Exact amplitude ratio and finite-size corrections for the $M\times N$ square lattice Ising model

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Let f, U, and C represent, respectively, the free energy, the internal energy, and the specific heat of the critical Ising model on the $M\times N$ square lattice with periodic boundary conditions, and f_∞ represents f for fixed M/N and $N\to\infty$. We find that f, U, and C can be written as $N(f-f_\infty)=\Sigma_{i=1}^{\infty}f_{2i-1}/N^{2i-1}$, $U=-\sqrt{2}+\Sigma_{i=1}^{\infty}u_{2i-1}/N^{2i-1}$, and $C=8\ln N/\pi+\Sigma_{i=0}^{\infty}c_i/N^i$, i.e., Nf and U are odd functions of N^{-1}. We also find that $u_{2i-1}/c_{2i-1}=1/\sqrt{2}$ and $u_{2i}/c_{2i}=0$ for $1\leqslant i<\infty$ and obtain closed form expressions for f, U, and C up to orders $1/N^5$, $1/N^5$, and $1/N^3$, respectively, which implies an analytic equation for c_5.

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I. INTRODUCTION

The Ising model has been used to represent critical phenomena in ferromagnets, binary alloys, binary fluids, gas-liquid mixture, etc., and is perhaps the most widely studied model of critical phenomena [1]. For analyzing the simulation or experimental data of finite critical systems [2], it is useful to appeal to theories of finite-size corrections [3] and finite-size scaling [4]. Such theories have attracted much attention in recent years [5–9] because of the fast advance in computers' computing power and algorithms for simulating or analyzing data. Theories of finite-size effects and of finite-size scaling, in general, have been most successful in deriving critical and noncritical properties of infinite systems from those of their finite or partially finite counterparts. Finite-size corrections and finite-size scaling for the $M\times N$ square lattice Ising model are of particular interest because the Ising model is very popular and such system is usually used to test the efficiency of algorithms for studying critical systems [10]. In the present paper, we present analytic results for finite-size effects in the Ising model on a large $M\times N$ square lattice at the critical point.

Finite-size scaling is the basis of the powerful phenomenological renormalization group method [11,12]. In the two-dimensional Ising model the finite-size effect on the renormalization transformation has been demonstrated to be rather benign [13], and the effects due to convergence to the fixed point and finite size are clearly distinguished [14]. The finite-size scaling theory predicts that near the critical point the singular part of the thermodynamic quantity of a finite system, say Q_s, has the scaling form

$$Q_s=L^{y_Q}Y_Q(L/\xi_\infty), \tag{1}$$

where L is system linear size, ξ_∞ is the correlation length of the bulk system, y_Q is a critical exponent, and Y_Q is the scaling function. The scaling ansatz mentioned above ignores the possible logarithmic corrections. In the case of planar Ising model, which displays a logarithmic singularities in the specific heat behavior due to a relation between scaling exponents in the renormalization group theory [11], the scaling form (1) must then be replaced by a more general form [4,15–17]

$$Q_s=L^{y_Q}Y_Q(L/\xi_\infty)+L^{y_Q}\ln LX_Q(L/\xi_\infty), \tag{2}$$

which in the case of the specific heat (C) becomes

$$C_s=Y_C(L/\xi_\infty)+\ln LX_C(L/\xi_\infty). \tag{3}$$

The results of this paper to be presented below show that the leading term of C_s is $8\ln L/\pi$, all other finite-size corrections to the specific heat are always integer powers of L^{-1}, which also imply that the scaling function X_C in Eq. (3) is constant and equal to $8/\pi$. Very recently, Caselle *et al.* [18] have shown that this result can be predicted by conformal field theory under a number of general conjectures.

The relevance of the finite-size properties to the conformal field theory is another source of interest. Discussion of general properties of nonuniversal corrections to finite-size scaling and their relation to irrelevant operators in conformal field theory can be found in Ref. [19]. On the basis of conformal invariance, the asymptotic finite-size scaling behavior of the critical free energy f_N per site and the inverse correlation length ξ_N^{-1} of a $N\times\infty$ system is found to be [20]

$$\lim_{N\to\infty}N^2(f_N-f_\infty)=\frac{c\pi}{6}, \tag{4}$$

$$\lim_{N\to\infty}N\xi_N^{-1}=2\pi x, \tag{5}$$

where f_∞ is the free energy of the bulk system, c is the conformal anomaly number, and x is the scaling dimension. The corrections to Eqs. (4) and (5) can be calculated by the means of a perturbated conformal field theory [21,22] and can be expressed in terms of the universal structure constants (C_{nln}) of the operator product expansion [21]. Quite recently, Izmailian and Hu [9] studied the finite-size correction terms for the free energy and the inverse correlation length of critical Ising model on $N\times\infty$ lattices and obtained a new set of the universal amplitude ratios for the coefficients in the

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free energy and the inverse correlation length expansions. It was shown that such results could be understood from a perturbated conformal field theory.

Based on Onsager's solution, explicit calculations of the specific-heat finite-size scaling behavior have been reported by Ferdinand and Fisher [3] and by Kleban and Akinci [23]. In 1969, Ferdinand and Fisher [3] first studied finite-size corrections for a critical Ising model on $M\times N$ square lattices with periodic boundary conditions. They gave explicit expressions for the critical free energy f, internal energy U, and specific heat C per lattice site for a fixed $\xi=M/N$ and large N up to orders $1/N^2$, $1/N$, and $1/N$, respectively,

$$f=f_\infty+\frac{1}{\xi N^2}\left[\ln(\theta_2+\theta_3+\theta_4)-\frac{1}{3}\ln(4\theta_2\theta_3\theta_4)\right]+O\left(\frac{1}{N^3}\right), \tag{6}$$

$$U=-\sqrt{2}-\frac{1}{N}\,\frac{2\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}+O\left(\frac{1}{N^2}\right), \tag{7}$$

$$\begin{aligned}C=&\frac{8}{\pi}\ln N+\frac{8}{\pi}\left(\ln\frac{2^{5/2}}{\pi}+C_E-\frac{\pi}{4}\right)-\frac{16}{\pi}\,\frac{\sum_{i=2}^{4}\theta_i\ln\theta_i}{\theta_2+\theta_3+\theta_4}\\&-4\xi\left(\frac{\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}\right)^2-2\sqrt{2}\,\frac{\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}\,\frac{1}{N}\\&+O([\ln N]^3/N^2),\end{aligned} \tag{8}$$

where $\theta_i=\theta_i(0,q)(i=2,3,4)$ is elliptic theta functions of modulus $q=e^{-\pi\xi}$, C_E is the Euler constant, and f_∞ is the free energy in the thermodynamic limit $M,N\to\infty$.

