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Multiparameter generalization of nonextensive statistical mechanics

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We show that the stochastic interpretation of Tsallis's thermostatistics given recently by Beck [Phys. Rev. Lett **87**, 180601 (2001)] leads naturally to a multiparameter generalization. The resulting class of distributions is able to fit experimental results, which cannot be reproduced within Boltzmann's or Tsallis's formalism.

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Nonextensive statistical mechanics (NESM) introduced by Tsallis [1] has gained a considerable interest in several fields of physics because of its capability to describe a wealth of disparate phenomena (from anomalous diffusion, to turbulent systems, to astrophysical systems, etc.) within a single formalism, generalization of the standard statistical–mechanical one with the addition of the single free parameter (entropic index) q. Recently it has been shown how to relate q with the internal microscopic properties of the system under consideration. This has been done by Wilk and Wlodarczyk [2]: they have shown that, when $q \geqslant 1$, the NESM canonical distribution $\rho_q(H,\beta_0)$ for the system with Hamiltonian H can be written as an average of the usual Boltzmann-Gibbs factor over the inverse temperature β,

$$\rho_q(H,\beta_0) = \int_0^\infty d\beta \exp(-\beta H) f_q(\beta,\beta_0), \tag{1}$$

where $f_q(\beta,\beta_0)$ is a weight function whose meaning is that of a probability distribution function for β which is, therefore, no longer a fixed parameter; instead, the macroscopically visible value is just its average value β_0. Fluctuations in β are related to coherent fluctuations existing in small parts of the system with respect to the whole system, due to the existence of long range correlations.

Recently Beck [3] has been able to give an interpretation of the fluctuating β as a function of stochastically varying microscopic variables. In order to recover Tsallis's results, Beck was forced to impose some constraints over β or, equivalently, the microscopic dynamics of the system. In this paper we show that, following Beck's approach but relaxing these constraints, we are able to derive an entire new class of distributions, which reduce to Tsallis's distribution under suitable limits. We will show that some members of this class are able to reproduce experimental results that would be outside the reach of Tsallis's formalism.

To start with, we quote the same example used in Beck's paper: let us set $H = u^2/2$ and suppose that the generalized velocity u satisfies the Langevin equation

$$\dot{u} = -\gamma u + \sigma L(t) \tag{2}$$

with $L(t)$ Gaussian white noise of unit amplitude, σ strength of the noise, and γ friction coefficient. This is the Brownian particle problem [4]. For this case, it can be shown that the temperature $1/\beta$ is related to the microscopic parameters γ,σ by

$$\beta = \gamma/\sigma^2. \tag{3}$$

Beck shows that Tsallis's distribution can be recovered if β is characterized by a χ^2 distribution with n degrees of freedom [5]

$$\hat{f}_n(\beta,\beta_0) = \frac{\left(\frac{n}{2}\right)^{n/2}}{\beta_0 \Gamma\left(\frac{n}{2}\right)} \left(\frac{\beta}{\beta_0}\right)^{n/2-1} \exp\left(-\frac{n\beta}{2\beta_0}\right), \tag{4}$$

where $f_q(\beta,\beta_0) = \hat{f}_n(\beta,\beta_0)$ provided that $q = 1 + 2/(n+1)$. Such a distribution arises if β can be written as a sum of normal stochastic variables,

$$\beta = \sum_{i=1}^{n} X_i^2, \tag{5}$$

with $\langle X_i \rangle = 0$ and $\langle X_i^2 \rangle = \beta_0/n$, so that $\langle \beta \rangle = \beta_0$ and $\langle \beta^2 \rangle - \langle \beta^2 \rangle = \beta_0^2(2/n)$. The χ^2 distribution is a common distribution, occurring in many physical problems, and is central in the problem of estimating parameters from data [6].

Some points are worth stressing at this stage:

(i) The macroscopic parameter β is written in terms of other parameters more directly related to the microscopical dynamics of the system at hand, just as in Eq. (3). We just mention another example: in the study of fully developed turbulence, where u is a local velocity difference, $\beta = (\varepsilon \tau)^{-1}$, with ε spatially averaged energy dissipation rate and τ typical time for the energy transfer.

