs at temperatures T_h and T_c, respectively. When k_1

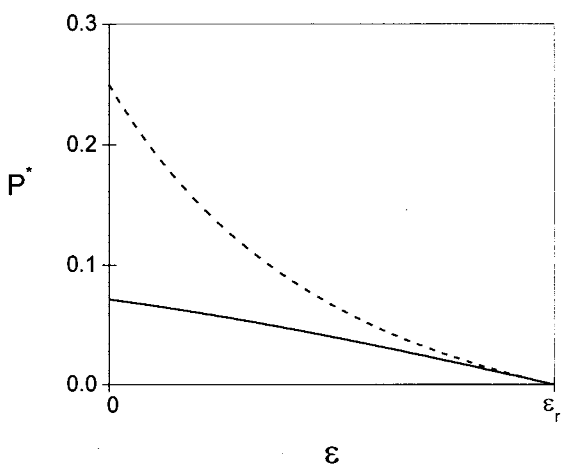

FIG. 4. The dimensionless power input $P^*(=P/B\beta_h)$ versus coefficient of performance ε. The values of the parameters $D\beta_h$ and β_c/β_h are the same as those used in Fig. 3.

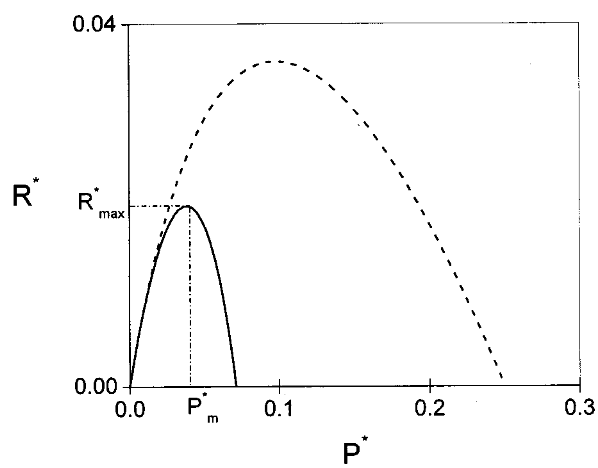

FIG. 5. The dimensionless cooling rate R^* versus dimensionless power input P^*. The values of the parameters $D\beta_h$ and β_c/β_h are the same as those used in Fig. 3.

$=k_2$, Eqs. (46) and (47) are identical with Eqs. (56) and (57) and the forms of Eqs. (48) and (49) are the same as those of Eqs. (58) and (59), respectively. If $k_1/4=B$ and $k_1 v/4=D$ is chosen further, Eqs. (48) and (49) are identical with Eqs. (58) and (59), respectively. The above results show clearly that in the high temperature limit, a quantum refrigeration cycle using spin-$\frac{1}{2}$ systems as the working substance and consisting of two isothermal and two isomagnetic field processes is equivalent to a magnetic Ericsson refrigeration cycle. Thus, such a cycle described in this paper may be referred to as the quantum Ericsson refrigeration cycle.

VII. DISCUSSION AND GENERALIZATIONS

(1) When the work substance is composed of a spin-J system ($J=1/2,1,3/2,2,\ldots,$) the mean value of the spin angular momentum is given by [28–30]

$$S=\langle \hat{S}_z\rangle=-JB_J(\beta\omega J), \tag{60}$$

where $-J\leqslant S\leqslant J$ and

$$B_J(x)=\left(\frac{2J+1}{2J}\right)\coth\left(\frac{2J+1}{2J}x\right)-\frac{1}{2J}\coth\left(\frac{x}{2J}\right)$$

is the Brillouin function.

At high temperatures, Eq. (60) may be simplified as

$$S=-\frac{J(J+1)}{3}\beta\omega. \tag{61}$$

Compared with Eq. (4), the heat amount of the various processes in the cycle may be obtained by multiplying the factor of $4J(J+1)/3$ in Eqs. (28)–(31). On the other hand, using the method in Sec. V, one can prove [17] that the time evolution of the spin angular momentum is determined by

$$\frac{dS}{dt}=-2a\{2S+\beta\omega[J(J+1)-M]\}, \tag{62}$$

where $M=[J(J+1)]/3$. From Eqs. (61) and (62), we can find that the times of the various processes are the same as Eqs. (34), (35), and (39). Thus, the coefficient of performance of the quantum cycle consisting of the spin-J systems is the same as that of the quantum cycle consisting of the spin-$\frac{1}{2}$ systems, while the cooling rate and power input are $4J(J+1)/3$ times of those of the quantum cycle consisting of the spin-$\frac{1}{2}$ systems, respectively.

