RAPID COMMUNICATIONS

PHYSICAL REVIEW E, VOLUME 65, 035602(R)

Statistical theory for incoherent light propagation in nonlinear media

B. Hall,* M. Lisak, and D. Anderson
Department of Electromagnetics, Chalmers University of Technology, SE-412 96 Göteborg, Sweden

R. Fedele
Dipartimento di Scienze Fisiche, Università Federico II di Napoli and INFN Sezione di Napoli, Complesso Universitario di MS Angelo, Via Cintia, I-80 126 Napoli, Italy

V. E. Semenov
Institute of Applied Physics, Russian Academy of Sciences, 46 Ulyanov Street, Nizhny Novgorod 603 600, Russia
(Received 26 April 2001; published 27 February 2002)

A statistical approach based on the Wigner transform is proposed for the description of partially incoherent optical wave dynamics in nonlinear media. An evolution equation for the Wigner transform is derived from a nonlinear Schrödinger equation with arbitrary nonlinearity. It is shown that random phase fluctuations of an incoherent plane wave lead to a Landau-like damping effect, which can stabilize the modulational instability. In the limit of the geometrical optics approximation, incoherent, localized, and stationary wave fields are shown to exist for a wide class of nonlinear media.

DOI: 10.1103/PhysRevE.65.035602 PACS number(s): 42.65.Tg, 05.45.Yv, 42.65.Sf

It is well known that the propagation of electromagnetic waves and beams in dispersive nonlinear media is associated with the phenomena of modulational instability and the formation of stationary wave structures, i.e., temporal and spatial envelope solitons, cf. [1]. The conventional treatment of these phenomena considers coherent wave structures. However, a recent series of experimental [2] and theoretical papers [3–6] have demonstrated the existence of incoherent solitons. In particular, spatially incoherent optical solitons have attracted much attention. Until now, three approaches have been developed to describe partially incoherent wave propagation in nonlinear media: (i) the propagation equation for the mutual coherence function [3], (ii) the coherent density method [4], and (iii) the self-consistent multimode theory [5]. In fact, it has been shown in Ref. [6] that these three approaches are formally equivalent.

In the present article, we propose a general, statistical theory for describing the dynamics of partially incoherent optical waves and beams in dispersive and nonlinear media. The approach is based on the Wigner transform method [7], which was introduced in statistical quantum mechanics to describe the dynamics of the quantum state of a system in the classical space language. A similar approach has been successfully applied in the theory of surface gravity waves [8] in plasma physics in connection with the theory of weak plasma turbulence [9], as well as with the description of electromagnetic wave propagation in a nonstationary, inhomogenous, and relativistic plasma [10]. Recently, the Wigner function method has been used to analyze the longitudinal dynamics of charged-particle beams in accelerators [11], and to study the dynamics of Bose-Einstein condensates in the presence of a chaotic external potential [12].

The scope and outline of this article is as follows. Starting from the nonlinear Schrödinger equation describing the evolution of the slowly varying wave amplitude in a dispersive medium with an arbitrary nonlinear response, we derive the Wigner-Moyal equation for the Wigner transform including the Klimontovich statistical average. This equation reduces, in the geometrical optics approximation, to the classical Liouville or Vlasov-like equation describing the conservation of optical quasiparticles in phase space, cf. Ref. [13]. To illustrate the usefulness of this approach, we consider an application to the case of one-dimensional (1D) propagation of partially incoherent light in a nonlinear Kerr medium, and investigate the stability of a constant amplitude plane wave against small harmonic perturbations. It is found that, in addition to the classical modulational instability (MI), a new linear Landau-like damping effect arises. This damping is due to the broadening of the Wigner spectrum, which is associated with the random phase fluctuations. Consequently, it is found that the partially incoherent character of the light may suppress the modulational instability, in agreement with the result of Refs. [14,15]. Finally, we illustrate an analog to the Bernstein-Greene-Kruskal (BGK) waves in plasma physics. In the limit of the geometrical optics approximation, we derive the Wigner distribution functions of a new class of stationary, self-trapped, and incoherent wave pulse structures, which may exist in a wide class of nonlinear media.

