Generalized theory and simulation of spontaneous and super-radiant emissions in electron devices and free-electron lasers

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A unified formulation of spontaneous (shot-noise) and super-radiant emissions in electron devices is presented. We consider an electron beam with an arbitrary temporal current modulation propagating through the interaction region of the electronic device. The total electromagnetic field is presented as a stochastic process and expanded in terms of transverse eigenmodes of the medium (free space or waveguide), in which the field is excited and propagates. Using the waveguide excitation equations, formulated in the frequency domain, an analytical expression for the power spectral density of the electromagnetic radiation is derived. The spectrum of the excited radiation is shown to be composed of two terms, which are the spontaneous and super-radiant emissions. For a continuous, unmodulated beam, the shot noise produces only incoherent spontaneous emission of a power proportional to the flux eI_0 (DC current) of the particles in the electron beam. When the beam is modulated or prebunched, a partially coherent super-radiant emission is also produced with power proportional to the current spectrum $|I(\omega)|^2$. Based on a three-dimensional model, a numerical particle simulation code was developed. A set of coupled-mode excitation equations in the frequency domain are solved self-consistently with the equations of particles motion. The simulation considers random distributions of density and energy in the electron beam and takes into account the statistical and spectral features of the excited radiation. At present, the code can simulate free-electron lasers (FELs) operation in various modes: spontaneous and self-amplified spontaneous emission, super-radiance and stimulated emission, in the linear and nonlinear Compton or Raman regimes. We employed the code to demonstrate spontaneous and super-radiant emission excited when a prebunched electron beam passes through a wiggler of an FEL.

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I. INTRODUCTION

Electron devices such as microwave tubes and free-electron lasers (FELs) utilize distributed interaction between an electron beam and electromagnetic radiation. Random electron distribution in the e beam due to its corpuscular nature causes fluctuations in current density, identified as *shot noise* in the beam current [1–4].

The shot noise current is characterized by a ''white'' power spectrum whose density is proportional to the average electron flux eI_0 of the beam (e is the electron charge and I_0 is the DC current). The electromagnetic fields excited by each electron add incoherently, resulting in *spontaneous emission* noise in the radiation. If the electron beam is modulated or prebunched, the fields excited by electrons become correlated, and coherent summation of radiation fields from individual particles occurs. If all electrons radiate in phase with each other, the generated radiation becomes coherent (*super-radiant emission*). The terminology of super radiance was suggested in [5] for radiation emitted in a quantum mechanical system during a transition between two energy levels of molecules in a gas of dimension small compared to a wavelength.

Electrons passing through a magnetic undulator emit a partially coherent radiation, which is called *undulator synchrotron radiation* [6]. In the classical analysis, each wiggling electron is a point source, which can be treated as a moving radiating dipole. An individual electron moving in an undulator emits a wave packet of electromagnetic radiation, which is in synchronism with the electron velocity. If a continuous (unmodulated and unbunched) electron beam advances through a periodic field of a wiggler, the radiation fields radiated by electrons, which enter the undulator at random, add up incoherently. Since the radiation process takes place in the absence of externally applied electromagnetic radiation, it is termed spontaneous emission [7]. A number of approaches were employed for the analysis the FEL spontaneous emission in free space [8–15] and in waveguides [16]. In high-gain FELs, utilizing sufficiently long undulators, the spontaneous emission radiation excited in the first part of the undulator is amplified along the reminder of the interaction region (*self-amplified spontaneous emission*) [17–22]. Super-radiant emission emerges if the electrons are injected into the undulator in a single short bunch (shorter than the oscillation period of the emitted radiation) [23–28] or enter as a periodic train of bunches at the frequency of the emitted radiation [29–32].

In this paper, we analyze and find the total spontaneous and super-radiant emission for an electron beam of arbitrary current modulation passing through the interaction region of the electron device. A unified expression, describing the power spectral density of spontaneous and super-radiant emissions in terms of the device's interaction transfer function, is derived. The approach is valid in both, the linear and the nonlinear regimes. The analytical derivation leads to identification of two related terms describing the spectrum of

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the (incoherent) spontaneous and of the coherent spontaneous (super-radiant) emission.