In 1983, Kleban and Akinci [23] gave a very accurate and relatively simple approximate closed form expression for leading specific-heat correction term that results from retaining only the two largest eigenvalues of the transfer matrix. This approximation is already good at $\xi=1$ and becomes exponentially better with increasing ξ, and they interpreted their results in terms of domain-wall energies.

In this paper we study the same system as [3] and find that f, U, and C can be written as $N(f-f_\infty)=\sum_{i=1}^{\infty}f_{2i-1}/N^{2i-1}$, $U=-\sqrt{2}+\sum_{i=1}^{\infty}u_{2i-1}/N^{2i-1}$, and $C=8\ln N/\pi+\sum_{i=0}^{\infty}c_i/N^i$, i.e., Nf and U are odd functions of N^{-1}. We also find that $u_{2i-1}/c_{2i-1}=1/\sqrt{2}$ and $u_{2i}/c_{2i}=0$ for $1\leqslant i<\infty$ and obtain analytic equations for f, U, and C up to orders $1/N^5$, $1/N^5$, and $1/N^3$, respectively, which implies an analytic equation for c_5.

We have also shown that Kleban and Akinci approximation is in excellent agreement with our exact results for the leading correction terms of the free energy (f_1), the internal energy (u_1), and the specific heat (c_0,c_1) at $\xi\geqslant 1$. For the next correction terms the error introduced by the two-eigenvalues approximation is maximum at $\xi=1$ $(M=N)$. With increasing ξ the exact and approximate values approach exponentially and approximation becomes already good at $\xi=1.65$ for the correction terms f_3,u_3,c_2,c_3 and at $\xi=1.85$ for the correction terms f_5,u_5.

This paper is organized as follows. In Sec. II, we write the free energy f, the internal energy U, and the specific heat C of the Ising model in terms of P_1, P_2, P_3, P_4, Q_1, Q_2, and Q_3 defined in this section. In Sec. III, we present asymptotic expansions for f, U, and C. In Sec. IV, we discuss some problems for further studies. Some mathematical details used in the derivation of equations in Sec. III are given in the Appendix.

II. ISING MODEL

Consider an Ising ferromagnet on an $M\times N$ lattice with periodic boundary conditions (i.e., a torus). The Hamiltonian of the system is

$$\beta H=-J\sum_{\langle ij\rangle}s_is_j, \tag{9}$$

where $\beta=(k_BT)^{-1}$, the Ising spins $s_i=\pm 1$ are located at the sites of the lattice and the summation goes over all nearest-neighbor pairs of the lattice. The partition function $Z_{MN}(T)$ of a finite $M\times N$ square Ising lattice wrapped on a torus can be written as

$$Z_{MN}(T)=\frac{1}{2}(2\sinh 2J)^{(1/2)MN}\sum_{i=1}^{4}Z_i, \tag{10}$$

where the partial partition functions Z_i are defined by

$$Z_1=\prod_{r=0}^{N-1}2\cosh\frac{M}{2}\gamma_{2r+1}=P_1\exp\left[\frac{M}{2}\sum_{r=0}^{N-1}\gamma_{2r+1}\right], \tag{11}$$

$$Z_2=\prod_{r=0}^{N-1}2\sinh\frac{M}{2}\gamma_{2r+1}=P_2\exp\left[\frac{M}{2}\sum_{r=0}^{N-1}\gamma_{2r+1}\right], \tag{12}$$

$$Z_3=\prod_{r=0}^{N-1}2\cosh\frac{M}{2}\gamma_{2r}=P_3\exp\left[\frac{M}{2}\sum_{r=0}^{N-1}\gamma_{2r}\right](1+e^{-M\gamma_0}), \tag{13}$$

$$Z_4=\prod_{r=0}^{N-1}2\sinh\frac{M}{2}\gamma_{2r}=P_4\exp\left[\frac{M}{2}\sum_{r=0}^{N-1}\gamma_{2r}\right](1-e^{-M\gamma_0}), \tag{14}$$

with

$$P_1=\prod_{r=0}^{N-1}(1+e^{-M\gamma_{2r+1}}),\quad P_2=\prod_{r=0}^{N-1}(1-e^{-M\gamma_{2r+1}}),$$

$$P_3=\prod_{r=1}^{N-1}(1+e^{-M\gamma_{2r}}),\quad P_4=\prod_{r=1}^{N-1}(1-e^{-M\gamma_{2r}}), \tag{15}$$

and γ_r is implicitly given by

$$\cosh\gamma_r=\frac{\cosh^2 2J}{\sinh 2J}-\cos\frac{r\pi}{N}. \tag{16}$$

At the critical point J_c of the square lattice Ising model, where $J_c=\frac{1}{2}\ln(1+\sqrt{2})$, one then obtains

$$\gamma_r^{(cr)}=2\psi_{sq}\left(\frac{r\pi}{2N}\right),\quad \text{with}\quad \psi_{sq}(x)=\ln(\sin x+\sqrt{1+\sin^2 x}). \tag{17}$$

The free energy, the internal energy per spin, and the specific heat per spin can be obtained from the partition function Z_{MN}

$$f=\frac{1}{MN}\ln Z_{MN}=\frac{1}{2}\ln 2\sinh 2J+\frac{1}{MN}\ln\frac{1}{2}\left(\sum_{i=1}^{4} Z_i\right), \tag{18}$$

$$U=-\frac{1}{MN}\frac{d}{dJ}\ln Z_{MN} =-\coth 2J-\frac{1}{MN}\left(\sum_{i=1}^{4} Z_i'\right)\Bigg/\left(\sum_{i=1}^{4} Z_i\right), \tag{19}$$

$$C=\frac{1}{MN}\frac{d^2}{dJ^2}\ln Z_{MN} =-\frac{2}{\sinh^2 2J}+\frac{1}{MN}\left\{\left(\sum_{i=1}^{4} Z_i''\right)\Bigg/\left(\sum_{i=1}^{4} Z_i\right) -\left[\left(\sum_{i=1}^{4} Z_i'\right)\Bigg/\left(\sum_{i=1}^{4} Z_i\right)\right]^2\right\}, \tag{20}$$

where the primes denote differentiation with respect to J. At the critical point $(T=T_c)$ the partial partition functions Z_i and their first and second derivatives are given by

$$Z_1=P_1e^A,\quad Z_2=P_2e^A,\quad Z_3=2P_3e^B,\quad Z_4=0; \tag{21}$$

$$Z_1'=0,\quad Z_2'=0,\quad Z_3'=0,\quad Z_4'=4MP_4e^B; \tag{22}$$

$$\frac{Z_1''}{MN}=Q_1Z_1,\quad \frac{Z_2''}{MN}=Q_2Z_2,\quad \frac{Z_3''}{MN}=Q_3Z_3,\quad Z_4''=-\sqrt{2}Z_4', \tag{23}$$

where

$$A=\frac{M}{2}\sum_{r=0}^{N-1}\gamma_{2r+1}^{(cr)},\quad B=\frac{M}{2}\sum_{r=0}^{N-1}\gamma_{2r}^{(cr)}, \tag{24}$$

