Rotation number, stochastic resonance, and synchronization of coupled systems without periodic driving

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This article investigates the influence of noise in a two-dimensional square array of coupled nonlinear oscillators without periodic driving. Array enhanced stochastic resonance under global as well as local noise perturbation is shown to exist under a crucial condition: the value of the rotation number of the deterministic system being zero. Meanwhile, the stochastic synchronization phenomenon is displayed in a wide range of noise intensity whether noise is added globally or locally. Furthermore, for every oscillator, the peak frequency is shown to agree with the rotation number much better than in the uncoupled system.

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I. INTRODUCTION

Over the last two decades, the phenomenon of stochastic resonance (SR) has aroused considerable interest both in theory and application. One of the recent focuses is about SR in coupled systems due to their potential applications in physics, chemistry, and biology (see the review [1,2]). In this aspect, some preliminary results were first presented by Benzi and co-workers [3]. Later, spatiotemporal order and array-enhanced SR was introduced by Linder *et al.* [4] and the work was further carried to two-dimensional cases [5]. In these authors' investigations, the happening of SR was due to the cooperation of periodic driving force, white noise, and the coupling. In the literature, this phenomenon of SR has been extensively studied in different contexts [6–11].

In applications, however, systems sometimes are only subject to a constant driving force plus white noise. Then what will happen in such systems? For a single-oscillator system, noise-induced SR-like phenomenon has been well manifested in [12–16]. But the situation in systems with many degrees of freedom are only rarely considered [17,18]. In this article, by investigating the influence of noise on N^2 identical overdamped oscillators with nearest neighbor coupling on a net lattice without external periodic driving, we present a bona fide SR of an autonomous system subjected only to white noise in the following sense: (i) With the assistance of the coupling, the noise can drive all oscillators together from their quiescent to resonant state, in which all oscillators behave similarly if their power spectra and quality factors (defined below) are considered. (ii) For every single oscillator, the quality factor undergoes a bell-shaped maximum with the increase of the noise intensity. This maximum is shown to be much higher than that in the uncoupled case, i.e., the single oscillator case. Here a simple but crucial condition for SR is presented. Beside SR, other increasing responses of the coupled system are also manifested, such as the good agreement of the peak frequency of the power spectrum with the rotation number as well as synchronization phenomenon among oscillators and spatiotemporal patterns due to the inherent periodicity of the movement, etc. In our considerations, global as well as local noise perturbation is taken into account. In literature, some researchers referred to global noise as spatially correlated and local noise as spatially uncorrelated. In our paper, the noise is always uncorrelated from site to site, and the terminology of global noise perturbation means that every site is perturbed by white noise while local noise perturbation means that the noises are applied only to some sites.

The model studied here in dimensionless form is characterized as

$$\dot{u}_{i,j}=b_{i,j}-\sin u_{i,j}+K(u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j})+D_{i,j}\xi_{i,j}(t),$$

$$\vec{u}=(u_{i,j},i,j=1,2,\ldots,N)\in T_{N^2}\triangleq S^1\times\underbrace{S^1\times\ldots\times S^1}_{N^2} \tag{1}$$

in which $u_{i,j}$ is the phase angle of the osicallator on lattice (*i,j*), $b_{i,j}\geqslant 0$ is the control parameter, $K>0$ is the coupling coefficient, and $\xi_{i,j}(t)>0$ is the Gaussian white noise satisfying: $\langle\xi_{i,j}(t)\rangle=0$, $\langle\xi_{i,j}(t)\xi_{i',j'}(t')\rangle=\delta(i-i')\delta(j-j')\delta(t-t')$. We use free boundary conditions at the edges of the array. Such a system was introduced to describe the real physical phenomena such as the motion of squid arrays; they were also used to model oscillating chemical reactions and neural networks in biology. Here the array size is taken as 10×10 and the coupling $K=1$.