The approach proposed here sheds new light on the physics behind the recently reported results in this field. It also allows the formulation of conceptually new problems regarding the dynamics of partially incoherent waves and beams in nonlinear media. In addition, using the Wigner transform for studying this kind of phenomena establishes new connections to quantum mechanics, plasma physics, mesoscopic physics, image and signal processing, and mathematics. Furthermore, the Wigner transform, apart from serving as a very convenient mathematical tool, is widely used in its own right in the field of optics, as a supplement to the envelope wave function [16].

As our starting point, we assume that the 3D wave propagation in a dispersive (or diffractive) nonlinear medium is described by a system of coupled model equations for the

*Electronic address: bjorn.hall@elmagn.chalmers.se

slowly varying complex wave amplitude $\psi(t,\mathbf{r})$ and the nonlinear response function of the medium $n(t,\mathbf{r})$,

$$i\left(\frac{\partial}{\partial t}+\mathbf{v}_g\cdot\boldsymbol{\nabla}\psi\right)\psi+\frac{\beta}{2}\nabla^2\psi+n\psi=0,$$
$$\tau_m\frac{\partial n}{\partial t}+n=\kappa G(\langle\psi^*\psi\rangle). \quad (1)$$

For convenience, we use t and $\mathbf{r}$ as the evolution and spatial dispersive variables, respectively; $\mathbf{v}_g$ is the group velocity, β is the diffraction or second order dispersion coefficient, κ is the nonlinear coefficient, and the function $G(\langle\psi^*\psi\rangle)$ characterizes the nonlinear properties of the medium. The bracket $\langle\cdots\rangle$ denotes the statistical ensemble average. The relaxation time of the medium response function, τ_m, is assumed to be much longer than the characteristic time of the statistical wave intensity fluctuations, τ_s. Assuming also that $\tau_p \gg \tau_m$, where τ_p is the characteristic time scale of the (deterministic) wave amplitude variation, we can approximate the medium response function as $n\approx\kappa G(\langle\psi^*\psi\rangle)$. The system (1) then reduces to a generalized nonlinear Schrödinger equation,

$$i\frac{\partial\psi}{\partial t}+\frac{\beta}{2}\nabla^2\psi+\kappa G(\langle\psi^*\psi\rangle)\psi=0, \quad (2)$$

where the coordinate system has been transformed to the reference system moving with the group velocity $\mathbf{v}_g$, and $\mathbf{r}$ is the reduced distance according to $\mathbf{r}\rightarrow\mathbf{r}-\mathbf{v}_g t$.

We stress that the independent variables in Eq. (2) should be adopted to suit the particular situation. For instance, to describe the 2D spatial optical solitons one would use z as the evolution variable and restrict the nabla operator to the transverse dimensions, while an equation similar to Eq. (2) with z as evolution variable and $\partial^2/\partial t^2$ instead of the nabla operator would describe temporal 1D solitons.

Replacing $i\partial/\partial t$ by ω and $-i\boldsymbol{\nabla}$ by $\mathbf{p}$ yields the corresponding nonlinear dispersion relation,

$$\omega=\frac{\beta}{2}p^2-\kappa G(\langle\psi^*\psi\rangle). \quad (3)$$

We will now apply the Wigner transform method [7]. The s-dimensional Wigner transform, including the Klimontovich statistical average, is defined as

$$\rho(\mathbf{p},t,\mathbf{r})=\frac{1}{(2\pi)^s}\int_{-\infty}^{+\infty}d^s\boldsymbol{\xi}\,e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\langle\psi^*(\mathbf{r}+\boldsymbol{\xi}/2,t)\psi(\mathbf{r}-\boldsymbol{\xi}/2,t)\rangle, \quad (4)$$