$$Q_1=\frac{1}{2N}\sum_{r=0}^{N-1}\gamma_{2r+1}''^{(cr)}\tanh\frac{M\gamma_{2r+1}^{(cr)}}{2}, \tag{25}$$

$$Q_2=\frac{1}{2N}\sum_{r=0}^{N-1}\gamma_{2r+1}''^{(cr)}\coth\frac{M\gamma_{2r+1}^{(cr)}}{2}, \tag{26}$$

$$Q_3=4\xi+\frac{1}{2N}\sum_{r=1}^{N-1}\gamma_{2r}''^{(cr)}\tanh\frac{M\gamma_{2r}^{(cr)}}{2}, \tag{27}$$

and P_i are given by Eq. (15) with $\gamma_r=\gamma_r^{(cr)}$ and $\gamma_r''^{(cr)}$ denote the second derivative of γ_r with respect to J at the critical point $J=J_c$. Then the exact expression for the free energy, the internal energy, and the specific heat of a finite Ising model at critical point $(T=T_c)$ can be written as

$$f=\frac{1}{2}\ln 2+\frac{1}{MN}A+\frac{1}{MN}\ln\frac{P_1+P_2+2P_3\exp(B-A)}{2}, \tag{28}$$

$$U=-\sqrt{2}-\frac{4}{N}\frac{P_4}{2P_3+(P_1+P_2)\exp(A-B)}, \tag{29}$$

$$C=\sqrt{2}U-\xi\left(\frac{4P_4}{2P_3+(P_1+P_2)\exp(A-B)}\right)^2 +\frac{2Q_3P_3+(Q_1P_1+Q_2P_2)\exp(A-B)}{2P_3+(P_1+P_2)\exp(A-B)}. \tag{30}$$

III. ASYMPTOTIC EXPANSIONS

We consider only sequences of lattices in which $\xi=M/N$ remains positive and finite as the thermodynamic limit $M,N\to\infty$ is approached. Using Taylor's theorem we find that $M\gamma_r$ is even function of $1/N$ at the critical point

$$M\gamma_r^{(cr)}=\sum_{i=0}^{\infty}\frac{a_i}{N^{2i}}=\pi\xi r-\frac{\pi^3\xi}{12}\frac{r^3}{N^2}+\frac{\pi^5\xi}{96}\frac{r^5}{N^4}+\cdots. \tag{31}$$

Using Euler-Maclaurin summation formula [24] we can expand A and B up to arbitrary order

$$A=\frac{MN}{\pi}\int_0^{\pi}\psi_{sq}(x)dx+M\sum_{k=1}^{\infty}\frac{2B_{2k}}{(2k)!}(2^{2k-1}-1)\psi_{sq}^{(2k-1)}(0)\times\left(\frac{\pi}{2N}\right)^{2k-1} =\frac{2G}{\pi}MN+\frac{\pi\xi}{12}+\frac{7\pi^3\xi}{1440}\frac{1}{N^2}+\frac{31\pi^5\xi}{24\,192}\frac{1}{N^4} +\frac{10\,033\pi^7\xi}{9\,676\,800}\frac{1}{N^6}+\cdots, \tag{32}$$

$$B=\frac{MN}{\pi}\int_0^{\pi}\psi_{sq}(x)dx-M\sum_{k=1}^{\infty}\frac{2B_{2k}}{(2k)!}\left(\frac{\pi}{N}\right)^{2k-1}\psi_{sq}^{(2k-1)}(0) =\frac{2G}{\pi}MN-\frac{\pi\xi}{6}-\frac{\pi^3\xi}{180}\frac{1}{N^2}-\frac{\pi^5\xi}{756}\frac{1}{N^4}-\frac{79\pi^7\xi}{75\,600}\frac{1}{N^6}-\cdots, \tag{33}$$

where B_{2i} are the Bernoulli numbers and $G=0.915\,965\cdots$ is Catalan's constant.

Let us now evaluate the products P_i for $i=1,2,3,4$. It is easy to see from Eqs. (15) and (31) that P_i contains only even power of $1/N$

$$P_i=\pi_i(0,\xi)\left(1+\sum_{j=1}^{\infty}\frac{p_{ij}}{N^{2j}}\right) \quad (i=1,2,3,4), \tag{34}$$

$$P_1=\pi_1(0,\xi)\left(1+\frac{p_{11}}{N^2}+\frac{p_{12}}{N^4}+\cdots\right),$$

$$P_2=\pi_2(0,\xi)\left(1+\frac{p_{21}}{N^2}+\frac{p_{22}}{N^4}+\cdots\right),$$

$$P_3=\pi_3(0,\xi)\left(1+\frac{p_{31}}{N^2}+\frac{p_{32}}{N^4}+\cdots\right),$$

$$P_4=\pi_4(0,\xi)\left(1+\frac{p_{41}}{N^2}+\frac{p_{42}}{N^4}+\cdots\right),$$

with

$$\pi_1(0,\xi)=\frac{\theta_3}{\theta_0},\quad \pi_2(0,\xi)=\frac{\theta_4}{\theta_0},\quad \pi_3(0,\xi)=\frac{\theta_0^2}{\theta_3\theta_4},$$

$$\pi_4(0,\xi)=\theta_0^2,\quad e^{\pi\xi/4}=\frac{2\,\theta_0^3}{\theta_2\theta_3\theta_4},$$

where $\theta_i=\theta_i(0,q)$ is elliptic theta functions of modulus $q=e^{-\pi\xi}$. The explicit expressions for the coefficients p_{i1} and p_{i2} for $i=1,2,3,4$ are given in the Appendix.

One readily sees from Eqs. (28), (29), and (32)–(34) that the finite-size estimates of the free energy (Nf) and the internal energy (U) must be odd functions of N^{-1}.