Even without any noise, system (1) displays interesting dynamical behavior. Numerical simulations show that whenever the coupling $K>0$, it may alternatively rest near a stable state or has a unique running periodic solution. And if there are fixed points, then the number of which in one period is about e^{N^2} [19]. Taking T_{N^2} (the production of N^2 unit circle S^1) as its phase space, then the rotation numbers for

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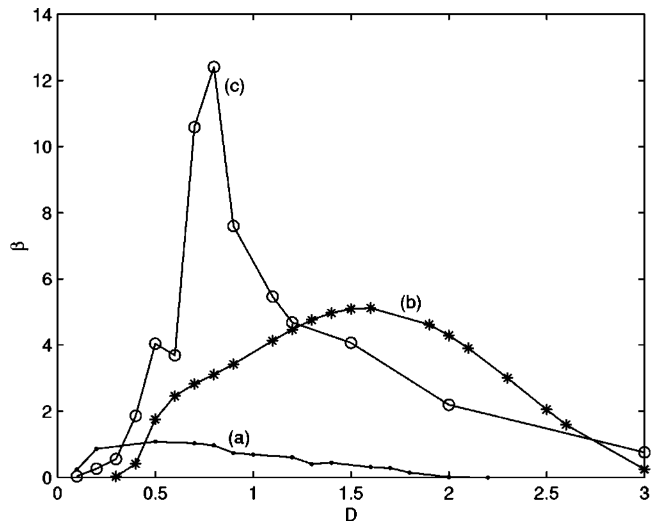

FIG. 1. The quality factor β vs D of lattice (5,5) for (a) $b_{i,j}=0.98$, $K=0$ (single oscillator case), (b) $b_{i,j}=0.98$, $K=1$, and (c) $b_{i,j}=0.98+0.2\times(-1)^i$, $K=1$ $(i,j=1,2,\ldots,N)$. Here $D_{i,j}=D$ $(i,j=1,2,\ldots,N)$.-

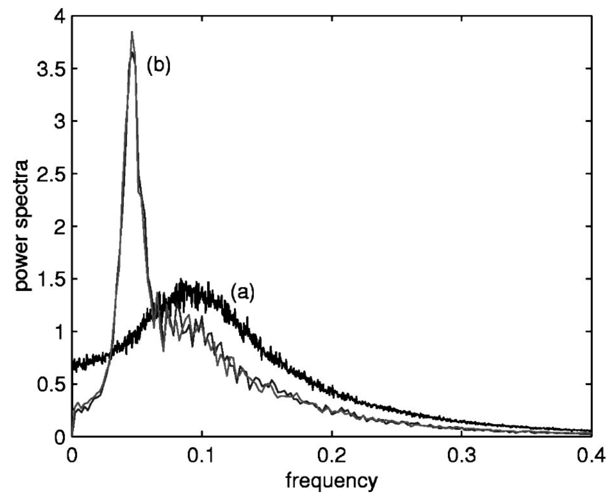

FIG. 2. The power spectrum of (a) $\{\sin u(t)\}$ for a single oscillator system and of (b) $\{\sin u_{1,5}(t)\}$ and $\{\sin u_{2,7}(t)\}$ for the coupled system. Here $b_{i,j}=b=0.98$, $D_{i,j}=D=0.5$ $(i,j=1,2,\ldots,N)$.

different components of the solution are either zero or nonzero. Further calculations show that they are equal even for different values of $b_{i,j}$s. Thus, we define the rotation number of the deterministic system as $R_{det}=\lim_{t\to\infty}u_{1,1}(t)/2\pi t$. In the following it will be seen that for the cases $R_{det}=0$ and $R_{det}\neq 0$, the added noise plays a very different role for the behavior of the corresponding system.

II. STOCHASTIC RESONANCE UNDER GLOBAL NOISE PERTURBATION

To measure the happening of SR, we average 200 times of the power spectrum of the time series of $\{\sin u_{i,j}(t)\}$ and calculate the quality factors which are defined as $\beta=h\omega_p/W$, where h is the height of the spectrum peak, ω_p is the peak frequency, and W is the width of the spectrum at the height of $h/\sqrt{e}$ [12].

