Parameter evaluation from time sequences using chaos synchronization

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Unknown parameters in nonlinear equations are estimated from chaotic time sequences using chaos synchronization. The method is based on a random optimization method. The parameters are randomly searched for in a sequential manner as the degree of the chaos synchronization is increased. The method is applied for the parameter evaluation in the Lorenz equation and the Lang-Kobayashi model for the chaotic semiconductor laser.

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Chaos appears in time evolutions of some kinds of nonlinear equations. It is an inverse problem to predict a governing nonlinear equation from the chaotic time sequence. This inverse problem is important but in general very difficult [1]. If the form of the governing nonlinear equation is assumed, the inverse problem is reduced to a simpler problem to evaluate parameters in the nonlinear equation from the chaotic time sequences. There are many attempts for the parameter estimation in ordinary differential equations and partial differential equations [2,3]. These methods are mainly based on the regression analysis between the temporal derivatives of the state variables and some polynomials of the variables and their spatial derivatives. We propose another method to estimate the parameters using chaos synchronization and a random optimization method. The temporal derivatives of the state variables are not necessary in this parameter estimation method.

We assume a nonlinear equation subject to a chaotic force $\mathbf{x}_0(t)$.

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(\mathbf{x},\{a\})+D(\mathbf{x}_0-\mathbf{x}), \tag{1}$$

where $D(\mathbf{x}_0-\mathbf{x})$ denotes a coupling term and $\{a\}$ denotes a parameter set of the nonlinear equation. If $\mathbf{x}_0$ obeys the same form of nonlinear equation:

$$\frac{d\mathbf{x}_0}{dt}=\mathbf{f}(\mathbf{x}_0,\{a_0\}), \tag{2}$$

with the same parameter set $\{a_0\}=\{a\}$, the chaotic time sequence $\mathbf{x}(t)$ is expected to be synchronized by the external force $\mathbf{x}_0(t)$ for $D>D_c$. The synchronization has been intensively studied in chaotic systems [4–6]. If the parameter values $\{a\}$ of the assumed equation are different from the original parameter values $\{a_0\}$, the complete synchronization cannot occur, however, the difference $|\mathbf{x}-\mathbf{x}_0|$ is expected to be small, if the difference of the two parameter sets is small and D is sufficiently large. We measure the degree of the chaos synchronization by the difference of the two time sequences. We search for the parameter values $\{a_0\}$ of the nonlinear equation with a random optimization method, as the degree of the chaos synchronization becomes stronger. If the perfect chaos synchronization is attained, the obtained parameter values are expected to be the desired parameters for the nonlinear equation which generates the chaotic sequence.

Our algorithm for the random optimization is such that

(1) We assume a nonlinear equation with a parameter set $\{a\}$. We also perform a numerical simulation of Eq. (1) with the chaotic force term and the coupling term with sufficiently large D.

(2) We calculate the difference of the two time sequences such as

$$U=\int_0^T|\mathbf{x}-\mathbf{x}_0|^2dt.$$

(3) Each parameter a in the parameter set $\{a\}$ is randomly modified as

$$a'=a+r,$$

where r is a random number which obeys the Gaussian distribution with small but fixed variance σ. We perform a numerical simulation of Eq. (1) with the modified parameter set $\{a'\}$ and obtain a time sequence $x'(t)$.

(4) The difference of the two time sequences is calculated as

$$U'=\int_0^T|\mathbf{x}'-\mathbf{x}_0|^2dt.$$

for the randomly modified values of the parameter set.

(5) If the difference U' is smaller than U, the parameter set is changed from $\{a\}$ to $\{a'\}$. On the other hand, if the difference U' is larger than U, the parameter set is unchanged and kept to be $\{a\}$.

(6) The processes (1)–(5) are repeated until the difference U becomes sufficiently small. This is a kind of random optimization method and is similar to the Metropolis method with temperature 0 [7,8]. Similar optimization methods were used for more complicated problems such as neural network models and the traveling salesman problem [9,10]. If there are many local minima in U as a function of parameters, it may be better to include some stochastic processes as the

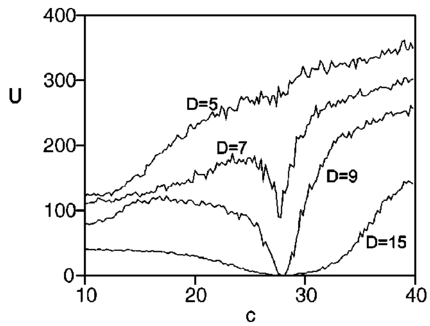

FIG. 1. Degree U of the chaos synchronization as a function of c and D for the Lorenz equation.