$$N(f-f_\infty)=\sum_{i=1}^{\infty}\frac{f_{2i-1}}{N^{2i-1}}, \tag{35}$$

$$U=-\sqrt{2}+\sum_{i=1}^{\infty}\frac{u_{2i-1}}{N^{2i-1}}. \tag{36}$$

Substituting Eqs. (32)–(34), (A14), and (A15) in Eqs. (28) and (29) we can write the expansions of the free energy (Nf) and the internal energy (U) at the critical point ($T=T_c$) up to $1/N^5$ order. The final result is

$$N(f-f_\infty)=\frac{\xi^{-1}}{N}\left[\ln(\theta_2+\theta_3+\theta_4)-\frac{1}{3}\ln(4\,\theta_2\theta_3\theta_4)\right]-\frac{\pi^3}{N^3}\,\frac{8\,\theta_2\theta_3\theta_4[\theta_3^3(\theta_2^3+\theta_4^3)-\theta_2^3\theta_4^3]-7(\theta_2^9+\theta_3^9+\theta_4^9)}{1440(\theta_2+\theta_3+\theta_4)}$$
$$+\frac{1}{N^5}\,\frac{\xi K^8}{189\pi^2}\,\frac{f_{51}+\left(\frac{E'}{K'}-\frac{E}{K}\right)f_{52}}{\theta_2+\theta_3+\theta_4}+O\left(\frac{1}{N^7}\right), \tag{37}$$

with

$$f_{52}=\theta_2(-32+48k^2-78k^4+31k^6)+\theta_3(31-78k^2+48k^4-32k^6)+\theta_4(31-15k^2-15k^4+31k^6), \tag{38}$$

$$f_{51}=\theta_2(-32+80k^2-38k^4+21k^6)+\theta_3(31-88k^2+88k^4)+\theta_4(31-67k^2+25k^4-21k^6)$$
$$+\frac{21}{8}\,\frac{\theta_3\theta_4k^4(1+k'^2)^2+\theta_2\theta_3k'^4(1+k^2)^2+\theta_2\theta_4(k'^2-k^2)^2}{\theta_2+\theta_3+\theta_4}, \tag{39}$$

and

$$U=-\sqrt{2}-\frac{2}{N}\,\frac{\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}+\frac{2}{N^3}\,\frac{\theta_2\theta_3\theta_4(\theta_2^9+\theta_3^9+\theta_4^9)}{(\theta_2+\theta_3+\theta_4)^2}\,\frac{\pi^3\xi}{96}+\frac{2}{N^5}\,\frac{\theta_2\theta_3\theta_4}{(\theta_2+\theta_3+\theta_4)^2}\,\frac{\xi^2K^8}{3\pi^2}\left[U_{51}+\left(\frac{E'}{K'}-\frac{E}{K}\right)U_{52}\right]+O\left(\frac{1}{N^7}\right) \tag{40}$$

with

$$U_{52}=\theta_3(k'^2-k^2)-\theta_2k^4(1+k'^2)+\theta_4k'^4(1+k^2), \tag{41}$$

$$U_{51}=\theta_3\frac{23-64k^2+64k^4}{24}+\theta_2k^4\frac{16+8k^2-k^4}{24}+\theta_4(1-k^2)^2\frac{23-6k^2-k^4}{24}+\frac{\theta_3\theta_4k^4(1+k'^2)^2+\theta_2\theta_3k'^4(1+k^2)^2+\theta_2\theta_4(k'^2-k^2)^2}{12(\theta_2+\theta_3+\theta_4)}, \tag{42}$$

where $f_\infty=-0.5\ln 2-2G/\pi$ and $\theta_2,\theta_3,\theta_4$ are elliptic functions,

$$\theta_2=\sqrt{\frac{2kK(k)}{\pi}},\quad \theta_3=\sqrt{\frac{2K(k)}{\pi}},\quad \theta_4=\sqrt{\frac{2k'K(k)}{\pi}}, \tag{43}$$

with $K(k)$ and $E(k)$ the elliptic integrals of the first and second kind, respectively. For simplicity we denote $K\equiv K(k)$, $K'\equiv K'(k)$, $E\equiv E(k)$, and $E'\equiv E'(k)$.

It is interesting to compare our results for the free energy and the internal energy with Kleban and Akinci two-eigenvalues approximation. Keeping only two largest eigenvalues λ_0 and λ_1 of the transfer matrix, the partition function of the Ising model can be written as

$$Z_{MN}=\lambda_0^M+\lambda_1^M \tag{44}$$

with

$$\lambda_0=(2\sinh 2J)^{N/2}\exp\left(\frac{1}{2}\sum_{r=0}^{N-1}\gamma_{2r+1}\right), \tag{45}$$

$$\lambda_1=(2\sinh 2J)^{N/2}\exp\left(\frac{1}{2}\sum_{r=1}^{N}\gamma_{2r}\right), \tag{46}$$

where γ_k is implicitly given by Eq. (16).

To write the critical free energy f and critical internal energy U in the form of Eqs. (35) and (36), we must evaluate Eqs. (45) and (46) asymptotically. These sums can be handled by using the Euler-Maclaurin summation formula [24]. After a straightforward calculation, we have obtained

$$N(f-f_\infty)=\frac{f_1^{(app)}}{N}+\frac{f_3^{(app)}}{N^3}+\frac{f_5^{(app)}}{N^5}+\cdots, \tag{47}$$

$$U=-\sqrt{2}+\frac{u_1^{(app)}}{N}+\frac{u_3^{(app)}}{N^3}+\frac{u_5^{(app)}}{N^5}+\cdots \tag{48}$$

with

$$f_1^{(app)}=-\frac{\pi}{12}-\frac{1}{\xi}\ln(1+e^{-\pi\xi/4}), \tag{49}$$

$$f_3^{(app)}=\frac{\pi^3}{2880}[1-15\tanh(\pi\xi/8)], \tag{50}$$

$$f_5^{(app)}=\frac{\pi^5}{48384}[1-63\tanh(\pi\xi/8)]-\frac{\pi^6\xi}{73728}\operatorname{sech}^2(\pi\xi/8), \tag{51}$$

$$u_1^{(app)}=-1+\tanh(\pi\xi/8), \tag{52}$$

$$u_3^{(app)}=\frac{\pi^3\xi}{192}\operatorname{sech}^2(\pi\xi/8), \tag{53}$$

$$u_5^{(app)}=\frac{\pi^5\xi}{768}\operatorname{sech}^2(\pi\xi/8)-\frac{\pi^6\xi^2}{36\,864}\operatorname{sech}^2(\pi\xi/8)\times\tanh(\pi\xi/8). \tag{54}$$

The expressions of the coefficients given by Eqs. (49)–(54) are much simpler than their exact counterparts given by Eqs. (37)–(42). Nevertheless, one can see from Figs. 1 and 2 that two-eigenvalues approximation proposed by Kleban and Akinci is already good at $\xi=1$ for the leading corrections terms in the free energy (f_1) and the internal energy (u_1) and becomes exponentially better with increasing ξ. The error introduces by the two-eigenvalues approximation is maximum at $\xi=1$ $(M=N)$. With increasing ξ the exact and approximate values approach exponentially and approximation becomes already good at $\xi=1.65$ for the correction terms f_3,u_3 and at $\xi=1.85$ for the correction terms f_5,u_5. We consider the case $\xi\geqslant1$ only. By symmetry, the same results hold for $\xi'=1/\xi\leqslant1$.