First, let us give a brief review of the uncoupled case $(K=0)$, where Eq. (1) is reduced to N^2 independent Langevin equations, and each individual one can be written as

$$\dot{u}=b-\sin u+D\xi(t). \tag{2}$$

In our previous paper [16], we have shown that when $0<b\leqslant 1$, system (2) displays a SR-like phenomenon because the quality factor undergoes a bell-shaped maximum [see Fig. 1(a)].

Now let us see the situation when the neighbors are coupled together. Here we set all $b_{i,j}$'s equal to 0.98. Without any noise, it is not difficult to see that system (1) has two homogenous stationary solutions: $u^s_{i,j}=\arcsin b_{i,j}$ (stable) and $u^u_{i,j}=\pi-\arcsin b_{i,j}$ (unstable) $(i,j=1,2,\ldots,N)$. So we guess that the quality factor for every oscillator will undergo a similar process as that in the uncoupled case. Numerical simulations confirm this speculation [see Fig. 1(b)]. And it turns out that the profile of the power spectrum as well as the $D-\beta$ curve is independent of the lattice sites. Beside this, the coupled system exhibits much better characteristics. First of all, for every oscillator, the height of the spectrum peak is increased, meanwhile the width of the spectrum becomes much narrower [compare Figs. 2(a) and 2(b)]. As a result, the maximum of the quality factor is significantly (about five times) higher than that of the uncoupled case [compare Figs. 1(a) and 1(b)]. Secondly, as shown in Fig. 2(b), all the oscillators have exactly the same peak frequency (even though the values of $D_{i,j}$ are different). This means that all of the oscillators happen to be in resonance together. From these two facts we say that even without external periodic driving, bona fide SR actually occurs in system (1) as a mutually cooperative phenomenon. The reason that causes such good array-enhanced response will be discussed in Sec. V.

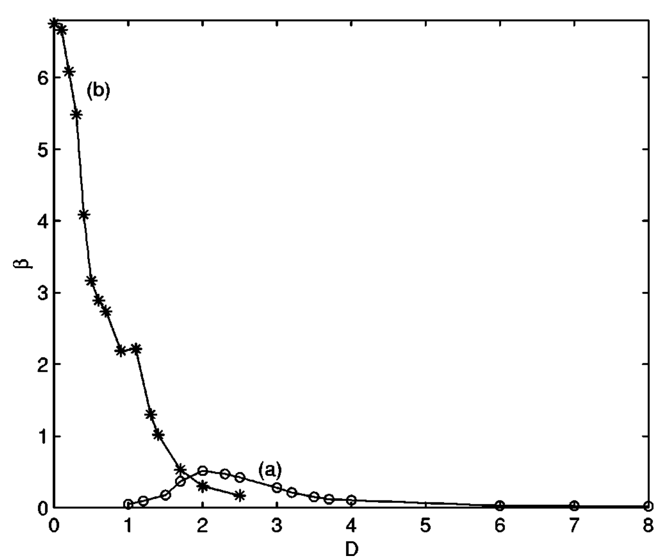

FIG. 3. The quality factor β vs D of lattice (5,5) for (a) $b_{5,5}=20$, $b_{i,j}=0$ $(i\neq 5$ or $j\neq 5)$, and (b) $b_{5,5}=98$, $b_{i,j}=0$ $(i\neq 5$ or $j\neq 5)$. Here $D_{i,j}=D$ $(i,j=1,2,\ldots,N)$.