which satisfies $\langle\psi^*(\mathbf{r},t)\psi(\mathbf{r},t)\rangle=\int_{-\infty}^{+\infty}d^s\mathbf{p}\,\rho(\mathbf{p},t,\mathbf{r})$. Applying this transform, with dimensionality $s=3$, to Eq. (2), we obtain the following Wigner-Moyal equation for the evolution of the Wigner distribution function,

$$\frac{\partial\rho}{\partial t}+\beta\mathbf{p}\cdot\frac{\partial\rho}{\partial\mathbf{r}}+2\kappa G(\langle|\psi|^2\rangle)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\mathbf{r}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{p}}\right)\rho=0. \quad (5)$$

The arrows in the sine differential operator indicate that the derivatives act to the left and right, respectively. The sine operator is defined by its Taylor expansion,

$$\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\mathbf{r}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{p}}\right)=\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)!2^{2l}}\frac{\overleftarrow{\partial}^{2l+1}}{\partial\mathbf{r}^{2l+1}}\cdot\frac{\overrightarrow{\partial}^{2l+1}}{\partial\mathbf{p}^{2l+1}}. \quad (6)$$

For those situations where the geometrical optics approximation is valid, we can neglect the higher order derivatives in Eq. (6). This approximation is valid for $\Delta\mathbf{p}\cdot\Delta\mathbf{r}\gg 2\pi$, where $\Delta\mathbf{p}$ is the local width of the Wigner spectrum and $\Delta\mathbf{r}$ is the width of the medium response function $n(t,\mathbf{r})$. We emphasize that this is a rather strong limitation, which can only be valid in the long-wavelength limit. Retaining only the first term in the expansion (6), we obtain a Vlasov-like equation,

$$\frac{\partial\rho}{\partial t}+\beta\mathbf{p}\cdot\frac{\partial\rho}{\partial\mathbf{r}}+\kappa\frac{\partial G(\langle|\psi|^2\rangle)}{\partial\mathbf{r}}\cdot\frac{\partial\rho}{\partial\mathbf{p}}=0. \quad (7)$$

Equation (7) implies the conservation of the number of optical quasiparticles in phasespace. Using the Liouville theorem we recover the canonical Hamilton equations of motion for the quasiparticle with mass β, or, equivalently, the ray equations of the geometrical optics approximation, where $\mathbf{r}$ and $\mathbf{p}$ are the canonical variables and ω is the Hamilton equation defined by Eq. (3),

$$\dot{\mathbf{r}}=\frac{\partial\omega}{\partial\mathbf{p}}=\beta\mathbf{p},$$
$$\dot{\mathbf{p}}=-\frac{\partial\omega}{\partial\mathbf{r}}=\kappa\frac{\partial G(\langle\psi^*\psi\rangle)}{\partial\mathbf{r}}. \quad (8)$$

Note that Eq. (7) is similar to the radiation transfer equation used in [3,17,18].

As an example of the applicability of the Wigner transform approach, we consider the MI of a 1D plane wave with a constant amplitude in a nonlinear Kerr-like medium, for which $G(\langle|\psi|^2\rangle)=\langle|\psi|^2\rangle$. For the case of *coherent* light propagating in a 1D Kerr medium, it is well known from conventional stability analysis that a perturbation of the monochromatic stationary solution $\psi(x,t)=\psi_0\exp(i\kappa\psi_0^2 t)$ experiences a modulational instability when $\beta\kappa>0$ and $K^2<4\kappa\psi_0^2/\beta$, where K is the wave number of the perturbation. The instability growth rate is given by

$$\Omega=i\frac{\beta\kappa}{2}\left(\frac{4\kappa\psi_0^2}{\beta K^2}-1\right)^{1/2}. \quad (9)$$