To calculate the specific heat we must also evaluate asymptotically the sums appearing in the expression (30) for C, namely, Q_1, Q_2, and Q_3. Since the analysis follows the same general lines as in the cases of the free energy and the internal energy, we will not present the details of calculations and we quote here only results, namely, at the critical point $T=T_c$ the asymptotic expansion of the sums Q_1, Q_2, and Q_3 can be written as

$$Q_i=\frac{8}{\pi}\ln N+\sum_{j=0}^{\infty}\frac{q_{ij}}{N^{2j}}\quad\text{for}\quad(i=1,2,3), \tag{55}$$

where q_{i0} and q_{i1} (for $i=1,2,3$) are given by

$$q_{10}=\frac{8}{\pi}\left(C_E+\ln\frac{2^{5/2}}{\pi}-2\ln\theta_3\right), \tag{56}$$

$$q_{20}=\frac{8}{\pi}\left(C_E+\ln\frac{2^{5/2}}{\pi}-2\ln\theta_4\right), \tag{57}$$

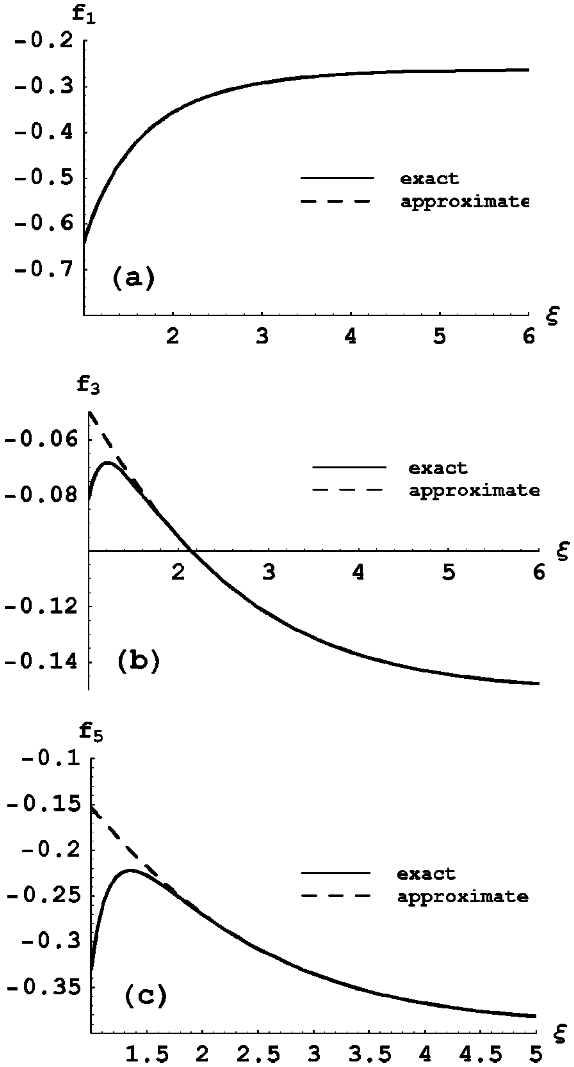

FIG. 1. Finite-size free energy correction terms (a) f_1, (b) f_3, and (c) f_5 as functions of the aspect ratio ξ, which are defined by Eqs. (37), (38), (39), and (43). Solid curves: exact values; dashed curves: two-eigenvalue approximations of Eqs. (49)–(51). The exact and approximate values approach exponentially as ξ increases.

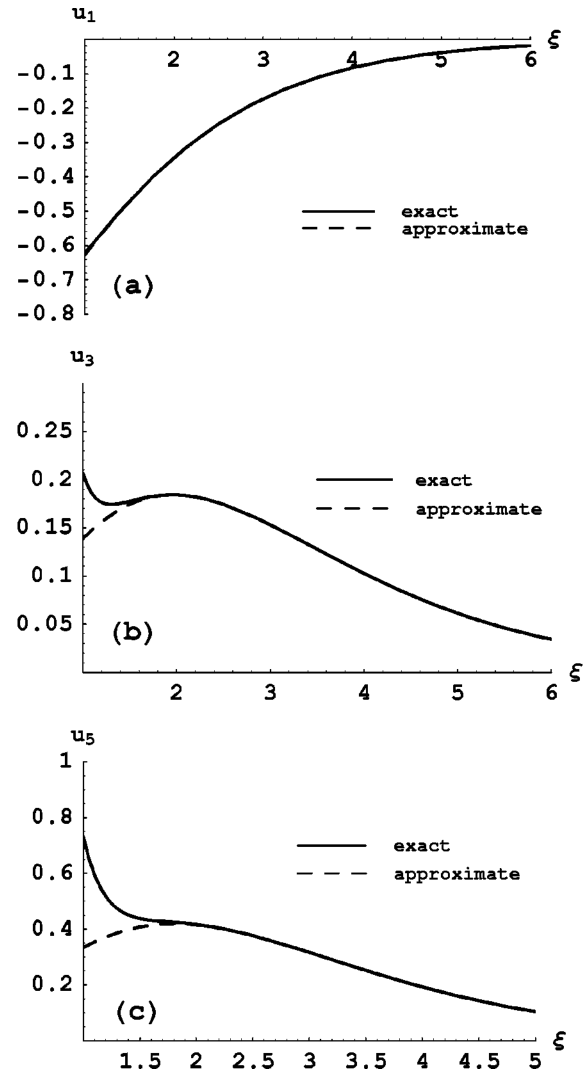

FIG. 2. Finite-size internal energy correction terms (a) u_1, (b) u_3, and (c) u_5 as functions of the aspect ratio ξ, which are defined by Eqs. (40)–(43). Solid curves: exact values; dashed curves: two-eigenvalue approximations of Eqs. (52)–(54). The exact and approximate values approach exponentially as ξ increases.

$$q_{30}=\frac{8}{\pi}\left(C_E+\ln\frac{2^{5/2}}{\pi}-2\ln\theta_2\right), \tag{58}$$

$$q_{11}=-\frac{8K^4\xi}{9\pi^2}\left[1+(1-2k^2)\left(\frac{E'}{K'}-\frac{E}{K}\right)\right], \tag{59}$$

$$q_{21}=-\frac{8K^4\xi}{9\pi^2}\left[1-3k^2+(1+k^2)\left(\frac{E'}{K'}-\frac{E}{K}\right)\right], \tag{60}$$

$$q_{31}=\frac{8K^4\xi}{9\pi^2}\left[2-3k^2+(2-k^2)\left(\frac{E'}{K'}-\frac{E}{K}\right)\right]. \tag{61}$$

It is easy to see from Eqs. (29), (30), (34), and (55) that the asymptotic expansion of the specific heat, can be written as

$$C=\frac{8}{\pi}\ln N+\sum_{i=0}^{\infty} c_i/N^i. \tag{62}$$

Except for the leading term, all other corrections in the asymptotic expansion of the specific heat are proportional to $1/N^i$, without multiplicative logarithms. This result imply immediately that scaling function X_C in Eq. (3) is constant and equal to $8/\pi$.