(ii) It is obvious that, if β is a stochastic variable, *a fortiori* the microscopic quantities $\gamma, \sigma, \ldots,$ must also be stochastic variables, therefore, characterized by their own probability distribution functions (PDFs).

(iii) Relations of the kind (5) impose severe constraints upon the PDFs of the microscopic variables. For example, to recover Eq. (4) starting from Eq. (3) there is the trivial

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choice: $\gamma\ \chi^2$ distributed and σ^2 a constant; it is difficult (and perhaps impossible) to devise other distributions which lead to Eq. (4).

The main idea of this paper is that if β is a function of some more fundamental stochastic control variables, then the far more logical path is the following: to guess statistical distributions for the microscopic quantities and, from them, to work out the corresponding distribution for β. Since β may have infinite functional dependences from microscopic variables, we can expect the PDF of β to have a large range of analytical forms, depending on a large number of parameters [we expect as many of them as the number of microscopic variables that control $\beta=\beta(Y,Z,\ldots)$].

Some simple rules, however, still allow to drastically reduce the class of likely distributions. First, although the PDF for each of the variables $Y,Z,\ldots,$ may be arbitrary, the same reasoning of Eqs. (4) and (5) still holds, that is, the χ^2 distribution for each variable is a very convenient choice. For example, the χ^2 distribution can tranform into a delta distribution, thus, allowing for well deterministic, nonstochastic quantities in the limit $n\to\infty$. Hence, we will suppose all the stochastic variables to be χ^2 distributed, possibly with different degrees of freedom. In second place, a simplicity principle suggests that the most frequently occurring cases should be those where β is some simple combination of a small number of variables. Some examples are given in the above expressions [e.g., Eq. (3)]. The simplest function of all is the sum of stochastic variables $\beta=Y+Z+\cdots$. However, with the previous choice for the PDFs of $Y,Z,\ldots,$ it is possible to show that it is a trivial case, since it reduces to a χ^2 distribution [5]. The next nontrivial cases, thus, are those involving products and ratios of one or two control variables: $Y,Z,\ Y/Z,\ 1/(YZ),\ldots$.

Our aim now is to compute a few examples of PDFs of β and to compare the results with the Tsallis's formalism. We will do the computation for the case of β ratio of two stochastic variables: $\beta=Y/Z$. This is particularly convenient since (i) it generalizes the example given by Beck [Eq. (3)]; (ii) it is a particular case of $\beta=1/(\varepsilon\tau)$, when Y and either ε or τ are constants.

The probability distribution function for the two χ^2 independent variables Y, Z of degree n, m respectively, is given by

$$\hat{f}_n(Y,Y_0)\hat{f}_m(Z,Z_0)=\frac{\left(\frac{n}{2Y_0}\right)^{n/2}\left(\frac{m}{2Z_0}\right)^{m/2}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}Y^{n/2-1}Z^{m/2-1}\times\exp\left(-\frac{nY}{2Y_0}\right)\exp\left(-\frac{mZ}{2Z_0}\right). \quad (6)$$

[Γ is the factorial function $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$.] We set $\beta=Y/Z,\ \beta_0=Y_0/Z_0$ and regard β and Z as independent variables; after integration over Z, we get

$$\hat{f}_{n,m}(\beta,\beta_0)=\frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n}{m}\right)^{n/2}\frac{(\beta/\beta_0)^{n/2-1}}{\left[1+\frac{n}{m}\frac{\beta}{\beta_0}\right]^{(n+m)/2}}\frac{1}{\beta_0}, \quad (7)$$

which is known as F distribution in statistics. This is the main result of the work, since the statistical properties of the system are determined through the two-parameters canonical distribution, generalization of Eq. (1),

$$\rho_{n,m}(H,\beta_0)=\int_0^\infty d\beta\exp(-\beta H)f_{n,m}(\beta,\beta_0). \quad (8)$$

