Exactly solved dynamics for an infinite-range spin system. II. Antiferromagnetic interactions

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In a previous paper [E. Milotti, Phys. Rev. E **63**, 026116 (2001)]. I have shown how to derive both thermodynamical and dynamical properties of an infinite-range Ising spin system with binary ferromagnetic interactions from the master equation for magnetization obtained from a simple spin dynamics. The same method can be adapted to different spin interactions: here I discuss the case of antiferromagnetic interactions. This model permits a study of the static properties of the antiferromagnetic lattice, and it displays very clearly the differences between the antiferromagnetic and the ferromagnetic case with long-range interactions. The dynamical behavior of the antiferromagnetic system is simpler, and the magnetization always relaxes exponentially.

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In recent years many efforts have been concentrated on the study of the relaxational dynamics of complex systems, like spin glasses, glass-forming liquids, and other kinds of disordered systems. There are common features that emerge from these systems, in particular, it seems that the simple exponential dynamics makes place to some more complex power-law dynamics in the vicinity of critical points.

In a previous paper [1] I have studied a simple dynamical model for an Ising spin system with long-range ferromagnetic interactions and I have shown that the assumed spin dynamics (which essentially amounts to a single spin flip per time step) leads to a partial differential equation for the magnetization. This equation gives in turn an ordinary differential equation that describes the time evolution of the magnetization of the spin system: this ordinary differential equation is nonlinear, and displays a clear bifurcation, which accounts for the thermodynamic properties of the spin system. In addition the same equation gives a neat power-law relaxation near the critical temperature, while otherwise it gives an ordinary exponential relaxation.

In this paper 1 consider the case of antiferromagnetic interactions, i.e., I take the Hamiltonian

$$H=\frac{J}{N}\sum_{i,j}\sigma_1\sigma_j-h\sum_i\sigma_i, \tag{1}$$

and I use the same formalism of the previous paper. Just as it happens for ferromagnetic interactions, the properties of the antiferromagnetic system can be well understood with the formalism described in Ref. [1], but it turns out that now there is no direct relationship with bifurcation theory, and that the antiferromagnetic behavior is conceptually very different from the ferromagnetic behavior.

The method used in Ref. [1] can be easily adapted to the antiferromagnetic case: if a spin configuration σ has n "down" spins and $N-n$ "up" spins, then the magnetization in these long-range models is $M_n\equiv M(\sigma)=(1/N)(N-2n)$, and the energy is

$$\begin{aligned}E_n\equiv E(\sigma)&=-\frac{J}{N}n(N-n)+\frac{J}{2N}n(n-1)\\&\quad+\frac{J}{2N}(N-n)(N-n-1)-h(N-2n)\\&=\frac{J}{2N}[(N-2n)^2-N]-h(N-2n).\end{aligned} \tag{2}$$

Following Ref. [1] it can be shown that for large n the probability $P(x,t)$ of finding a fraction $x=n/N$ of the spins in the down direction at time t obeys the quasilinear partial differential equation,

$$\frac{1}{c}\frac{\partial P}{\partial t}+[1-2x-2f(x)]\frac{\partial P}{\partial x}=2[1+f'(x)]P(x,t), \tag{3}$$

where

$$f(x)=\frac{1-\{(1-x)\exp[2J\beta(1-2x)-2\beta b]+x\exp[-2J\beta(1-2x)+2\beta h]\}}{\frac{1-x}{x}\exp[2J\beta(1-2x)-2\beta h]-\frac{x}{1-x}\exp[-2J\beta(1-2x)+2\beta h]}, \tag{4}$$

and then, using the method of characteristics [2] we find the ordinary differential equation for x,

$$\frac{1}{c}\frac{dx}{dt}=1-2x-f(x). \tag{5}$$

Just as in the ferromagnetic long-range model [1], Eq. (5) describes the thermodynamic properties of the system as well as its dynamics, because the derivative (5) must vanish when the system reaches thermal equilibrium, i.e., the following system of equations must be satisfied:

$$y = 1 - 2x, \tag{6}$$

$$y = f(x),$$

where y is the magnetization in the large-N limit, and this system can be solved numerically for all values of the external field h.

The antiferromagnetic case is very different from the ferromagnetic case: while in the ferromagnetic case there was a supercritical pitchfork bifurcation [3], now in the symmetrical $h=0$ case, there is just one solution for all temperatures $[x=1/2, f(x)=0]$. Moreover it is easy to see that at very low temperature,

$$f(x) \approx -x \quad \text{if } x < \frac{1}{2}\left(1 - \frac{h}{j}\right), \tag{7}$$

and

$$f(x) \approx 1 - x \quad \text{if } x > \tfrac{1}{2}\left(1 - \frac{h}{J}\right), \tag{8}$$

while $f(0) = f(1) = 0$ at all temperatures.