It is also clear that the contribution to odd (N^{-2i-1}) order in the specific-heat expansion give only first term in right-hand side of the Eq. (30). Thus, we can obtained immediately that the ratio u_{2i+1}/c_{2i+1} of subdominant (N^{-2i-1}) finite-size corrections term in the internal energy and the specific-heat expansions are constant, namely,

$$u_{2i+1}/c_{2i+1}=1/\sqrt{2} \tag{63}$$

as well that $u_{2i}/c_{2i}=0$ for $1\leqslant i<\infty$.

Let us now evaluate the first few terms in the specific-heat expansion. Substituting Eqs. (32)–(34), (55), (56)–(61), (A14), and (A15) in Eq. (30) we have finally obtained the expansion of the specific heat (C) at the critical point ($T=T_c$),

$$C=\frac{8}{\pi}\ln N+\frac{8}{\pi}\left(\ln\frac{2^{5/2}}{\pi}+C_E-\frac{\pi}{4}\right)-4\xi\left(\frac{\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}\right)^2 -\frac{16}{\pi}\frac{\sum_{i=2}^{4}\theta_i\ln\theta_i}{\theta_2+\theta_3+\theta_4}-2\sqrt{2}\frac{\theta_2\theta_3\theta_4}{\theta_2+\theta_3+\theta_4}\frac{1}{N}+\frac{c_2}{N^2} +\frac{1}{N^3}\frac{\pi^3\xi}{24\sqrt{2}}\frac{\theta_2\theta_3\theta_4(\theta_2^9+\theta_3^9+\theta_4^9)}{(\theta_2+\theta_3+\theta_4)^2}+O\left(\frac{1}{N^4}\right)+O\left(\frac{1}{N^5}\right), \tag{64}$$

with

$$c_2=\frac{\pi^3\xi^2}{12}\frac{\theta_2^2\theta_3^2\theta_4^2(\theta_2^9+\theta_3^9+\theta_4^9)}{(\theta_2+\theta_3+\theta_4)^3}+\frac{\pi^2\xi}{9}\frac{\theta_3^4\theta_4^4(2\theta_2-\theta_3-\theta_4)}{\theta_2+\theta_3+\theta_4} -\frac{\pi^2\xi}{6}\frac{\theta_2\theta_3\theta_4}{(\theta_2+\theta_3+\theta_4)^2}\left[(\theta_3^4+\theta_4^4)\theta_2^3\ln\frac{\theta_3}{\theta_4} -(\theta_2^4+\theta_3^4)\theta_4^3\ln\frac{\theta_2}{\theta_3}+(\theta_2^4-\theta_4^4)\theta_3^3\ln\frac{\theta_2}{\theta_4}\right] -\frac{\pi}{9}\frac{\theta_2^5+\theta_3^5+\theta_4(\theta_2^4+\theta_3^4)-2\theta_2\theta_3(\theta_2^3+\theta_3^3)}{\theta_2+\theta_3+\theta_4} \times(1-2\xi\theta_3^2E). \tag{65}$$

Equation (63) imply that the amplitude of the term $O(1/N^5)$ in Eq. (64), i.e., c_5, is $\sqrt{2}u_5$ where u_5 is the amplitude of the N^{-5} correction terms in the internal energy expansion Eq. (40).

In two-eigenvalues approximation the specific heat can be written as

$$C=\frac{8}{\pi}\ln N+c_0^{(app)}+\frac{c_1^{(app)}}{N}+\frac{c_2^{(app)}}{N^2}+\frac{c_3^{(app)}}{N^3}+\cdots \tag{66}$$

with

$$c_0^{(app)}=\frac{8}{\pi}\left(\ln\frac{2^{5/2}}{\pi}+C_E-\frac{\pi}{4}\right)+\xi\,\mathrm{sech}^2(\pi\xi/8) +\frac{8\ln 2}{\pi}[-1+\tanh(\pi\xi/8)], \tag{67}$$

$$c_1^{(app)}=\sqrt{2}[-1+\tanh(\pi\xi/8)], \tag{68}$$

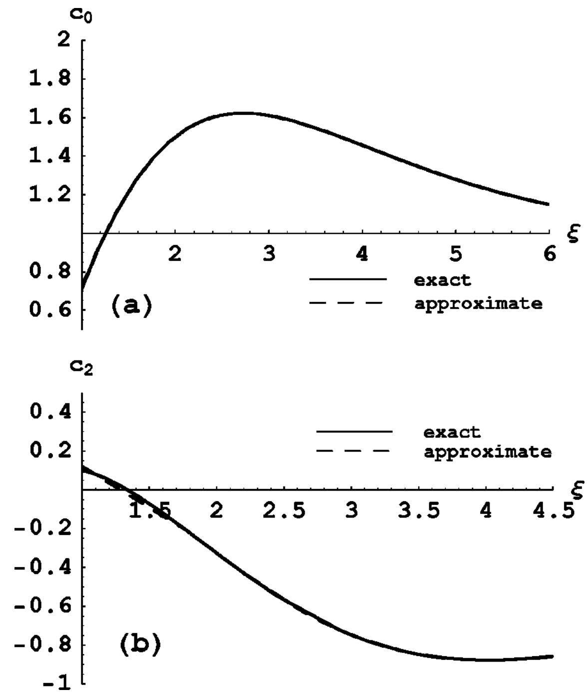

FIG. 3. Finite-size specific-heat correction terms (a) c_0 and (b) c_2 as functions of the aspect ratio ξ, which are defined by Eqs. (64) and (65). Solid curves: exact values; dashed curves: two-eigenvalue approximations of Eqs. (67) and (69). The exact and approximate values approach exponentially as ξ increases.

$$c_2^{(app)}=-\frac{\pi}{9}+\frac{\pi}{6}[-1+\tanh(\pi\xi/8)]+\frac{\pi^2\xi\ln 2}{24}\mathrm{sech}^2(\pi\xi/8) -\frac{\pi^3\xi^2}{96}\mathrm{sech}^2(\pi\xi/8)\tanh(\pi\xi/8), \tag{69}$$

$$c_3^{(app)}=\frac{\pi^3\xi}{96\sqrt{2}}\mathrm{sech}^2(\pi\xi/8). \tag{70}$$

We plot the aspect-ratio (ξ) dependence of the finite-size specific-heat correction terms c_0 and c_2 in Fig. 3. The exact and approximate values approach exponentially as ξ increases. Note, that the ratios of correction terms u_1/c_1 and u_3/c_3 are constant and given by Eq. (63). In Fig. 4 we plot the aspect-ratio dependence of the error introduced by two-eigenvalue approximation for the correction terms in the free energy, internal energy, and specific-heat asymptotic expansions. The deviation of the two-eigenvalues approximation from exact result is about one percentage at $\xi=1$ for the leading correction terms f_1,u_1,c_0, at $\xi=1.65$ for the second correction terms f_3,u_3,c_2, at $\xi=1.85$ for the third correction terms f_5,u_5, and diminishes very rapidly as ξ increases.

