Stability of periodic paraxial optical systems

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Based on ray propagation of paraxial geometric optics, we show that any stable periodic paraxial system or optical resonator becomes unstable in presence of stochastic perturbations of the the periodic sequence along which the rays are propagated. The exponential divergence with distance of ray displacements from the optical axis bears a close connection to the phenomenon of Anderson localization in disordered systems. The stability of the periodic focusing system is restored when finite aperture effects are accounted for and complex paraxial optics is used to describe wave propagation.

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The study of the propagation of paraxial rays and waves through periodic focusing systems plays an important role in the understanding of basic properties of optical resonators, laser beams, and graded-index fibers. An extended literature exists indeed on this subject (for an early review see, for instance, Ref. [1]), and detailed treatments, based on the well-known *ABCD* matrix propagation method of paraxial ray and wave optics, can be found nowadays in many textbooks (see, for instance, Refs. [2], [3]). In the framework of the geometric ray optics, periodic paraxial systems with purely real ray matrices are classified as either stable or unstable, depending on the value of the semitrace $(A+D)/2$ of the *ABCD* matrix for the single sequence. A system is stable for $|A+D|<2$; in this case the rays in the system oscillate back and forth about the optical axis, and the maximum excursions of ray displacement r_n and ray slope r'_n from optical axis are bounded. Conversely, a system with $|A+D|\geqslant 2$ is unstable and rays become more and more dispersed from the optical axis, the further they pass through the sequence. Through this simple classification scheme, the focusing properties of a stable system may be, nevertheless, drastically modified when effects of perturbations of optical elements, albeit small, are taken into account. In particular, it was shown that misalignment effects of the optical elements in a periodic system may lead to secular growth of ray displacement from optical axis during propagation, making the system unstable [3–5]. This circumstance has been especially investigated in case of continuous lensguides that describe, e.g., curved or tilted graded-index fibers [3,5]. Even in the absence of misalignment effects, it was recently shown [6] that *periodic* or *quasiperiodic* perturbations of *ABCD* matrix elements in a periodic paraxial stable system can lead to secular growth of ray displacements owing to a parametric resonance effect.

In this paper, we study the effects of stochastic perturbations of *ABCD* matrix elements in a periodic paraxial stable system and show that, owing to a rather general phenomenon analogous to Anderson localization in disordered systems [7,8], the Lyapunov growth rate of ray displacement is positive, which is a signature of instability. Under a different viewpoint, the dynamics of ray propagation as ruled by paraxial *ABCD* ray matrices turns out to be analogous to that of a periodically kicked harmonic oscillator, and the instability induced by the disorder thus corresponds to divergence of trajectories of the stochastic Hamiltonian system [9–11]. When the effects of soft (Gaussian) apertures are taken into consideration and complex paraxial wave propagation is considered, it is shown that the field remains confined and oscillates around the perturbationally stable eigenmode of the sequence.

We consider ray and wave propagation through a discrete periodic focusing system composed of a periodic and aligned sequence, at planes $z_1, z_2, \ldots, z_n, \ldots,$ of a geometrically stable optical system, consisting of several paraxial elements (such as lenses, spherical dielectric interfaces, free-space propagation, ducts, etc.), with an unperturbed ray matrix *ABCD*. A notable example is represented by an optical resonator, where the back and forth ray propagation between the two cavity mirrors mimics the propagation through an iterated periodic focusing system that repeats indefinitely the sequence of optical elements inside the resonator. As we allow for small perturbations of matrix parameters from one block to the next, we denote by A_n, B_n, C_n, and D_n, the matrix elements of the nth block in the chain and assume that they are uniformly bounded with index n [see Fig. 1(a)]. In case of ray propagation inside an optical resonator, the index