We employ the three-dimensional (3D) coupled-mode theory for an analytical derivation of spontaneous and super-radiant emissions in the linear regime of FEL operation, and also for the development of a three-dimensional particle simulation code. Unlike a previously developed steady-state numerical model [38], in which the interaction is assumed to be at a single frequency (or at its discrete harmonics), the present approach considers a continuum of frequencies, enabling the solution of nonstationary, wide-band interaction in radiation devices. Solution of the excitation equations in the space-frequency domain (and not in the space-time domain, as often carried out in numerical particle simulation codes), inherently takes into account dispersive effects arising from the cavity and the gain medium. Furthermore, it facilitates the consideration of the statistical and spectral features of the electromagnetic field excitation process, necessary in a study of noncoherent and partially coherent effects, such as spontaneous and super-radiant emissions, self-amplified emission and noise, in the linear and nonlinear regimes of the FEL operation.

II. MODAL REPRESENTATION OF THE ELECTROMAGNETIC FIELD

The analysis is based on modal expansion of the total electromagnetic field in terms of transverse eigenmodes of the medium (free space or waveguide) in which the radiation is excited [33]. The field of each transverse mode q in the angular frequency domain ω is given by

$$\tilde{\mathbf{E}}_q(\mathbf{r},\omega)=\tilde{C}_q(z,\omega)\tilde{\mathcal{E}}_q(x,y)e^{+jk_{zq}(\omega)z}, \tag{1}$$

where $\tilde{\mathcal{E}}_q(x,y)$ is the transverse profile (Hermite-Gaussian free-space mode or waveguide mode) and $k_{zq}(\omega)$ is its wave number. (Although the form of mode presentation given in Eq. (1) is not valid in the far-field free-space propagation, it is still applicable to most electron devices in which the interaction takes place within a Rayleigh length of the Hermite-Gaussian modes, where the diffraction is small). $\tilde{C}_q(z,\omega)$ is the propagating mode amplitude satisfying the excitation equation

$$\frac{d}{dz}\tilde{C}_q(z,\omega)=-\frac{1}{2\mathcal{P}_q}e^{-jk_{zq}(\omega)z}\int\int \mathbf{J}(\mathbf{r},\omega)\cdot\mathcal{E}_q^*(x,y)dx\,dy. \tag{2}$$

$\mathbf{J}(\mathbf{r},\omega)$ is the Fourier transform of the current density defined in the positive frequency domain $\omega>0$, and $\mathcal{P}_q=1/2\,\mathrm{Re}\int\int_{\mathrm{c.s.}}[\tilde{\mathcal{E}}_{q\perp}\times\tilde{\mathcal{H}}^*_{q\perp}]\ \hat{z}\,dx\,dy$ is the normalization power of the propagating mode q.

According to the Wiener-Khinchine theorem, the power spectral density carried by the propagating mode during a temporal period T is found by averaging the ensemble of the radiation fields emitted by the electrons in the beam pulse [34]

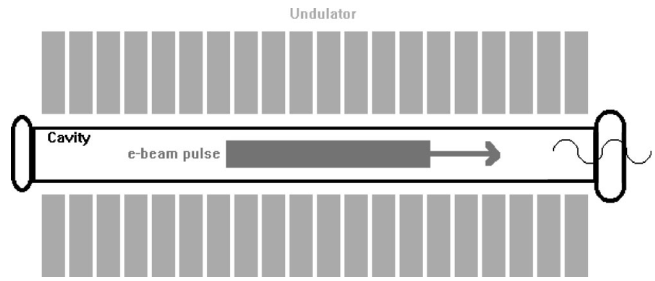

FIG. 1. Schematic illustration of a pulsed beam free-electron laser.

$$\frac{dP_q(z)}{d\omega}=\frac{1}{2\pi}\frac{1}{T}\overline{|\tilde{C}_q(z,\omega)|^2}\mathcal{P}_q \tag{3}$$

defined for $\omega>0$. The power spectrum is the Fourier transform of the space-time correlation function of the electromagnetic radiation, describing its coherence properties [35–37].

III. THE EXCITATION CURRENT

Consider an electron beam with a time-dependent current $i(t)$ entering the interaction region of an electron device. Figure 1 is an example illustrating an electron bunch passing through the undulator of a free-electron laser. The current density of k electrons in the bunch, each moving at an instantaneous velocity $\mathbf{v}_i$ is given in the space-time domain by

$$\mathbf{J}(\mathbf{r},t)=-e\sum_{i=1}^{k}\mathbf{v}_i\,\delta(x-x_i)\,\delta(y-y_i)\,\delta(z-z_i(t)), \tag{4}$$

where e is the electron charge, k is the number of particles in an electron beam pulse, and (x_i,y_i,z_i) are the coordinates of the ith electron's position at a time t.