To investigate the effect of the incoherence, we assume a Wigner distribution function of the form $\rho(p,t,x)=\rho_0(p)+\rho_1\exp[i(Kx-\Omega t)]$, with $\rho_0\gg|\rho_1|$. Here, $\rho_0(p)$ is the background distribution function corresponding to the plane wave with a complex amplitude $\psi=\psi_0\exp[i\kappa\psi_0^2 t+i\phi(x)]$, implying $\int_{-\infty}^{+\infty}\rho_0(p)\,dp=\psi_0^2$. The incoherence is modeled by the randomly varying phase term $\phi(x)$. Studying the linear evolution of the perturbation ρ_1, we obtain from the linearized Wigner-Moyal equation (5) the following dispersion relation:

$$1+\frac{\kappa}{\beta}\int_{-\infty}^{+\infty}\frac{\rho_0(p+K/2)-\rho_0(p-K/2)}{K(p-\Omega/\beta K)}dp=0. \qquad (10)$$

The corresponding dispersion relation following from the linearized Vlasov-like equation (7) has the form

$$1+\frac{\kappa}{\beta}\int_{-\infty}^{+\infty}\frac{d\rho_0/dp}{(p-\Omega/\beta K)}dp=0, \qquad (11)$$

which can also be directly obtained from Eq. (10) by taking the limit of small K. The relation (11) is similar to the dispersion relation for electron plasma waves, which is well known to contain the effect of the Landau damping. In general, the kinetic integrals in Eqs. (10) and (11) can be represented as the sum of a principal value and a residue contribution, where the latter leads to a Landau-like damping of the perturbation. This stabilizing effect is not an ordinary, dissipative damping. It is rather an energy-conserving self-action effect within a partially incoherent wave field, which causes a redistribution of the Wigner spectrum because of the interaction between different parts of the spectrum. This spectral redistribution counteracts the MI. Similar phenomena occur in connection with nonlinear propagation of electron plasma waves interacting with intense electromagnetic radiation [9,17], nonlinear interaction between random phase photons and sound waves in electron-positron plasmas [18], and the longitudinal dynamics of charged-particle beams in accelerators [11].

It is interesting to note that for a coherent wave, i.e., a delta-shaped background distribution $\rho_0(p)=\psi_0^2\delta(p)$, and for $\beta\kappa>0$, Eq. (10) gives exactly the modulational instability growth rate defined by Eq. (9). For the same case, the dispersion relation (11) yields $\Omega=i(\beta\kappa)^{1/2}\psi_0 K$, which is identical to the growth rate of the MI as given by the expression (9) in the limit of long wavelengths, i.e., when $K^2\ll 4\kappa\psi_0^2/\beta$.

To illustrate the incoherent case, we assume that $\phi(x)$ is described by the following autocorrelation function:

$$\langle\exp[-i\phi(x+y/2)+i\phi(x-y/2)]\rangle=\exp(-p_0|y|), \qquad (12)$$

where p_0^{-1} is the correlation length. The corresponding Wigner function has a Lorentzian shape,

$$\rho_0(p)=\frac{\psi_0^2}{\pi}\frac{p_0}{p^2+p_0^2}, \qquad (13)$$

and the dispersion relation (10) yields the following exact result for $4\kappa\psi_0^2/\beta K^2>1$:

$$\frac{\Omega}{\beta K}=i\frac{K}{2}\left(\frac{4\kappa\psi_0^2}{\beta K^2}-1\right)^{1/2}-ip_0, \qquad (14)$$

which is similar to the result obtained in Ref. [14].

Equation (14) clearly shows the stabilizing effect of the Landau-like damping due to the finite width p_0 of the Lorentzian spectrum (or the finite correlation length of the wave phase). In fact, if the width of the Lorentzian spectrum, p_0, satisfies the relation $p_0>p_c\equiv K_c/2$, where $K_c\equiv(4\kappa\psi_0^2/\beta)^{1/2}$ is the cutoff wavelength of the MI, the instability is completely suppressed for all wave numbers K of the perturbation. In other words, if the Landau-like damping, induced by the broadening of the Wigner spectrum due to the partial incoherence of the wave, is strong enough, it can overcome the coherent growth associated with the MI. This result has also been verified experimentally; see, e.g., Ref. [15].