The main feature of Eq. (7) is that the exponential term of Eq. (4) has disappeared, replaced by a power-law term. One should expect this term to depress high-energy tails in Eq. (8). In order to have an insight on the trends of Eq. (7), let us consider some interesting limits. First of all, we observe that, in the limit $m\to\infty$,

$$\hat{f}_{n,\infty}(\beta,\beta_0)=\frac{\left(\frac{n}{2}\right)^{n/2}}{\beta_0\Gamma\left(\frac{n}{2}\right)}\left(\frac{\beta}{\beta_0}\right)^{n/2-1}\exp\left(-\frac{n\beta}{2\beta_0}\right). \quad (9)$$

We recover the χ^2 distribution [Eq. (4)] since, in the limit of infinite degrees of freedom, the distribution for Z shrinks to a delta distribution, so we are actually dealing with just one stochastic variable Y. It is completely new the limit $n\to\infty$ (that is, we are computing the PDF of the variable $1/Z$), for which we get

$$\hat{f}_{\infty,m}(\beta,\beta_0)=\frac{\left(\frac{m}{2}\right)^{m/2}}{\beta_0\Gamma\left(\frac{m}{2}\right)}\left(\frac{\beta_0}{\beta}\right)^{m/2+1}\exp\left(-\frac{m\beta_0}{2\beta}\right). \quad (10)$$

In order to give visual insight, we plot in Fig. 1 some examples of these distributions. The qualitative shape is rather similar. The occupation factors are computed through Eq. (8). We give explicit expressions for the cases corresponding to the two limits $n\to\infty,\ m\to\infty$,

$$\rho_{n,\infty}(H,\beta_0)=\frac{1}{\left[1+\frac{2}{n}\beta_0 H\right]^{n/2}}, \quad (11a)$$

$$\rho_{\infty,m}(H,\beta_0)=\frac{(2m\beta_0 H)^{m/4}}{2^{m/2-1}\Gamma\left(\frac{m}{2}\right)}K_{m/2}(\sqrt{2m\beta_0 H}), \quad (11b)$$

where K is the modified Bessel function of order $m/2$. The general case of arbitrary n,m can be explicitly written down, but it is not revealing since it involves complex combinations of hypergeometric function, difficult to visualize. We plot in

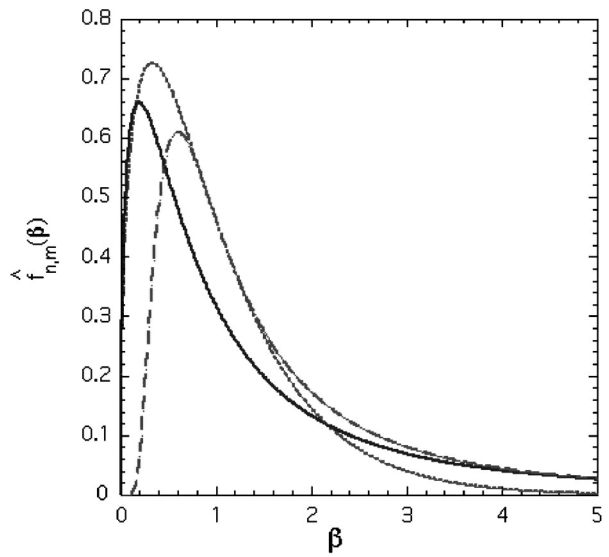

FIG. 1. Probability distribution $\hat{f}_{n,m}(\beta)$ from Eq. (7), with $\beta_0=1$. Solid line, $\hat{f}_{3,3}$; dotted line, $\hat{f}_{3,\infty}$; dashed line, $\hat{f}_{\infty,3}$.