The number k of electrons in the pulse is a random variable satisfying the Poisson distribution. The probability that the e-beam pulse contains k electrons is

$$p(k)=\frac{1}{k!}(\bar{k})^k e^{-\bar{k}}, \tag{5}$$

where

$$\bar{k}=\frac{1}{e}\int_{-\infty}^{+\infty}i(t)dt \tag{6}$$

is the expected number (statistical average) of electrons in the pulse with variance $\overline{k^2}-(\bar{k})^2=\bar{k}$ (equal to the average).

It is convenient to take z, the coordinate of axial propagation, as the independent variable, and write the time of particle arrival at z as

$$t_i(z)=t_{0i}+\int_0^z\frac{1}{v_{zi}(z')}dz'. \tag{7}$$

t_{0i} is the time that a particle i entered at $z=0$ and $v_{zi}(z)$ is its axial velocity along the path of motion. The electron arrival times t_{0i} at the entrance to the interaction region are independent random variables having a probability density function

$$p(t_{0i}) = \frac{1}{\overline{k}e} i(t_{0i}). \tag{8}$$

To find the electromagnetic field radiated from the moving charge, it is first required to calculate the Fourier transform of the current density as given in Eq. (4)

$$\mathbf{J}(\mathbf{r},\omega) = \int_{-\infty}^{+\infty} \mathbf{J}(\mathbf{r},t) e^{+j\omega t} dt = -e \sum_{i=1}^{k} \frac{\mathbf{v}_i}{v_{zi}} \delta(x-x_i)\,\delta(y-y_i) e^{+j\omega t_i(z)}, \tag{9}$$

and substitute it into the excitation Eq. (2), resulting in

$$\frac{d}{dz}\tilde{C}_q(z,\omega) = \frac{e}{2\mathcal{P}_q} \sum_{i=1}^{k} \frac{1}{v_{zi}} \mathbf{v}_i \cdot \tilde{\mathcal{E}}_q^*(x_i,y_i) e^{+j[\omega t_i(z) - k_{zq} z]}. \tag{10}$$

IV. SPONTANEOUS AND SUPER-RADIANT EMISSIONS

The solution of the excitation equation at point z in the interaction region is found by integration of Eq. (10), assuming $\tilde{C}_q(0,\omega)=0$

$$\tilde{C}_q(z,\omega) = e \sum_{i=1}^{k} \mathcal{H}_{q_i}(z,\omega) e^{+j\omega t_{0i}}, \tag{11}$$

where the electron-field excitation "transfer function" is defined by

$$\mathcal{H}_{q_i}(z,\omega) \equiv \frac{1}{2\mathcal{P}_q} \int_0^z \frac{1}{v_{zi}} \mathbf{v}_i \cdot \tilde{\mathcal{E}}_q^*(x_i,y_i) \times \exp\left[j\left(\omega \int_0^{z'} \frac{1}{v_{zi}(z'')} dz'' - k_{zq} z' \right) \right] dz'. \tag{12}$$

The function $\mathcal{H}_{q_i}(z,\omega)$ fully describes the interaction of the ith electron with the electromagnetic field along its path of motion. It expresses both linear (low and high gain) and nonlinear (saturation) regimes of the electron device operation. Ensemble of $\mathcal{H}_{q_i}(z,\omega)$ of the total k electrons in the e beam is a stochastic process taking into account statistical distributions in phase space (spatial distribution, energy, and angular spreadings).

Solution of the mode amplitude $\tilde{C}_q(z,\omega)$ given in Eq. (11) [using expression (12)] together with equation of motion

$$\frac{d}{dz}(\gamma_i \mathbf{v}_i) = -\frac{e}{m} \frac{1}{v_{z_i}} [\mathbf{E}(\mathbf{r}_i,t) + \mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i,t)] \tag{13}$$

enables one to calculate spontaneous and super-radiant emissions, including stimulated interaction resulting in amplification of the excited radiation as occurring in self-amplified spontaneous emission (SASE).

The spectral density of the radiation power emitted by a stream of electrons during a temporal period T is calculated by substitution of Eq. (11) into Eq. (3)

$$\frac{dP_q(z)}{d\omega} = \frac{1}{2\pi} \frac{e^2}{T} \left[\overline{\sum_{i=1}^{k} |\mathcal{H}_{q_i}(z,\omega)|^2 + \sum_{i=1}^{k} \sum_{i' \neq i} \mathcal{H}_{qi}(z,\omega) \mathcal{H}_{q_{i'}}^*(z,\omega) e^{j\omega(t_{0i} - t_{0i'})}} \right] \mathcal{P}_q. \tag{14}$$