Figure 1 shows plots of $f(x)$ for different values of h/J at a fixed temperature; notice that if $h=0$ the solution of Eq. (6) is $x = \frac{1}{2}$ (this is true at all temperatures), and that in this case the derivative of $f(x)$ at $x = \frac{1}{2}$ is just $J\beta$, so that at low temperature the function $f(x)$ is very steep near $x = \frac{1}{2}$.

The plots in Fig. 1 show that the steepness of the region of $f(x)$ with positive derivative is determined by the temperature of the system, while the position of this region is determined by the the magnetic field value, and moreover at very low temperature the function near $x = \frac{1}{2}(1 - h/J)$ is always very steep, so that the magnetization from the solution of the system of Eqs. (6) is just

$$M \approx 1 - 2x = \frac{h}{J}, \tag{9}$$

i.e., the magnetic susceptibility at low temperature is $\chi = M/h = 1/J = \text{constant}$. If $h/J > 1$ there is just one fixed solution $(x=0)$, so that $M=1$ and $\chi = 1/h$.

Both the magnetization and the static susceptibility can be computed numerically, and then one we also obtain the heat capacity per unit spin, which is

$$\frac{1}{N} C(T, h/J) = \frac{1}{N} \left.\frac{\partial E}{\partial T}\right|_{h/J} = \frac{1}{N} \left.\frac{\partial E}{\partial M}\right|_{h/J} \left.\frac{\partial M}{\partial T}\right|_{h/J}$$

$$= J\left(M(T, h/J) - \frac{h}{J} \right) \left.\frac{\partial M}{\partial T}\right|_{h/J}. \tag{10}$$

Since the magnetization is always 0 for $h=0$, then the heat capacity at zero field vanishes for all temperatures, and indeed the flatness of the heat capacity at zero field is one of the unrealistic features of the assumed long-range interaction, as already remarked in [4]. In a spin system with nearest-neighbor interactions the zero magnetization state can be either a high-temperature state in which all the spins

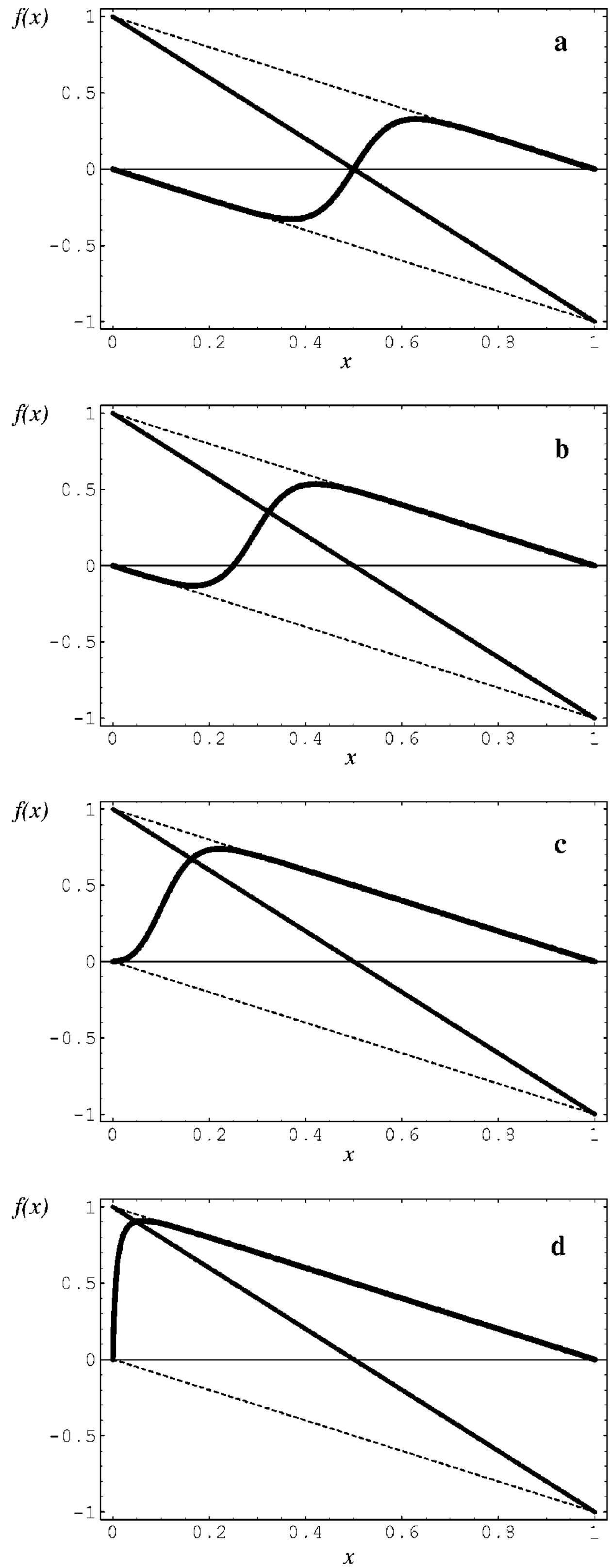

FIG. 1. Plots of the function $f(x)$ (thick solid line) at fixed temperature $(2J/kT = 10)$ and different values of the magnetic field: (a) $h=0$; (b) $h=0.5$; (c) $h=1$; (d) $h=1.5$. The central straight line (thin solid line) is the function $y = 1 - 2x$, and the upper and lower straight lines (dashed lines) are the functions $-x$ and $1-x$ [see Eqs. (7) and (8)].