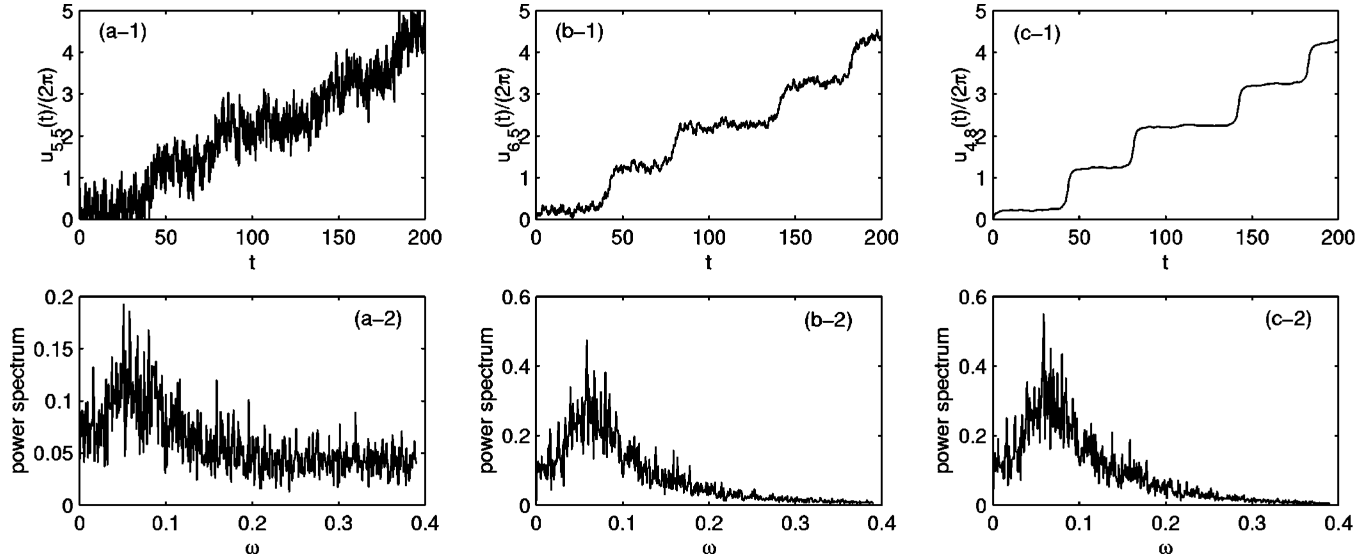

FIG. 4. The trajectories of (a) $\{u_{5,5}(t)\}$, (b) $\{u_{6,5}(t)\}$, and (c) $\{u_{4,8}(t)\}$ and their corresponding power spectra for $b_{i,j}=0.9$ $(i,j=1,2,\ldots,N)$ and $D_{5,5}=9$, $D_{i,j}=0$ $(i\neq 5$ or $j\neq 5)$.

Now let the control parameters be different. Here we take $b_{i,j}=0.98+(-1)^i 0.2$ $(i,j=1,2,\ldots,N)$ such that $\Sigma b_{i,j}/N^2=0.98$. The corresponding curve for the quality factor versus D is plotted in Fig. 1(c). Interestingly, we find that the peak of this bell-shaped curve is about two times higher than that in the case where all $b_{i,j}$'s equal 0.98. This means that a certain degree of heterogeneity of the control parameters can improve the effect of SR. Naively, one may take the term $\{b_{i,j}-\Sigma b_{i,j}/N^2\}$ as a random term of zero mean and look upon its role as a surrogate of noise which may result in the boost of the quality factor. Nevertheless, rigorous mechanical explanation is still needed.

It is known that without the coupling, if only one control parameter is larger than 1 (here we set $b_{5,5}=20$ and others are all zero), then only the middle oscillator can rotate on a unit circle while others rest near their stable fixed points. In such a situation, for every oscillator, SR could not happen because there is no threshold for the middle oscillator and the potentials for all other elements are symmetry. However, when coupling between the neighbors exists, things become intriguing. In the deterministic case, the coupled system becomes quiescent, while when noise is applied, all the oscillators happen to resonate together at a certain frequency and the quality factors of most oscillators (except some boundary ones) undergo a process of first increasing, reaching a maximum, and then decreasing again with the increase of the noise intensity [see Fig. 3(a)]. This reveals the phenomenon of SR. And numerical simulations show that this SR effect is much better than that in the case when all the oscillators share equal constant forces, i.e, $b_{i,j}=0.2$. In fact, for the later case, it is hard for SR to happen. This further confirms the result in the above paragraph that the effect of SR can be