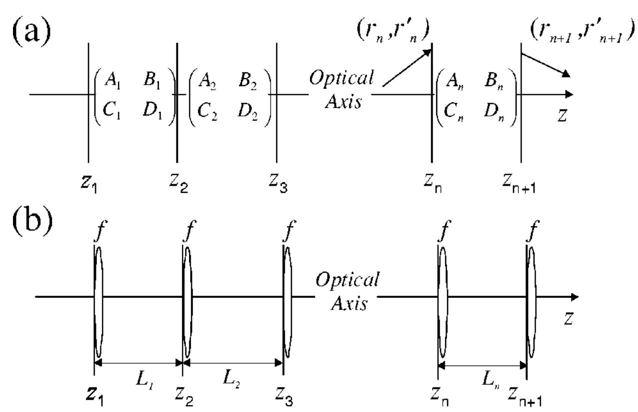

FIG. 1. (a) Schematic representation of ray propagation through a periodic optical system with modulated parameters. (b) An example of a disordered periodic lensguide composed of a sequence of lenses of focal length f placed at distances $L_n = L + \epsilon_n$. The condition $L<4f$ is assumed to ensure geometric stability of the unperturbed periodic lensguide.

n denotes the cavity round-trip number, and perturbation of matrix elements may account for time variation of resonator parameters, such as moving mirrors or time-varying lenses. In the following, we will focus our attention mainly on the case where the matrix element C along the chain is not perturbed; this case applies, for instance, to any repeated sequence of aligned optical elements, including lenses, dielectric interfaces, and ducts that are irregularly spaced along the chain. A notable example, shown in Fig. 1(b), is that of a chain of lenses, of focal length f, irregularly spaced at distances $L_n = L + \epsilon_n$, where L is the mean separation distance and ϵ_n are stochastic variables with zero mean. Light propagation through the chain can be described in terms of either ray or wave propagation by use of the well-known ray matrix techniques and generalized Huygens integrals [1–3]. Let us first consider the simplest ray propagation based on geometric optics rules; this approach may be used whenever the matrix elements along the chain are real valued, i.e., when transverse gain or loss variations in the system are negligible. In this case, any light ray propagating along the chain is characterized by its ray displacement r_n and ray slope r'_n from the optical axis at plane $z = z_n$ [see Fig. 1(a)]; in the paraxial approximation, these evolve according to the simple linear mapping rule [1–3]

$$\begin{cases} r_{n+1} = A_n r_n + B_n r'_n, \\ r'_{n+1} = C_n r_n + D_n r'_n. \end{cases} \tag{1}$$

The optical sequence is said to be *geometrically stable* if any initial paraxial ray propagates along the chain remaining close the optical axis, i.e., if $\mathrm{Lim\,Sup}_{n\to\infty} |r_n|, |r'_n| < \infty$. For the unperturbed periodic chain, the geometric stability criterion is readily derived by, e.g., direct calculation of the matrix elements for the cascading of n blocks using Sylvester's theorem and by an inspection of the asymptotic behavior of matrix elements as n goes to infinity [1]; it then turns out that stability for the unperturbed periodic chain is ensured, provided that the angle ϕ_0, defined by the relation $\cos\phi_0 = (A + D)/2$, is real valued, i.e., for $|A + D| < 2$. To study the stability for the perturbed periodic chain, it is worth deriving a second-order linear difference equation describing propagation of the ray slope r'_n solely; assuming C_n to be independent of n and using the identity $A_n D_n - B_n C_n = 1$, from Eq. (1) one readily obtains

$$r'_{n+1} + r'_{n-1} = 2\cos\phi_0 r'_n + \epsilon f(n) r'_n. \tag{2}$$

where $\epsilon f(n) \equiv A_{n-1} - A + D_n - D$ accounts for the disorder in the chain. The ray displacement r_n can then be calculated, if needed, according to $r_n = (r'_{n+1} - D_n r'_n)/C$. The asymptotic growth rate for the ray slope r'_n can be derived by evaluation of the Lyapunov exponent λ for the discrete equation (2), which is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{2n} \ln(r'^2_{n+1} + r'^2_n). \tag{3}$$