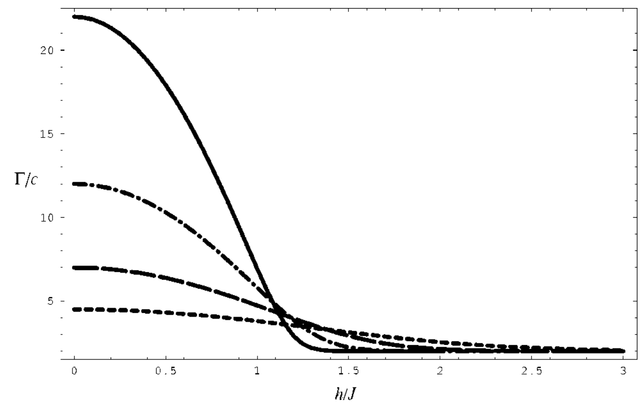

FIG. 2. Plot of the decay constant Γ vs h/J for different temperatures: $2J/kT=5$ (short dashes), $2J/kT=10$ (long dashes), $2J/kT=20$ (dashed dotted), and $2J/kT=40$ (solid curve).

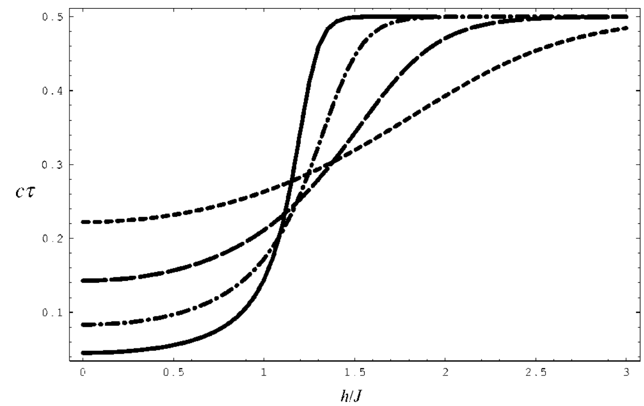

FIG. 3. Plot of the relaxation time τ vs h/J for different temperatures: $2J/kT=5$ (short dashes), $2J/kT=10$ (long dashes), $2J/kT=20$ (dashed dotted), and $2J/kT=40$ (solid curve).

are at random or a low-temperature one in which there are two ordered checkerboards of spins with opposite orientation and minimum total energy. In the present system with long-range interactions there is no such difference between high-temperature and a low-temperature zero magnetization states. This is no longer true when an external magnetic field is present: in this case the low-temperature configuration has a nonzero magnetization and can be distinguished from the high-temperature, random configuration.

The numerical calculations confirm the known results for all the static quantities of the antiferromagnetic system (thoroughly discussed in [4]), and in addition the ac susceptibility may also be derived.

It turns out that the dynamics of this antiferromagnetic system is very simple, i.e., if we let x_0 be the solution of the system of Eqs. (6) then, in the neighborhood of x_0 Eq. (5) becomes

$$\frac{1}{c}\frac{dx}{dt}\approx-[2+f'(x_0)](x-x_0). \tag{11}$$

and, therefore, the magnetization undergoes a simple exponential relaxation—with decay constant $\Gamma=c[2+f'(x_0)]$—for all temperatures and all field values, unlike the ferromagnetic case; Fig. 2 shows the decay constant for some selected cases, and Fig. 3 shows the corresponding relaxation time $\tau=1/\Gamma$. Notice that at low temperature there is a rather sharp transition from a fast relaxation at low field values to a slow relaxation at high field values: the transition between the two regimes takes place at $h/J=1$ [this is related to the limiting behavior (7) and (8)], and the lower the temperature, the faster the relaxation at low field values.

The approach described in Ref. [1] and applied here to the antiferromagnetic long-range case had already been tried long ago by Griffiths, Weng, and Langer [5], and further developed by other authors [6,7], however, here and in [1] both the dynamics and its connection with bifurcation theory have been clarified.

Both the ferromagnetic and the antiferromagnetic long-range Hamiltonians have been well studied in the standard thermodynamic framework [4,8,9], and the dynamical approach to magnetic systems has been pioneered by several other researchers in addition to the ones mentioned above (see, e.g., [10,11]), however, the method described in this and in the companion paper [1] is straightforward and intuitive.

Finally it is worthwhile to notice that in the antiferromagnetic case there is no phase transition and no Néel point, because there is no bifurcation in the dynamics, and the magnetization always follows a simple exponential relaxation law.

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