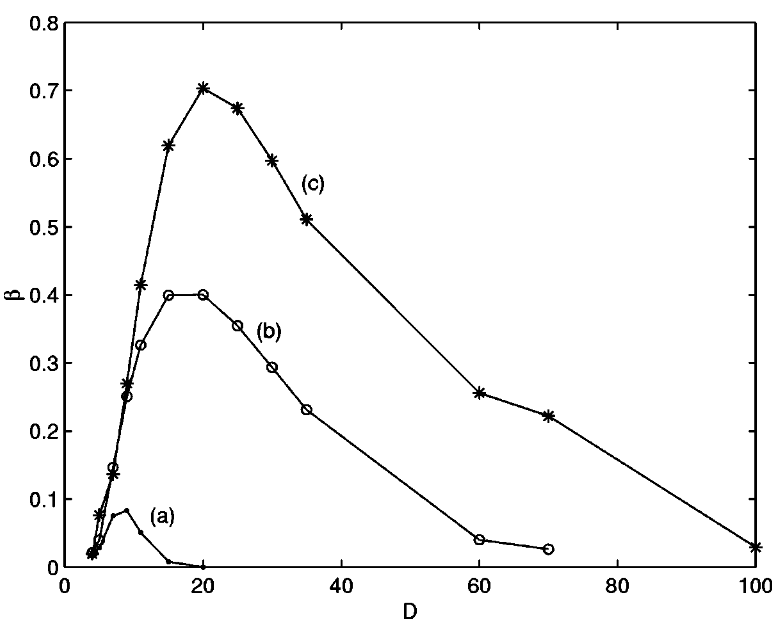

FIG. 5. The quality factor β vs $D_{5,5}$ of lattice (a) (5,5), (b) (6,5), and (c) (4,8) for $b_{i,j}=0.9$ $(i,j=1,2,\ldots,N)$, $D_{i,j}=0$ $(i\neq 5$ or $j\neq 5)$.

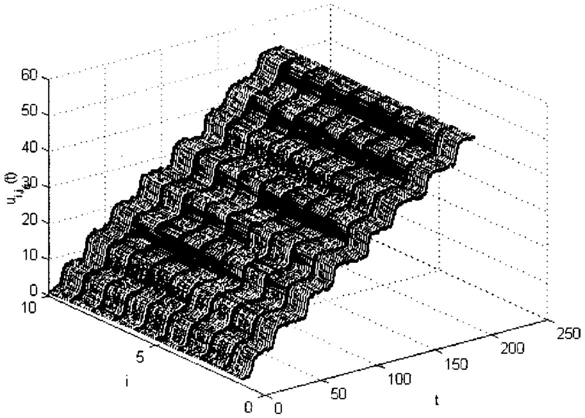

FIG. 6. The meshes of $\{u_{i,j_0}(t)\}$ with $j_0=3$ being fixed for global noise perturbation.

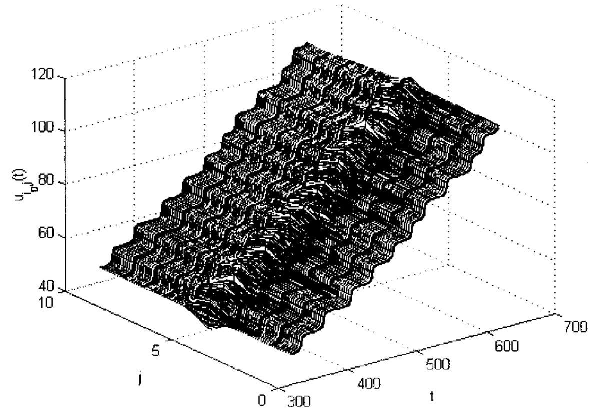

FIG. 7. The meshes of $\{u_{i_0,j}\}$ with $i_0=5$ being fixed for local noise perturbation.

improved by increasing the heterogeneity of the control parameters.