Metropolis method with finite temperature. We have not included such additional stochastic processes in this paper for the sake of simplicity.

We have applied the above algorithm to estimate parameters in some model equations. The first example is the Lorenz equation [11]. The model equation is

$$\frac{dx}{dt}=a(-x+y)+D[x_0(t)-x],$$

$$\frac{dy}{dt}=-xz+cz-y,$$

$$\frac{dz}{dt}=xy-bz, \tag{3}$$

where a, b, and c are the unknown parameters and D is a coupling constant. Many types of coupling terms are possible for the chaos synchronization, but we have assumed the above form of coupling for the sake of simplicity. The chaotic time sequence $x_0(t)$ was generated by the same Lorenz equation with $a=10=a_0$, $b=8/3=b_0$, and $c=28=c_0$. Direct numerical simulations show that the complete chaos synchonization $x(t)=x_0(t)$ occurs for $D>D_c=7.95$ at $a=10$, $b=8/3$, and $c=28$ in the forced Lorenz Eq. (3).

The difference U was numerically calculated as $\int_0^T[x(t)-x_0(t)]^2dt$ for $a=10$, $b=8/3$, and $T=40$ by changing c and D. The integral time T is larger than a typical period of the chaotic oscillation in the Lorenz equation. Figure 1 displays the averaged value of U with respect to many time sequences. The difference U takes a minimum value 0 at $c=c_0$ for $D=15$ and 9, since the coupling constant D is above the critical value 7.95. As D is increased from 7, the value of U at $c=c_0=28$ is decreased and becomes 0 for $D>D_c=7.95$. The point $c=c_0$ is a local minimum point for $D=9$ and 7, that is, c is expected to approach c_0, if the initial value of c is chosen to be close to c_0 and the above random optimization process is applied. For $D=5$, there is no definite local minimum point in the range [10,40], therefore, c is expected to decrease in a monotonic manner in the optimization process. These results suggest that the above random optimization may succeed in obtaining the parameter c_0, if the coupling constant D is sufficiently large.

We have searched for three parameters a_0, b_0, and c_0 with the random optimization method. The variance σ of the Gaussian distribution for the random modification is assumed to be 0.01 for a, b, and c. The initial values of a, b, and c are $a=3$, $b=1$, and $c=5$. The Lorenz equation has a stable stationary solution at the parameter values, that is, the initial values are sufficiently apart from the parameter values

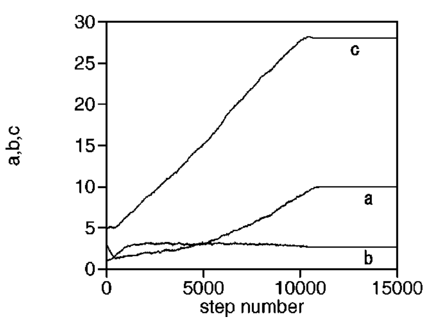

FIG. 2. Time evolution of a, b, and c by the random optimization process for the Lorenz chaos.

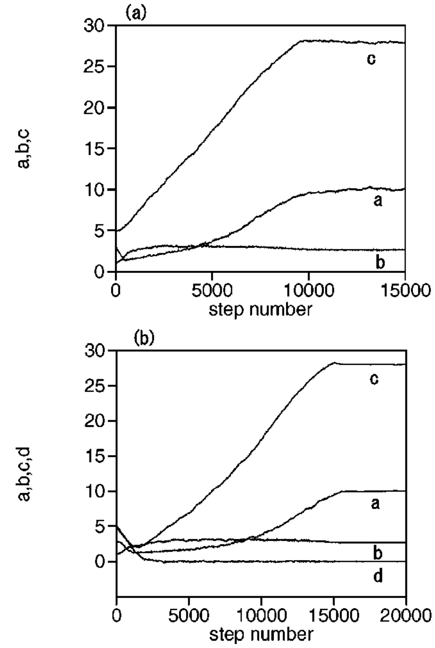

FIG. 3. (a) Time evolution of a, b, and c by the random optimization process with noisy signals for the Lorenz equation. (b) Time evolution of a, b, c, and d by the random optimization process for a generalized Lorenz equation.