The properties of the discrete equation (2) in presence of stochastic disorder $\epsilon f(n)$ has been studied by several authors since long time, as it describes the dynamics of many one-dimensional disordered systems [8]; in particular, Eq. (2) was first considered in solid-state physics to study localization of electron energy states in disordered one-dimensional crystals in the tight-binding approximation (Anderson localization [7]). The positiveness of the Lyapunov exponent (3), which in the Anderson model corresponds to exponential localization of electronic states, in our context is the condition for *geometric instability* of the periodic sequence. In a different but closely related perspective, the dynamics of the ray slope r'_n in a stochastically perturbed periodic sequence can be shown to be analogous to that of displacement of a linear harmonic oscillator with unitary mass and angular frequency in which the stochastic perturbation $\epsilon f(n)$ causes a periodic and instantaneous variation of the momentum with a period equal to ϕ_0. In fact, if one considers a periodically kicked harmonic oscillator described by the Hamiltonian $H(x,p) = (1/2)(p^2 + x^2) + (1/2)x^2 \Sigma_n \epsilon g(n) \delta(t - n\phi_0)$, where ϕ_0 is the period of kicks and $\epsilon g(n)$ their amplitudes, after integration of the equations of motion in the interval $t = n\phi_0$, $t = (n+1)\phi_0$ one obtains for the mass displacements $x_n = x(n\phi_0)$ at times $t = n\phi_0$ the same equation as given in Eq. (2) with $f(n) = -g(n)\sin\phi_0$ (for details see Refs. [10], [11]). In absence of perturbations ($\epsilon = 0$), any ray periodically oscillates close to the optical axis in the same way as the mass displacement x_n oscillates around its equilibrium position; in this case $\lambda = 0$ and the system is stable. The successive kicks may, however, destroy the coherent ray oscillations leading to a positive asymptotic growth that corresponds, for the Hamiltonian system, to divergence of the trajectory in the phase space (x, p). The positiveness of Lyapunov exponent in case of not correlated disorder, i.e., when $\epsilon f(n)$ are independent and equally distributed random variables with zero mean, is a very general result that follows basically from an ergodic theorem by Furstenberg for the product of an increasing number of random independent matrices [12]. However, the derivation of an explicit expression for λ is a challenging task requiring the determination of some invariant distribution functions that are solutions of integral equations; for a detailed discussion of this problem and analysis of exact and perturbative techniques for the determination of the exponential growth rate we refer the reader to specific works [8,9,13,14]. Here we just mention that a quite general expression for λ has recently been derived in Ref. [9] by considering the Hamiltonian map associated to the harmonic oscillator with periodic kicks; it reads,

$$\lambda = \frac{1}{2} \int d\eta P(\eta) \int_0^{2\pi} d\theta \rho(\theta) \ln[D(\eta,\theta)], \tag{4}$$

where $D(\eta,\theta) \equiv 1 - \eta[\sin(2\theta)/\sin\phi_0] + \eta^2(\sin\theta/\sin\phi_0)^2$, $P(\eta)$ is the probability density of the distribution of $\epsilon f(n)$, and $\rho(\theta)$ is the invariant measure of the one-dimensional map for the phase $\theta = \mathrm{atan}(x/p)$. In particular, for a relatively strong disorder or for weak disorder ($\epsilon \to 0$) far from both band edges $\cos\phi_0 = \pm 1$ and band center $\cos\phi_0 = 0$ of stability region, the phase θ undergoes strong rotation and the invariant phase measure can be taken almost uniform, i.e., $\rho(\theta) = 1/2\pi$. In such cases, one can explicitly derive an ex-

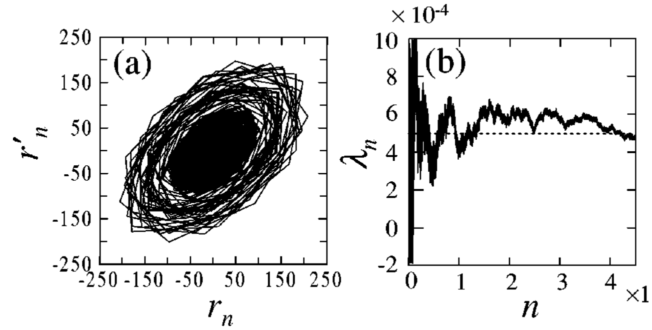

FIG. 2. (a) An example of diverging trajectory in the phase space (r_n, r'_n) of light rays propagating through the disordered periodic lensguide of Fig. 1(b). Parameter values are $L=1$, $f/L=1.0877$ (corresponding to $\phi_0=1$), $\Lambda/L=0.2$. (b) Corresponding behavior of λ_n versus n. For the sake of clarity in (a) the trajectory is limited to n up to 8000. The dashed horizontal line in (b) is the value of Lyapunov exponent λ as calculated by Eq. (4).

pression for λ after integration of Eq. (4), once the probability density $P(\eta)$ is assigned. We also mention that positiveness of Lyapunov exponent may occur as well in the presence of correlated disorder, though this is not always the case [10].