Finally, we will demonstrate the existence of a class of self-trapped and partially incoherent wave pulse solutions to the stationary Vlasov-like equation

$$\beta p\frac{\partial\rho}{\partial x}+\kappa\frac{\partial G(\langle|\psi|^2\rangle)}{\partial x}\frac{\partial\rho}{\partial p}=0, \qquad (15)$$

for an arbitrary nonlinear function $G(\langle|\psi|^2\rangle)$. According to the Jeans theorem, cf. [13], the solution of Eq. (15) can be expressed as an arbitrary function of the Hamiltonian $H=\beta p^2/2-\kappa G(\langle|\psi|^2\rangle)$. Thus, we have $\rho=\rho_s(H)$, where ρ_s is an arbitrary function. The quasiparticles are trapped in the nonlinear potential for $-\gamma\leqslant p\leqslant\gamma$, where $\gamma=(2\kappa G(\langle|\psi|^2\rangle)/\beta)^{1/2}$ and $\kappa\beta>0$, and consequently the condition $\int_{-\gamma}^{+\gamma}dp\,\rho_0(p)=\langle|\psi|^2t\rangle$ leads to an integral equation having the following solution for ρ_s (cf. [19]):

$$\rho_s(H)=\frac{1}{\pi}\left(\frac{\beta}{2}\right)^{1/2}\int_0^{-H}\frac{dH'}{(-H-H')^{1/2}}\frac{dF(H')}{dH'}, \qquad (16)$$

where $F(\Theta)\equiv G^{-1}(\Theta)$ with $\Theta\equiv G(\langle|\psi|^2\rangle)=(\beta p^2/2-H)/\kappa$. For instance, for the Kerr nonlinearity we have $F(\Theta)\equiv\Theta$ and Eq. (16) yields $\rho_s(H)=(2/\pi\kappa)\sqrt{-\beta H/2}$. Equation (16) describes the Wigner distributions of quasiparticles trapped in a collectively produced, stationary, partially incoherent, and localized wave structure. These structures are similar to the large-amplitude BGK waves in a plasma; see [19].

In conclusion, we have proposed a general, statistical approach for the theoretical analysis of the propagation of partially incoherent optical waves and beams in dispersive media with arbitrary nonlinearities. The approach is based on the Wigner transform method including the Klimontovich statistical average. The derived Wigner-Moyal equation determining the evolution of the Wigner distribution function represents a generalization of the Vlasov-like equation, which is only valid within the geometrical optics approximation. The Wigner-Moyal equation clearly shows that the number of optical quasiparticles is *not* conserved in phase space beyond the validity of the geometrical optics approximation. Using the Wigner-Moyal equation, we have carried out a linear stability analysis for small perturbations on a constant, 1D, and partially incoherent background in a nonlinear Kerr medium. The theory reproduces the exact expression for the MI growth rate of a coherent wave, but also includes a linear Landau-like damping effect associated with the broadening of the Wigner spectrum due to partial wave incoherence. This damping effect explains the previously reported incoherent suppression of the modulational instability.

The authors are grateful to D. N. Christodoulides, M. Segev, and A. Hasegawa for valuable discussions.

[1] V.I. Karpman and E.M. Krushkal, Sov. Phys. JETP **28**, 277 (1969); T. Tanuti and H. Washimi, Phys. Rev. Lett. **22**, 454 (1969).

[2] M. Mitchell *et al.*, Phys. Rev. Lett. **77**, 490 (1996); M. Mitchell and M. Segev, Nature (London) **387**, 880 (1997); Z. Chen *et al.*, Science **280**, 889 (1998).