In the above considerations, all the values of $\Sigma b_{i,j}/N^2$ are smaller than 1. Enlightened by the discussion in our previous paper [16], one may expect that SR will happen when $\Sigma b_{i,j}/N^2 \leqslant 1$. However, this is not the case. To illustrate this, let $b_{5,5}=98$ and the other control parameters all be zero. Then $\Sigma b_{i,j}/N^2 < 1$, but the quality factor of every oscillator only decreases with the increase of the noise intensity [see Fig. 3(b)]. In fact, there exists a critical value for $b_{5,5}$ [it is about 27.5] such that for $b_{5,5} \leqslant 27.5$ there exists SR phenomenon, while for $b_{5,5} > 27.5$ no SR could happen. These two different noise-induced effects are actually caused by the corresponding deterministic rotation number R_{det}. In the case $b_{5,5} \leqslant 27.5$, the distributed energy on other lattice points due to the middle one is insufficient to motivate them to escape away from their rest states and the middle oscillator itself is also tied to its neighbor, so $R_{det}=0$. But when noise is included, there is a finite probability for the middle oscillator and consequently, via the coupling for every oscillator to rotate on a circle with some degree of coherence. At an optimal noise intensity, such rotations of oscillators display the best degree of coherence and SR phenomenon then occurs. However, when the control parameter increases to a certain value, the constant driving force on the middle oscillator is so large that even without any noise, it can drive other elements to run together via the coupling, hence $R_{det} > 0$. Since there has already been an oscillatory output in the unperturbed case, adding the noise is of no help to improve the coherent motion of the system, therefore no SR could occur. Here we see the crucial role played by R_{det}, the rotation number of the system without noise: For the phenomenon of SR to happen, the value of R_{det} should be zero. Otherwise, the added noise can only play a destructive role and therefore no SR could occur.

III. STOCHASTIC RESONANCE UNDER LOCAL NOISE PERTURBATION

In the above section we apply the noise to every lattice point; nevertheless, it should also be interesting to study the situation when there is only local noise perturbation, i.e., only some sites are perturbed by white noise. For illustration, we set $b_{i,j}=0.9$ $(i,j=1,2,\ldots N)$ and only let the middle oscillator subject to the noise. In such a situation, it requires relatively larger noise perturbation to motivate the running motion than in the globally perturbed case. In Fig. 4 we plot the trajectories of $\{u_{5,5}(t)\}$, $\{u_{6,5}(t)\}$, $\{u_{4,8}(t)\}$ and the corresponding power spectra of $\{\sin u_{5,5}(t)\}$, $\{\sin u_{6,5}(t)\}$, $\{\sin u_{4,8}(t)\}$ for $D_{5,5}=9$, $D_{i,j}=0$ $(i,j=1,2,\ldots,N)$, respectively. One can see that though we add the noise only on one element, all the oscillators undergo skipping motions. But the motions of the middle oscillator are badly disturbed by the noise and there is only low peak in the power spectrum. For its neighbor, the oscillator on lattice (6,5), the disturbance of noise is not as large as that on the middle element, where there is a significantly higher spectrum peak. As for other oscillators, say the one on lattice (4,8), the noise disturbance is further decreased; in fact, it is the coupling which plays a dominant role to motivate the oscillator to go skipping cycles. Consequently, for this site, there is a higher spectrum peak than the middle two. In Fig. 5 we plot the curves of the quality factors vs D of these oscillators, where one can clearly see the different effects induced by the noise.

We also investigate the case when only one lattice has nonzero control parameter and only this element is subject to noise. Numerical simulations show that though it is hard for SR to happen on this lattice and its neighbors because too strong a noise perturbation is needed to motivate the running motion, there are still SR on other elements.