It is of interest to compare this finding with other results. Equations (37), (40), and (64) are consistent with Ferdinand and Fisher's similar expansions [3] up to orders $1/N^2$, $1/N$, and $1/N$, respectively. Others terms in our equations, except the term of $O(1/N^3)$ for U [6], are new. For $\xi=1$, we have $u_3=0.206\,683\,145\cdots$ and $u_5=0.730\,182\,312\,347\cdots$ that are

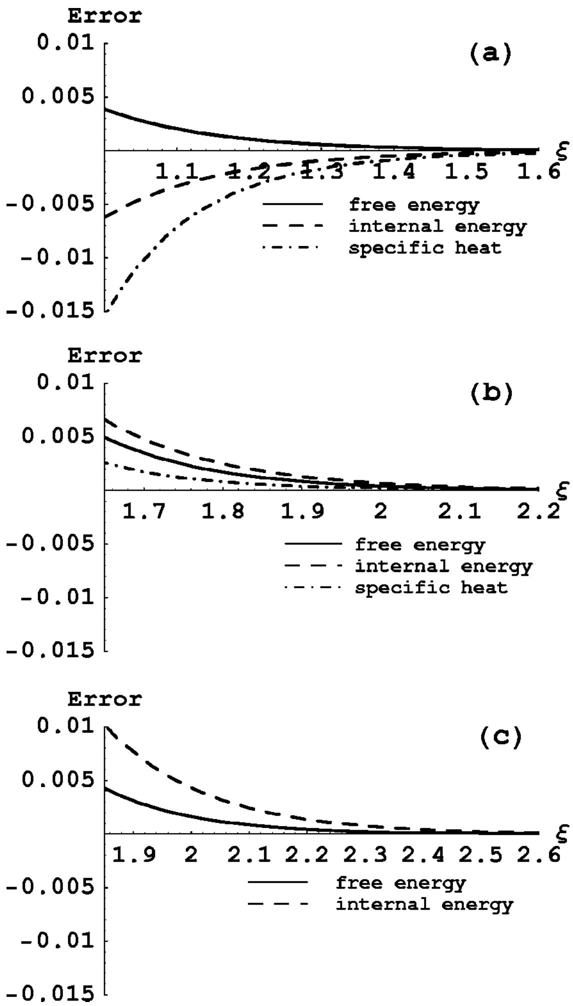

FIG. 4. The error introduced by two-eigenvalue approximation for the correction terms in the free energy, internal energy, and specific-heat asymptotic expansions. (a) The error for the leading correction terms f_1, u_1, and c_0 as functions of ξ. (b) The error for the second correction terms f_3, u_3, and c_2 as functions of ξ. (c) The error for the third correction terms f_5 and u_5 as functions of ξ. The vertical axes represent the error defined by $\text{Error}=(a^{exact}-a^{approx})/a^{exact}$, where a stand for the correction terms in the free energy, internal energy, and specific-heat asymptotic expansions. Solid curves: free energy; dashed curves: internal energy; dot-dashed curves: specific heat. The deviation of the two-eigenvalues approximation from exact result is about one percentage at $\xi=1$ for the leading correction terms, at $\xi=1.65$ for the second correction terms, at $\xi=1.85$ for the third correction terms and diminishes very rapidly as ξ increases.

quite consistent with numerical data $u_3=0.206\,683\,133$ and $u_5=0.730\,182\,312\,35$ obtained by Salas and Sokal [8].

IV. DISCUSSION

The results of this paper inspire several problems for further studies: (i) can one obtain an exact asymptotic expansion for the thermodynamic functions up to arbitrary order, as it can be done for the Ising model on $N\times\infty$ square, honeycomb, and plane triangular lattices [9]. (ii) It is of interest to know whether the amplitude ratio of Eq. (63) can be extended to honeycomb and plane triangular lattices, i.e., whether the ratio is universal. (iii) If, so, how do such amplitudes behave in other models, for example, in the three-state Potts model?

Note added. After the completion of this paper, we learned that similar results have been independently obtained by Salas [26].

ACKNOWLEDGMENTS

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APPENDIX

Let us now evaluate the coefficients p_{i1} and p_{i2} for $i=1,2,3,4$. After little algebra, following the general lines of the Ferdinand and Fisher paper [3], we can obtain the following expression for the coefficients p_{i1} and p_{i2}:

$$p_{11}=-\frac{1}{3}\pi^3\xi\left[\frac{1}{4}\sum_{j=1}^{\infty}(-1)^j\frac{\cosh\pi\xi j}{\sinh^2\pi\xi j}+\frac{3}{2}\sum_{j=1}^{\infty}(-1)^j\frac{\cosh\pi\xi j}{\sinh^4\pi\xi j}\right],$$

$$p_{21}=-\frac{1}{3}\pi^3\xi\left[\frac{1}{4}\sum_{j=1}^{\infty}\frac{\cosh\pi\xi j}{\sinh^2\pi\xi j}+\frac{3}{2}\sum_{j=1}^{\infty}\frac{\cosh\pi\xi j}{\sinh^4\pi\xi j}\right],\tag{A1}$$

$$p_{31}=-\frac{1}{3}\pi^3\xi\left[\sum_{j=1}^{\infty}\frac{(-1)^j}{\sinh^2\pi\xi j}+\frac{3}{2}\sum_{j=1}^{\infty}\frac{(-1)^j}{\sinh^4\pi\xi j}\right],$$

$$p_{41}=-\frac{1}{3}\pi^3\xi\left[\sum_{j=1}^{\infty}\frac{1}{\sinh^2\pi\xi j}+\frac{3}{2}\sum_{j=1}^{\infty}\frac{1}{\sinh^4\pi\xi j}\right],$$

$$p_{12}=\frac{1}{2}p_{11}^2+\frac{4\pi^6\xi^3}{81}\left[\frac{3}{\pi\xi}\Psi_1(\pi\xi)+\frac{d}{d(\pi\xi)}\Psi_1(\pi\xi)\right],$$

$$p_{22}=\frac{1}{2}p_{21}^2+\frac{4\pi^6\xi^3}{81}\left[\frac{3}{\pi\xi}\Psi_2(\pi\xi)+\frac{d}{d(\pi\xi)}\Psi_2(\pi\xi)\right],$$

$$p_{32}=\frac{1}{2}p_{31}^2+\frac{4\pi^6\xi^3}{81}\left[\frac{3}{\pi\xi}\Psi_3(\pi\xi)+\frac{d}{d(\pi\xi)}\Psi_3(\pi\xi)\right],\tag{A2}$$

$$p_{42}=\frac{1}{2}p_{41}^2+\frac{4\pi^6\xi^3}{81}\left[\frac{3}{\pi\xi}\Psi(\pi\xi)+\frac{d}{d(\pi\xi)}\Psi(\pi\xi)\right],$$