[3] G.A. Pasmanik, Zh. Éksp. Teor. Phys. **66**, 490 (1974) [Sov. Phys. JETP **39**, 234 (1974)]; V.V. Shkunov and D.Z. Anderson, Phys. Rev. Lett. **81**, 2683 (1998).

[4] D.N. Christodoulides *et al.*, Phys. Rev. Lett. **78**, 646 (1997); D.N. Christodoulides *et al.*, Opt. Lett. **22**, 1080 (1997); T.H. Coskun *et al.*, *ibid.* **23**, 418 (1998).

[5] A. Hasegawa, Phys. Fluids **18**, 77 (1975); M. Mitchell *et al.*, Phys. Rev. Lett. **79**, 4990 (1997); D.N. Christodoulides *et al.*, *ibid.* **80**, 2310 (1998); M.I. Carvalho *et al.*, Phys. Rev. E **59**, 1193 (1999); T.H. Coskun *et al.*, Phys. Rev. Lett. **84**, 2374 (2000); N. Akhmediev, W. Królikowski, and A.W. Snyder, *ibid.* **81**, 4632 (1998); W. Królikowski, N. Akhmediev, and B. Luther-Davies, Phys. Rev. E **59**, 4654 (1999); A.A. Sukhorukov and N.N. Akhmediev, Phys. Rev. Lett. **83**, 4736 (1999).

[6] D.N. Christodoulides, E.D. Eugenieva, T.H. Coskun, M. Segev, and M. Mitchell, Phys. Rev. E **63**, 035601 (2001).

[7] E. Wigner, Phys. Rev. **40**, 749 (1932); J.E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949); see also the textbooks in statistical quantum mechanics.

[8] I. E. Alber, Proc. R. Soc. London, Ser. A **636**, 525 (1978); D. R. Crawford, P. G. Saffman, and H. C. Yuen, Wave Motion **2**, 1 (1980); P. A. E. M. Janssen, Fluid Mech. **133**, 113 (1983).

[9] A.A. Venedov, *Theory of Turbulent Plasma*, Moskva, 1965 (Israel Program for Scientific Translations, Jerusalem, 1966).

[10] N.L. Tsintsadze and J.T. Mendonça, Phys. Plasmas **5**, 3609 (1998); Phys. Rev. E **62**, 4276 (2000).

[11] D. Anderson *et al.*, Phys. Lett. A **258**, 244 (1999); R. Fedele, D. Anderson, and M. Lisak, Phys. Scr. **T84**, 27 (2000); R. Fedele and D. Anderson, J. Opt. B: Quantum Semiclassical Opt. **2**, 207 (2000).

[12] S.A. Gardiner *et al.*, Phys. Rev. A **62**, 023612 (2000).

[13] A.I. Akhiezer, I.A. Akhiezer, R.V. Polovin, A.G. Sitenko, and K.N. Stepanov, *Plasma Electrodynamics* (Pergamon Press, New York, 1975).

[14] M. Soljacic *et al.*, Phys. Rev. Lett. **84**, 467 (2000).

[15] C. Anastassiou *et al.*, Phys. Rev. Lett. **85**, 4888 (2000).

[16] For a long list of references, see the feature issue: J. Opt. Soc. Am. A **17**, 2274 (2000), *Wigner Distributions and Phase Space in Optics*, edited by G.W. Forbes, V.I. Man'ko, H.M. Ozaktas, R. Simon, and K.B. Wolf.

[17] R. Bingham, J.T. Mendonça, and J.M. Dawson, Phys. Rev. Lett. **78**, 247 (1997).

[18] P.K. Shukla and L. Stenflo, Phys. Plasmas **5**, 1554 (1998).

[19] I.B. Bernstein, J.M. Green, and M.D. Kruskal, Phys. Rev. **108**, 546 (1957).