The above phenomena show that for SR, it is not necessary to let every oscillator be subjected to white noise. Suitable local noise perturbation can also have a global effect on the whole dynamics of the system, and to make most of the oscillators undergo SR.

IV. SYNCHRONIZATION PHENOMENA

The fact of the same peak frequencies in the power spectra of different elements also reveals the phenomenon of synchronization. Numerical simulations show that such phenomenon keeps both in global and local noise perturbation. To see this, we apply global and equal noise intensity on every oscillator and plot the meshes of $u_{i,j_0}(t)$ with j_0 being fixed

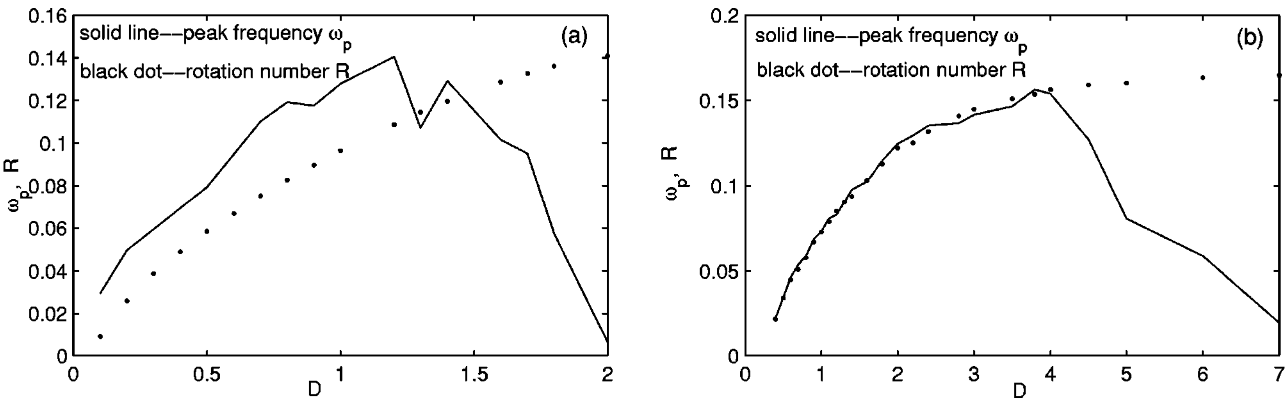

FIG. 8. The relationship between the peak frequency of the power spectrum and the rotation number for (a) a single oscillator system and (b) a coupled system. Here $b_{i,j}=b=0.98$, $D_{i,j}=D$ $(i,j=1,2,\ldots,N)$.

in Fig. 6. It shows that eventually every oscillator skips nearly at the same time, and as a result, the system shows good spatiotemporal structure, which is impossible in the uncoupled case. As for local noise perturbation, Fig. 4 shows that all the oscillators almost keep synchronization. And except the part close to the trajectory $u_{i_0,5}(t)$, there is still a good spatiotemporal structure (see Fig 7).

V. GOOD AGREEMENTS OF THE PEAK FREQUENCIES OF THE POWER SPECTRA WITH THE ROTATION NUMBERS

The coupled enhanced effect can also be reflected by another interesting quantity, the rotation number. In Ref. [16] we have shown that the rotation number of system (2) is an increasing function of noise. So if the values of $D_{i,j}$ $(i,j=1,2,\ldots,N)$ are different, the uncoupled oscillators will have different values of rotation number. But when there is a coupling between the neighbors, we find that the rotation numbers of all the oscillators are exactly the same even though $D_{i,j}$'s are not equal. Thus the rotation number of system (1) (denoted as R) in noisy background still makes sense.