As an example, we studied ray propagation in the irregularly spaced sequence of lenses shown in Fig. 1(b), assuming that the lens spacing deviations ϵ_n from the mean distance L are independent stochastic variables uniformly distributed between $-\Lambda/2$ and $\Lambda/2$. In this case, one has $A=1-L/f$, $B=L$, $C=-1/f$, $D=1$, and $\epsilon f(n)=-\epsilon_{n-1}/f$; stability of the unperturbed periodic lensguide is ensured for $L<4f$. Figure 2(a) shows an example of the evolution of light rays in the phase plane (r_n, r'_n), as obtained by direct numerical integration of Eq. (1) with initial conditions $r_1=1$, $r'_1=1$. The divergence of the trajectory in the phase plane is clearly shown in Fig. 2(b), where the behavior of $\lambda_n=(1/2n)\ln(r'^2_n+r'^2_{n+1})$ is depicted which provides an estimate of Lyapunov exponent as $n\to\infty$. The horizontal dashed curve in Fig. 2(b) is the value of Lyapunov exponent, as calculated from Eq. (4), assuming a uniform invariant phase measure $[\rho(\theta)=1/2\pi]$ and a uniform distribution function for $P(\eta)$; in this case an analytical calculation of integrals in Eq. (4) is possible and yields $\lambda=(1/2)\ln(1+\alpha^2)+(1/\alpha)\mathrm{atan}(\alpha)-1$, where $\alpha\equiv\Lambda/(4f\sin\phi_0)$.

The above analysis was based on simple ray optics propagation, which can be applied when finite aperture effects are negligible and ray matrix elements are real valued. It is expected, however, that the asymptotic divergence of rays induced by disorder will be prevented when finite apertures of the optical systems are properly taken into account. We can prove this property analytically by considering wave propagation in a periodic sequence of optical elements containing soft apertures with a Gaussian amplitude transmission in the transverse plane that mimic the effects of finite apertures of optical elements. In this case, wave propagation through the chain can be described in terms of complex *ABCD* matrices by means of generalized Huygens integrals [3] or by use of the generalized complex Q parameter as defined in Ref. [15]. Since we are interested in the stability property of the periodic chain, we adopt here the latter method, which is simpler involving simple algebraic relations between complex Q parameters at successive planes along the chain. In fact, for any paraxial field we can introduce a complex parameter Q as in Ref. [15] that evolves according to the *ABCD* law,

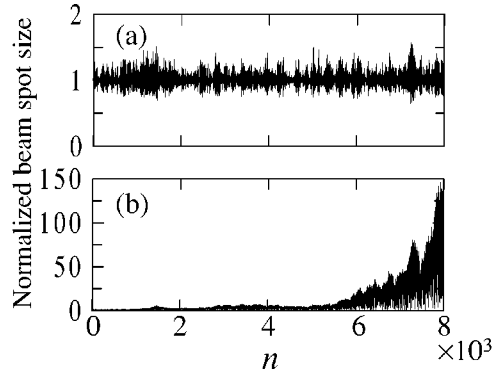

FIG. 3. Evolution of normalized beam spot size W_n/W_1 for a paraxial wave propagating through the disordered periodic lensguide of Fig. 1(b), as obtained by numerical iteration of Eq. (5), with Gaussian apertures for (a) $\Theta=1/100$, and (b) $\Theta=0$; the other parameter values are the same as in Fig. 2. The initial value for the complex Q parameter is taken equal to that of the stable eigenmode of the unperturbed sequence.