with

$$\Psi_1(x)=\frac{1}{128}\sum_{j=1}^{\infty}(-1)^j\cosh xj\left(\frac{1}{\sinh^2 xj}+\frac{60}{\sinh^2 xj}+\frac{120}{\sinh^6 xj}\right),$$

$$\Psi_2(x)=\frac{1}{128}\sum_{j=1}^{\infty}\cosh xj\left(\frac{1}{\sinh^2 xj}+\frac{60}{\sinh^4 xj}+\frac{120}{\sinh^6 xj}\right), \tag{A3}$$

$$\Psi_3(x)=\frac{1}{16}\sum_{j=1}^{\infty}(-1)^j\left(\frac{2}{\sinh^2 xj}+\frac{15}{\sinh^4 xj}+\frac{15}{\sinh^6 xj}\right),$$

$$\Psi(x)=\frac{1}{16}\sum_{j=1}^{\infty}\left(\frac{2}{\sinh^2 xj}+\frac{15}{\sinh^4 xj}+\frac{15}{\sinh^6 xj}\right).$$

Let us now introduce the following notation:

$$S_n(x)=\sum_{j=1}^{\infty}\frac{1}{\sinh^n xj}\quad \text{for}\quad n=2,4,6. \tag{A4}$$

Then the coefficients p_{i1} and p_{i2} can be rewritten in the more symmetrical way,

$$p_{11}=\frac{1}{3}\pi^3\xi\left[\frac{1}{8}R(\xi/2)-\frac{5}{4}R(\xi)+2R(2\xi)\right],$$

$$p_{21}=\frac{1}{3}\pi^3\xi\left[R(\xi)-\frac{1}{8}R(\xi/2)\right], \tag{A5}$$

$$p_{31}=\frac{1}{3}\pi^3\xi[R(\xi)-2R(2\xi)],$$

$$p_{41}=-\frac{1}{3}\pi^3\xi R(\xi),$$

and

$$\Psi_1(\xi)=-2\Psi(2\xi)+\frac{17}{16}\Psi(\xi)-\frac{1}{32}\Psi(\xi/2),$$

$$\Psi_2(\xi)=-\Psi(\xi)+\frac{1}{32}\Psi(\xi/2), \tag{A6}$$

$$\Psi_3(\xi)=2\Psi(2\xi)-\Psi(\xi)$$

with

$$R(x)=S_2(x)+\frac{3}{2}S_4(x), \tag{A7}$$

$$\Psi(x)=\frac{1}{16}[2S_2(x)+15S_4(x)+15S_6(x)]. \tag{A8}$$

Thus we have shown that the coefficients p_{i1} and p_{i2} can be expressed in terms of the only object, namely $S_n(x)$ for $n=2,4,6$. The $S_2(x)$ is given by (see [25], p. 721)

$$S_2(x)=\frac{1}{6}+\frac{2(2-k^2)}{3\pi^2}K^2(k)-\frac{2}{\pi^2}K(k)E(k), \tag{A9}$$

where $x=\pi K'(k)/K(k)$ with $K(k)$ and $E(k)$ the elliptic integrals of the first and second kind, respectively. The $S_4(x)$ and $S_6(x)$ are calculated to be

$$S_4(x)=-\frac{11}{90}-\frac{4(2-k^2)}{9\pi^2}K^2(k)+\frac{4}{3\pi^2}K(k)E(k)+\frac{8(1-k^2+k^4)}{45\pi^4}K^4(k), \tag{A10}$$

$$S_6(x)=\frac{191}{1890}+\frac{32(2-k^2)}{45\pi^2}K^2(k)-\frac{16}{15\pi^2}K(k)E(k)-\frac{8(1-k^2+k^4)}{45\pi^4}K^4(k)-\frac{32(2-3k^2-3k^4+2k^6)}{945\pi^6}K^6(k). \tag{A11}$$

Thus we are now in position to evaluate $R(x)$ and $\Psi(x)$ given by Eqs. (A7) and (A8), respectively. The result is

$$R(x)=-\frac{1}{60}+\frac{4(1-k^2+k^4)}{15\pi^4}K^4(k) \tag{A12}$$

$$\Psi(x)=\frac{1}{1008}-\frac{2(2-3k^2-3k^4+2k^6)}{63\pi^6}K^6(k). \tag{A13}$$

The expressions for $R(2x),R(x/2)$ and $\Psi(2x),\Psi(x/2)$ can be written as function of the modulus k by using properties of the elliptic functions.

Thus for the coefficients p_{i1} (for $i=1,2,3,4$) we have finally obtained

$$p_{11}=-\frac{7\pi^3\xi}{1440}+\frac{(7+8k^2-8k^4)\xi}{90\pi}K^4(k),$$

$$p_{21}=-\frac{7\pi^3\xi}{1440}+\frac{(7-22k^2+7k^4)\xi}{90\pi}K^4(k), \tag{A14}$$

$$p_{31}=\frac{\pi^3\xi}{180}+\frac{(-8+8k^2+7k^4)\xi}{90\pi}K^4(k),$$

$$p_{41}=\frac{\pi^3\xi}{180}-\frac{4(1-k^2+k^4)\xi}{45\pi}K^4(k).$$

After little algebra the expressions for p_{i2} (for $i=1,2,3,4$) can be written as

$$p_{12}=\frac{\xi^2K^8}{189\pi^2}\left[31-88k^2+88k^4+(31-78k^2+48k^4-32k^6)\times\left(\frac{E'}{K'}-\frac{E}{K}\right)\right]-\frac{31\pi^5\xi}{24192}+\frac{p_{11}^2}{2},$$

$$p_{22}=\frac{\xi^2K^8}{189\pi^2}\left[31-67k^2+25k^4-21k^6+(31-15k^2-15k^4+31k^6)\left(\frac{E'}{K'}-\frac{E}{K}\right)\right]-\frac{31\pi^5\xi}{24192}+\frac{p_{21}^2}{2}, \tag{A15}$$

$$p_{32}=-\frac{\xi^2K^8}{189\pi^2}\left[32-80k^2+38k^4-21k^6+(32-48k^2+78k^4-31k^6)\left(\frac{E'}{K'}-\frac{E}{K}\right)\right]+\frac{\pi^5\xi}{756}+\frac{p_{31}^2}{2},$$

$$p_{42}=-\frac{16\xi^2K^8}{189\pi^2}\left[2-5k^2+5k^4+(2-3k^2-3k^4+2k^6)\times\left(\frac{E'}{K'}-\frac{E}{K}\right)\right]+\frac{\pi^5\xi}{756}+\frac{p_{41}^2}{2},$$

where for simplicity we denote $K\equiv K(k)$, $K'\equiv K'(k)$, $E\equiv E(k)$, and $E'\equiv E'(k)$.

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