In addition, in [16] we have also shown that for a single oscillator system, i.e., system (2), there is a large deviation between the rotation number and the peak frequency of the power spectrum [see Fig. 8(a)]. There we argued that such deviation is caused by the wandering motion of the oscillator in the potential well like Fig. 9(a). But in this coupled system, numerical simulations show that these two quantities agree with each other very well in a wide range of noise intensity [see Fig. 8(b)]. The reason may lie in the fact that there are a large number of stable fixed points and unstable fixed points of system (1) in the deterministic case. Then for every oscillator, the equivalent potential well in the strip $[2k\pi,2(k+1)\pi](k\in Z)$ roughly looks like Fig. 9(b), where except for the state in the bottom of the potential, we call other stable states the metastable states which plays a role of intermedium. Compared with the potential like Fig. 9, once the particle reaches one metastable state in the equivalent potential like Fig. 9(b), it seldom comes back to the former metastable state because of the positivity of the control parameter and the normal distribution of the white noise. So in a coupled case, not only can the particle be easily motivated from one potential well to the next identical one, but also the wandering time of the particle in the potential well is greatly decreased compared to the time in the potential well like Fig. 9(a). Consequently, the agreement of the rotation number and the peak frequency of the power spectrum are greatly increased.

Figure 9 can also be taken to explain why there is a much better profile of the power spectra for the coupled system than that for the single oscillator system. As one can see that noise plays a much more obviously constructive role in a potential well like Fig. 10(b) than that in a potential well like Fig. 9(a). So in Fig. 2(b), besides the height of the spectrum peak is greatly increased, the noise background is also not so outstanding as that in Fig. 2(a).

VI. CONCLUDING REMARKS

Most authors think that the effect of SR requires three basic ingredients, namely a form of threshold, a weak periodic input, and noise. In this paper we have shown that even

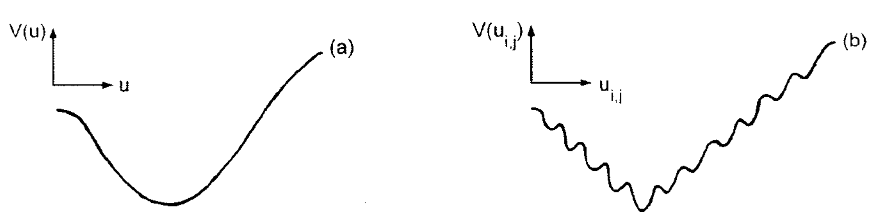

FIG. 9. (a) The potential well of a single oscillator system (2) in the strip $[2k\pi,2(k+1)\pi]$ $(k\in Z)$ and (b) the effective potential well of every oscillator of coupled system (1) in the strip $[2k\pi,2(k+1)\pi]$ $(k\in Z)$.

without periodic driving, the cooperation of the white noise and the coupling can also result in array-enhanced SR. Here, the coupling plays a crucial role to produce such results. Because of its existence, the coherent output of every single oscillator plays a role of the periodic input to its neighbors, thus there is an intrinsic periodic driving as one of the three necessary ingredients of the SR. And due to this inherent periodicity, all oscillators realize synchronization and the system shows good spatiotemporal pattern.

Secondly, we present a simple necessary condition for the happening of SR, i.e., the deterministic rotation number R_{det} being zero. We have shown that for $R_{det}>0$, no SR occurs, while for $R_{det}=0$, SR can happen. And in noisy background, the zero rotation number becomes nonzero. This is an interesting phenomenon. More intriguingly, compared with the single oscillator system, the rotation number of the coupled system agrees with the peak frequency of the power spectrum very well.

Thirdly, we have also found that by keeping the same average value of $\Sigma b_{i,j}/N^2$, the heterogeneity of the control parameters can improve the SR effect. Though the corresponding mechanism is not very clear, this fact will be of help for practical application. For a small average value of the control parameters (about $\Sigma b_{i,j}/N^2<0.275$), the best SR effect is obtained in the case of concentrating the driving force on one element. But for a large value of $\Sigma b_{i,j}/N^2$, it should be kept in mind that the value of R_{det} should be kept zero when one modulates the control parameters, otherwise SR will disappear.

ACKNOWLEDGMENTS

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