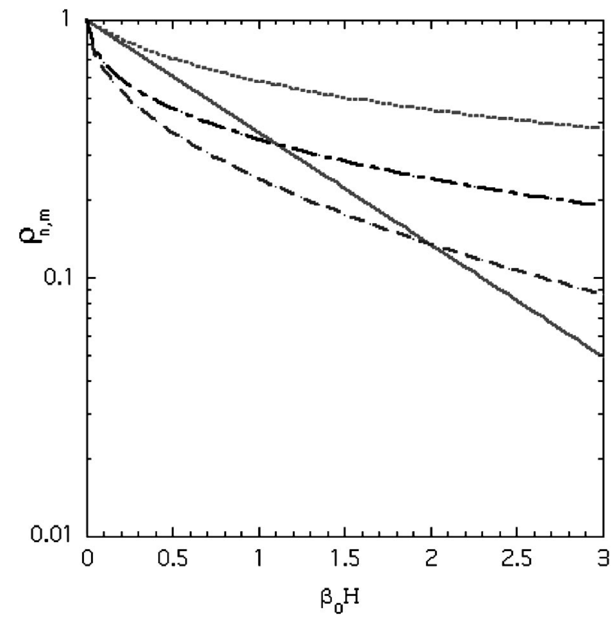

FIG. 2. Generalized canonical distributions $\rho_{n,m}$ as a function of the scaled energy $\beta_0 H$. Solid line, $n\rightarrow\infty, m\rightarrow\infty$ [this yields the usual Boltzmann-Gibbs (BG) case $\exp(-\beta_0 H)$]; dotted line, Eq. (11a) with $n=1$, corresponding to the Tsallis distribution (with $q=2$); dashed line, Eq. (11b) with $m=1$; dotted-dashed line, $n=2$, $m=1$.

Fig. 2 the standard Boltzmann-Gibbs factor together with the curves (11). In general, the new distributions are characterized by tails intermediate between Boltzmann's and Tsallis's statistics. We can obtain the probability distribution $P_{n,m}(u)$ for the generalized velocity u once an explicit form for $H=H(u)$ is given. By assuming the usual form $H=u^2/2$,

$$P_{n,\infty}(u)=\sqrt{\frac{\beta_0}{2\pi}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{1}{\left[1+\frac{\beta_0}{n}u^2\right]^{n/2}}, \tag{12}$$

$$P_{\infty,m}(u)=\frac{\beta_0^{(m+2)/4}|u|^{m/2}}{2^{(m-2)/4}\pi^{1/2}\Gamma\left(\frac{m+1}{2}\right)}K_{m/2}([\sqrt{m\beta_0}|u|). \tag{13}$$

Notice that the function K yields a typical $\exp(-cH^{1/2})$ or $\exp(-c'|u|)$ dependence. Such a law cannot be recovered within Tsallis's formalism, which predicts power-law dependences. Therefore, we take it as a signature of this new class of functions. It may be of interest to notice that the dependence on $|u|$ comes from the variable at the denominator of β, while the numerator provides a dependence on u^2. In the general case, both $|u|$ and u^2 terms do appear.

The question arises if such distributions do exist in nature. We are interested in fluctuations of some quantity; for independent fluctuations, the central limit theorem predicts a Gaussian PDF. If departures from Gaussianity are described in terms of Tsallis's statistics, only PDFs with power-law asymptotics may be included. On the basis of what was told before, we must look for PDFs with exponential tails. Actually, in literature several examples are presented of quantities whose PDFs are (at least on some ranges) exponential. We briefly mention the numerical computation of the velocity distribution function solution of the Enskog–Boltzmann equation for a granular gas [7]; other hints come from calculations of the large-scale probability density distribution in astrophysics [8] and from the numerical simulation of stresses in sheared granular materials [9]. A field where several well documented examples can be found is the study of turbulence in fluids. We refer in particular to papers [10–13]. The quantity we are interested in here is the PDF of the velocity difference between two spatial points. It is found both experimentally and numerically that this quantity shows an exponential tail. In particular, in paper [10] the departure from a Gaussian form is interpreted within a formalism very close to ours, where the average (8) is done using their equivalent of $\hat{f}_{n,m}(\beta,\beta_0)$ given by a log-normal function [see their Eqs. (3.1)–(3.4)]. The paper [12], furthermore, shows that the tails of this PDF can smoothly vary between the Cauchy form (which is a particular kind of Tsallis's distribution) to a Gaussian form passing through the exponential form, by varying a few control parameters. This is strikingly reminiscent of varying n,m parameters in our formalism.