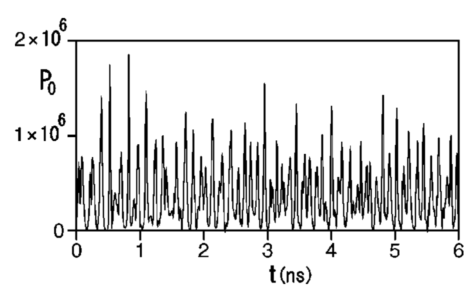

FIG. 4. Chaotic time sequence of power $P_0=E_0^2(t)$ for the Lang-Kobayashi equation at $\kappa_0=30\ \text{ns}^{-1}$ and $J=2.74\times10^{17}$.

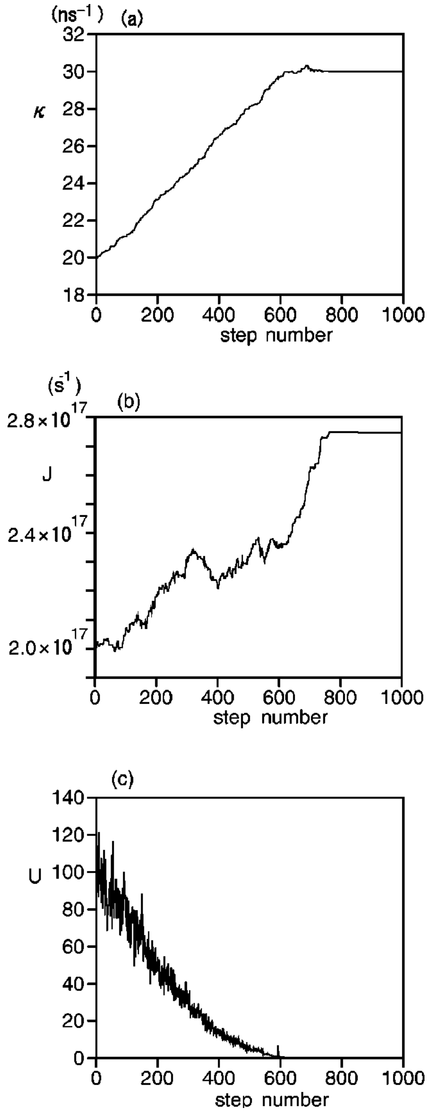

FIG. 5. Time evolution of κ (a), J (b), and U (c) by the random optimization process for the Lang-Kobayashi model.

of a_0, b_0, and c_0. We have checked that the optimization method is successful for several other initial values of a, b, and c. The difference U of the two time sequences $x(t)$ and $x_0(t)$ is calculated as $U=\int_0^T[x(t)-x_0(t)]^2dt$ with $T=40$. The difference U does not take a constant value even for fixed values of a, b, and c, since the time sequences are chaotic. A kind of additional stochasticity is naturally included in the random optimization process. Figure 2 displays the time evolution of a, b, and c. The desirable parameter values are obtained and nearly perfect chaos synchronization is attained after 11 000 steps. The parameter values at the 15 000th step are $c=28.001$, $a=9.997$, and $b=2.667$.

We have checked the robustness of this optimization method with several simulations. The input signal $x_0(t)$ is perturbed by some noises in some cases. We have performed the optimization algorithm in the case of noisy signals. The input signal $x_0(t)$ is assumed to be a chaotic time sequence by the Lorenz equation with $a_0=10$, $b_0=8/3$, and $c_0=28$ overlapped with the Gaussian white noise of variance 1. Figure 3(a) displays the time evolution of a, b, and c. After 11 000 steps, the parameters are close to the desirable values and are slightly fluctuating around the desirable values a_0, b_0, and c_0. It implies that the optimization method is applicable for the noisy signals.

We have assumed the exact form of the Lorenz equation in the former simulation. If we do not know the exact form of the equation, we prepare some additional terms at first. If the parameters of the additional terms become 0 in the process of the optimization, the exact form of the equation is recovered. We have performed the optimization method for a generalized Lorenz equation. The input signal x_0 is generated by the Lorenz equation with $a_0=10$, $b_0=8/3$, and $c_0=28$. We assume a generalized Lorenz equation with the forcing term

$$\frac{dx}{dt}=a(-x+y)+dz+D[x_0(t)-x],$$

$$\frac{dy}{dt}=-xz+cz-y,$$

$$\frac{dz}{dt}=xy-bz, \tag{4}$$

where dz is an additional term. The initial parameter values are $a=3$, $b=1$, $c=5$, and $d=5$. The time evolution of the parameter values is displayed in Fig. 3(b). The parameter d for the additional term becomes 0 and the exact Lorenz equation is recovered after 15 000 steps.