(2) When the regenerative time is negligible, $D=0$. Equations (46), (47), (53), and (54) are still true, while Eqs. (48)–(52) may be, respectively, simplified by

$$R=\frac{B\beta_h\varepsilon(\varepsilon_r-\varepsilon)}{\varepsilon_c(\varepsilon+1)(\varepsilon+2)}, \tag{63}$$

$$P=\frac{B\beta_h(\varepsilon_r-\varepsilon)}{\varepsilon_c(\varepsilon+1)(\varepsilon+2)}, \tag{64}$$

$$\varepsilon_m=\frac{-2+[4+2\varepsilon_r(\varepsilon_r+3)]^{1/2}}{\varepsilon_r+3}, \tag{65}$$

$$R_{\max}=\frac{B\beta_h(\varepsilon_r-\varepsilon_m)}{\varepsilon_c(\varepsilon_m+1)(\varepsilon_m+2)}, \tag{66}$$

and

$$P_m=\frac{B\beta_h(\varepsilon_r-\varepsilon_m)}{\varepsilon_c(\varepsilon_m+1)(\varepsilon_m+2)}. \tag{67}$$

(3) When the two isomagnetic field processes in the cycle are replaced by two adiabatic processes, the cycle becomes a quantum Carnot refrigeration cycle. In this case, $D=0$, $Q_{bc}=0$, $Q_{da}=0$, $S_b=S_c=S_1$, $S_d=S_a=S_2$, and S_1 and S_2 are the spin angular momentums in two adiabatic processes, respectively. In the adiabatic processes, there is not any heat exchange between the working substance and the external heat reservoirs. The time of the adiabatic processes is often assumed to be negligible compared with the time of the isothermal processes. Using Eqs. (10) and (18), we can calculate Q_1, Q_2, t_1, and t_2. In the high temperature limit, they are, respectively, given by

$$Q_1=-\frac{2}{\beta_1}(S_1^2-S_2^2), \tag{68}$$

$$Q_2=\frac{2}{\beta_2}(S_1^2-S_2^2), \tag{69}$$

$$t_1=\frac{\beta_1}{\beta_h-\beta_1}\frac{\ln(S_1/S_2)}{4a}, \tag{70}$$

and

$$t_2=\frac{\beta_2}{\beta_2-\beta_c}\frac{\ln(S_1/S_2)}{4a}. \tag{71}$$

From Eqs. (68)–(71), we can obtain the coefficient of performance, the cooling rate, and power input as

$$\varepsilon=\frac{\beta_1}{\beta_2-\beta_1}, \tag{72}$$

$$R=\frac{8a(S_1^2-S_2^2)/\beta_2}{\ln(S_1/S_2)[\beta_1/(\beta_h-\beta_1)+\beta_2/(\beta_2-\beta_c)]}, \tag{73}$$

and

$$P=\frac{8a(S_1^2-S_2^2)(1/\beta_1-1/\beta_2)}{\ln(S_1/S_2)[\beta_1/(\beta_h-\beta_1)+\beta_2/(\beta_2-\beta_c)]}. \tag{74}$$

Using the similar method mentioned above, one can prove that the fundamental optimum relations for a quantum Carnot refrigerator are, respectively, given by

$$R=\frac{2a(S_1^2-S_2^2)}{\ln(S_1/S_2)}\left(\frac{1}{\beta_c}-\frac{\varepsilon}{1+\varepsilon}\frac{1}{\beta_h}\right), \tag{75}$$

$$P=\frac{2a(S_1^2-S_2^2)}{\ln(S_1/S_2)}\left(\frac{1}{\varepsilon\beta_c}-\frac{1}{1+\varepsilon}\frac{1}{\beta_h}\right), \tag{76}$$

$$\beta_1 = \frac{2\varepsilon\beta_h\beta_c}{(1+\varepsilon)\beta_h + \varepsilon\beta_c}, \tag{77}$$

and

$$\beta_2 = \frac{2(1+\varepsilon)\beta_h\beta_c}{(1+\varepsilon)\beta_h + \varepsilon\beta_c}. \tag{78}$$

Using the fundamental optimum relations and the similar method mentioned above, one can further discuss the various optimum performance characteristics of a quantum Carnot refrigerator.

The results obtained above show clearly that when the irreversibilities existing in refrigerators are taken into account, it is very important to find the fundamental optimum relations of refrigerators. Using these relations, one can reveal the universal performance characteristics of refrigerators [31–33].

(4) The above discussion only refers to a single spin-J system. For the working substance consisting of many noninteracting spin-J systems, the coefficient of performance is still true, while the internal energy, work input, power input, and heat quantity can be obtained as long as the above results are simply multiplied by the total number of spin systems.

VIII. CONCLUSIONS

We have established the cycle model of a typical quantum refrigerator consisting of two isothermal and two isomagnetic field processes and using noninteracting spin-$\frac{1}{2}$ systems as the working substance. Based on the spin theory, motion equation of an operator, and semigroup formalism, we have analyzed the optimal performance characteristics of the quantum refrigeration cycles and derived the concrete expressions of several important parameters such as the coefficient of performance, cooling rate, power input, and temperatures of the working substance in two isothermal processes. Especially, the optimal performance of the quantum refrigerator in the high temperature limit is discussed in detail. The maximum cooling rate and the corresponding parameters are calculated. The optimally operating region of the quantum refrigeration is determined. These results derived in this paper are compared with those obtained from the cycle model of a magnetic Ericsson refrigerator. Many similarities between them are found. If some parameters are chosen reasonably, they are equivalent to each other.

The results obtained are further generalized, so that they are also suitable for the working substance consisting of noninteracting spin-J systems. Finally, the replacement of two isomagnetic field processes by two adiabatic processes gives directly the cycle model of a quantum Carnot refrigerator, so that the optimal performance of the quantum Carnot refrigerator can be derived simply from the present paper.

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