In more detail, we can quote two experimental studies from fusion plasma physics: in the first paper [14] a study of the density fluctuations existing in a thermonuclear fusion device is presented. The time behavior of the electron density n_e close to the boundary of the device was measured with high sampling frequency, thus, allowing one to compute the PDF of the fluctuation $\tilde{n}_e=n_e-\langle n_e\rangle$. It was found that the curve is highly asymmetrical, with the negative wing approximately Gaussian, and the positive one nearly exponential. In Fig. 3 we fit the experimental data with both Tsallis's

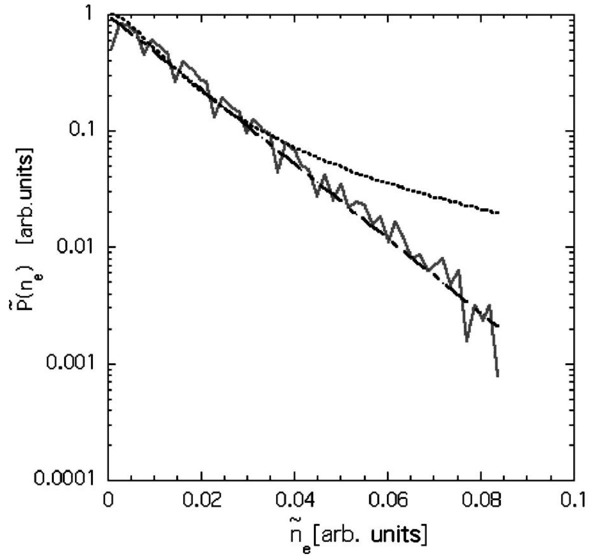

FIG. 3. Probability distribution $P(\tilde{n}_e)$ of the electronic density fluctuations $\tilde{n}_e$. Broken line, experimental data from Ref. [14] (only the side of positive fluctuations is shown); dotted line, best fit using Tsallis's distribution (12); dashed line, best fit with curve (13) and $m=1$.

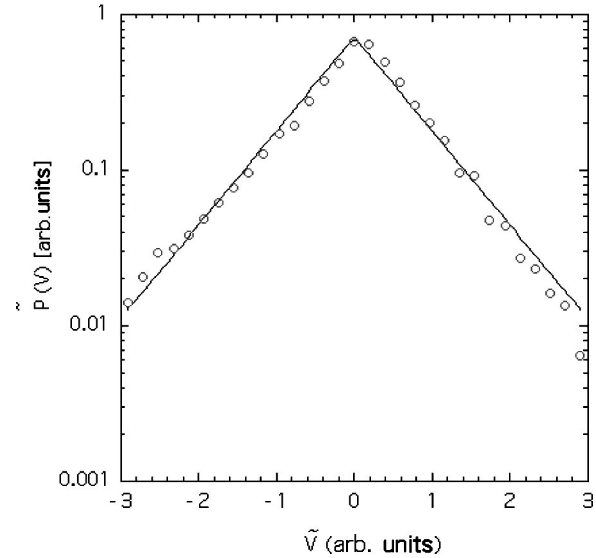

FIG. 4. Probability distribution $P(\tilde{V})$ of the electrostatic potential fluctuations $\tilde{V}$. Circles, experimental data from Ref. [15]; solid line, best fit with curve (13) and $m=1$.

and our curve, showing that the former curve cannot fit the tail of the experimental distribution. Rather closely related, we mention a second paper, dealing with a statistical analysis of electrostatic potential fluctuations, still in the edge of a plasma [15]. A wavelet analysis of the data allowed there to compute PDFs as function of the time scale of the fluctuations. A scaling law for PDFs was recovered by fitting them with stretched exponentials $P(X)\approx\exp(-b|X|^{\alpha})$. The parameter α is a function of the time scale, varying between 1 (exponential distribution) and 2 (Gaussian distribution). In Fig. 4 the case closest to an exponential is shown.

We think we have given in this work constructive evidence of the existence of generalized nonextensive distributions. The very simple PDFs we have computed, seemingly gave us the tools to describe complicated phenomena.

A crucial point is the choice of the microscopic variables, since one could always choose varying definitions for them so as to identify several different cases within the same classes of functions. Therefore, work in this direction should: (i) either show that trivial redefinitions of variables are not important for the final result, or (ii) find that some sets of variables are preferred with respect to all the others.

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