$$Q_{n+1}=\frac{A_nQ_n+B_n}{C_nQ_n+D_n}, \tag{5}$$

where Q_n is the value of Q taken at the plane z_n. The beam size W can be derived from the complex Q parameter by means of the simple relation $W^2=-M^2\lambda_0/\pi\,\mathrm{Im}(1/Q)$, where λ_0 is the wavelength of the optical field and M^2 is the beam quality factor ($M^2=1$ for a Gaussian beam). The propagation law (5) thus rules out the confinement properties of the periodic chain for *any* field distribution. We assume that the unperturbed periodic system admits of a stable confined mode and that this mode is perturbationally stable (see Ref. [16]; see also Ref. [3], Chap. 21). The fixed points of Eq. (5) without perturbations are given by $Q_0=(A-D)/(2C)\pm i(1/C)\sin\phi_0$. We assume that one of these two solutions corresponds to a confined and stable mode, i.e., that $\mathrm{Im}(Q_0)>0$ (confinement condition) and $\mathrm{Im}(\phi_0)<0$ (stability condition). Such two conditions are, for sure, simultaneously satisfied in any purely real periodic lensguide (either geometrically stable or unstable) whenever a Gaussian aperture is periodically inserted into the structure (see Sec. 21.4 of Ref. [3]). In this case, any initial field distribution not only remains confined during propagation along the periodic chain, but its transverse size W asymptotically reaches a stationary value W_0 that is independent of the initial field distribution. To study wave propagation in the perturbed disordered chain, it is worth introducing the complex R parameter at planes $z=z_n$ through the relation $R_n=CQ_{n-1}+D_{n-1}$; the evolution equation for R_n follows from Eq. (5) and reads,

$$R_{n+1}+\frac{1}{R_n}=2\cos\phi_0+\epsilon f(n) \tag{6}$$

One can prove that the solution to Eq. (6) admits of a regular expansion in power series of ϵ and that the asymptotic expansion is uniformly valid with respect to the index n [17]. This means that, for small values of ϵ, the effect of the perturbation $\epsilon f(n)$ is merely to introduce small deviations of Q from Q_0 during propagation. As an example, let us consider again the lensguide of Fig. 1(b) and introduce, close to each lens in the sequence, a Gaussian aperture of width w_a. The introduction of the Gaussian apertures corresponds to the change $1/f \rightarrow (1+i\Theta)/f$ in the *ABCD* matrix elements, where $\Theta \equiv \lambda_0 f/\pi w_a^2$ is a dimensionless parameter that provides a measure of the finite transverse size of the lensguide. In Fig. 3(a) we show the evolution of normalized beam spot size W_n/W_1 versus period index n, as obtained by numerical analysis of Eq. (5), for the same parameter values as in Fig. 2 and for $\Theta = 1/100$; for comparison, in Fig. 3(b) it is also shown the corresponding behavior that one would obtain for $\Theta = 0$, i.e., in absence of the Gaussian apertures. Notice that in the latter case the beam size is not confined around its steady-state value, which is a signature of the geometric instability predicted by the ray optics analysis.

In conclusion, we have shown that in stochastically perturbed periodic optical systems, light rays are always asymptotically diverging despite the stability of the unperturbed sequence and the smallness of perturbation. We have shown that the onset of geometric instability induced by stochastic perturbations is analogous to that of Anderson localization in disordered systems and related to the divergence of trajectories in noisy hamiltonian maps describing the dynamics of periodically kicked harmonic oscillators.

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[17] In fact, by setting $R_n = R_n^{(0)} + \epsilon R_n^{(1)} + \epsilon^2 R_n^{(2)} + \ldots$ in Eq. (6), a hierarchy of equations for successive corrections to R_n is obtained. The solution to the equation at leading order is attracted toward the stable fixed point $\exp(i\phi_0)$, so that we may assume $R_n^{(0)} = \exp(i\phi_0)$. This is because ϕ_0 is complex valued with $\mathrm{Im}(\phi_0) < 0$. The equations at higher orders $s \geqslant 1$ then read $R_{n+1}^{(s)} = \exp(-2i\phi_0) R_n^{(s)} + \sigma_n$, where σ_n depends on solutions at previous orders: $\sigma_n = \exp(-i\phi_0) f(n) R_n^{(s-1)} + \exp(-i\phi_0) \Sigma_{k=1}^{s-1} R_n^{(k)} \times R_{n+1}^{(s-k)}$. The solutions to these equations are given by $R_n^{(s)} = \Sigma_{k=0}^{n-1} \sigma_k \exp[-2i\phi_0(n-k-1)]$. It is easy to show that, if $\{\sigma_n\}$ is bounded, then $\{R_n^{(s)}\}$ is also bounded with respect to the index n. Since, at $O(\epsilon)$, $\sigma_n = f(n)$ is bounded for definiteness, for recurrence it then follows that $R_n^{(s)}$ is bounded at any order $\sim \epsilon^s$ and the asymptotic expansion is uniformly valid with respect to the index n.