We have applied the above method to the Lang-Kobayashi model [12]. The Lang-Kobayashi equation is a model for the semiconductor lasers. Chaotic output appears by the optical feedback in this model equation. The control of chaotic semiconductor lasers and the information transmission using laser systems are considered to be promising for the application of the chaotic dynamics [13]. The model equation is written as

$$\frac{dE_0}{dt}=\tfrac{1}{2}\{G_N[N_0(t)-N_s]-\gamma_c\}E_0(t)+\kappa_0E_0(t-\tau)\cos[\phi_0(t)-\phi_0(t-\tau)+\omega_0t],$$

$$\frac{d\phi_0}{dt}=\tfrac{\alpha}{2}\{G_N[N_0(t)-N_s]-\gamma_c\}-\kappa\frac{E_0(t-\tau)}{E_0(t)}\sin[\phi_0(t)-\phi_0(t-\tau)+\omega_0t],$$

$$\frac{dN_0}{dt}=J_0-\gamma_NN_0(t)-G_N[N_0(t)-N_s]E(t)^2, \tag{5}$$

where E_0 is the amplitude of the electrical field, ϕ_0 is the phase of the electrical field, N_0 is the carrier number inside the cavity, G_N is the gain parameter, α is the linewidth enhancement factor, γ_c is the decay constant of E by the photon life time, κ_0 is the feedback coefficient, τ is the external cavity roundtrip time, ω_0 is the angular frequency of the wave, N_s is the carrier number at transparency, γ_N is the decay constant for the carrier, $J_0 = I_0/e$ is the generation rate of the carrier by the bias current I and the bias current I is a control parameter. The parameter values are assumed in our numerical simulation to be $G_N = 1.5\times 10^{-8}$ ps^{-1}, $\alpha = 5$, $\gamma_c = 500$ ns^{-1}, $\kappa_0 = 30$ ns^{-1}, $\tau = 0.3$ ns, $\omega_0 \sim 1.2\times 10^3$ ps^{-1}, $N_s = 1.5\times 10^8$, and $\gamma_N = 0.5$ ns^{-1}. Figure 4 is a chaotic time sequence at $I = 44$ mA. Assuming that we do not know the parameter values of J and κ, we search for the parameter values of J and κ with the random optimization method. The model equation with chaotic forcing terms is written as

$$\frac{dE}{dt} = \tfrac{1}{2}\{G_N[N(t)-N_0]-\gamma_c\}E(t)+\kappa E(t-\tau)\cos[\phi(t) - \phi(t-\tau)+\omega_0 t]+D[E_0(t)-E(t)],$$

$$\frac{d\phi}{dt} = \frac{\alpha}{2}\{G_N[N(t)-N_0]-\gamma_c\}-\kappa\frac{E(t-\tau)}{E(t)}\sin[\phi(t) - \phi(t-\tau)+\omega_0 t]+D[\phi_0(t)-\phi(t)],$$

$$\frac{dN}{dt} = J-\gamma_N N(t)-G_N[N(t)-N_0]E(t)^2+D[N_0(t)-N(t)], \tag{6}$$

where all components are coupled and the coupling constant $D = 500$ ns^{-1}, and $E_0(t)$, $\phi_0(t)$, and $N_0(t)$ are chaotic time sequences which obey Eq. (5). Complete synchronization occurs at $J = J_0$ and $\kappa = \kappa_0$ for the coupling constant. We have measured the degree of chaos synchronization as $U = \int_0^T [E_0(t)-E(t)]^2 dt$ with $T = 1.2$ ns. The random optimization method is applied for this model. The initial values were set to be $\kappa = 20$ ns^{-1} and $J = 2\times 10^{17}$ s^{-1}. Figures 5(a) and 5(b) display the time evolution of κ and J. Figure 5(c) displays the time evolution of the distance U. The desirable parameters $\kappa_0 = 30$ ns^{-1} and $J = 2.74\times 10^{17}$ s^{-1} are obtained in this simulation.

In summary, we have proposed a random optimization method using chaos synchronization to evaluate parameters in nonlinear equations, and demonstrated the validity of the method with the Lorenz equation. We have checked the robustness of the method for the Lorenz equation. Even if some noises are overlapped to the chaotic signals, the parameter estimation is possible. Even if we do not know the exact form of the equation, the additional parameter becomes zero in the optimization process and the desirable equation can be recovered. We have applied the method to the Lang-Kobayashi model for the chaotic semiconductor laser, which may be important for the application of the chaotic dynamics.

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