After performance of statistical averaging of Eq. (14)

$$\frac{dP_q(z)}{d\omega} = \frac{1}{2\pi} \frac{e^2}{T} \{ \overline{k} \overline{|\mathcal{H}_{q_i}(z,\omega)|^2} + [\overline{k(k-1)}] \times [\overline{\mathcal{H}_{q_i}(z,\omega) \mathcal{H}_{q_{i'}}^*(z,\omega)}] \overline{e^{j\omega(t_{0i}-t_{0i'})}} \} \mathcal{P}_q. \tag{15}$$

The expected values

$$\overline{e^{+j\omega t_{0i}}} = (\overline{e^{-j\omega t_{0i'}}})^* = \int_{-\infty}^{+\infty} e^{+j\omega t_{0i}} p(t_{0i}) dt_{0i} = \frac{1}{\overline{k}e} \int_{-\infty}^{+\infty} i(t_{0i}) e^{+j\omega t_{0i}} dt_{0i} = \frac{1}{\overline{k}e} \mathcal{I}(\omega) \tag{16}$$

are given in terms of the Fourier transform $\mathcal{I}(\omega) = \int_{-\infty}^{+\infty} i(t) e^{+j\omega t} dt$ of the current. Note that according to Eq. (6), the statistical average of the number of electrons in the pulse can be written as $\overline{k} = 1/e\mathcal{I}(0)$, where $\mathcal{I}(0) \equiv \mathcal{I}(\omega=0) = \int_{-\infty}^{+\infty} i(t) dt$. Using these expressions in Eq. (15), the power spectrum of the radiation is given by

$$\frac{dP_q(z)}{dw} = \frac{1}{2\pi} \frac{1}{T} [e \overline{|\mathcal{H}_{q_i}(z,\omega)|^2} \mathcal{I}(0) + \overline{\mathcal{H}_{q_i}(z,\omega) \mathcal{H}_{q_{i'}}^*(z,\omega)} |\mathcal{I}(\omega)|^2] \mathcal{P}_q. \tag{17}$$

The first term in Eq. (17) is identified as the expression for the incoherent shot-noise (spontaneous-emission) spectrum

$$\frac{dP_q^{\mathrm{sp}}(z)}{df} = \frac{1}{T} e \overline{|\mathcal{H}_{q_i}(z,f)|^2} \mathcal{I}(0) \mathcal{P}_q, \tag{18}$$

while the second term corresponds to the spectral density of the super-radiant power

$$\frac{dP_q^{\rm sr}(z)}{df}=\frac{1}{T}\overline{\mathcal{H}_{q_i}(z,f)\mathcal{H}^*_{q_{i'}}(z,f)}|\mathcal{I}(f)|^2P_q. \tag{19}$$

In the linear, low-gain interaction regime $\overline{\mathcal{H}_{q_i}(z,f)\mathcal{H}^*_{q_{i'}}(z,f)}\simeq\overline{|\mathcal{H}_{q_i}(z,f)|^2}$ and the relation between the power spectral density of the spontaneous and super-radiant emissions can be written in the form

$$e\mathcal{I}(0)\frac{dP_q^{\rm sr}(z)}{df}=|\mathcal{I}(f)|^2\frac{dP_q^{\rm sp}(z)}{df}. \tag{20}$$

V. SPONTANEOUS EMISSION AND SUPER-RADIANCE IN FREE-ELECTRON LASERS

In electron passage through the periodic field of an undulator (see Fig. 1), its total velocity vector $\mathbf{v}_i$ consists of a transverse wiggling component of amplitude $\tilde{\mathcal{V}}_\perp^w$, which is due to the Lorentz force, in addition to a longitudinal axial velocity v_{zi}

$$\mathbf{v}_i(z)=\hat{\mathbf{z}}v_{zi}(z)+\mathrm{Re}\{\hat{\mathcal{V}}_\perp^w e^{-jk_wz}\}. \tag{21}$$

$k_w=2\pi/\lambda_w$ where λ_w is the wiggler period. The electron transverse trajectory in the wiggler is given by

$$\mathbf{r}_{\perp i}(z)=\overline{\mathbf{r}}_{\perp i}+\mathrm{Re}\{\overline{\mathbf{r}}_{\perp w}e^{-jk_wz}\}, \tag{22}$$

where $\overline{\mathbf{r}}_{\perp i}=(\overline{x_i},\overline{y_i})$ describes the average (over a wiggling period) transverse coordinates of the electron and $\tilde{\mathbf{r}}_{\perp w}=j\tilde{\mathcal{V}}_\perp^w/k_wv_{z0}$ is the amplitude of transverse displacement of the wiggling electron trajectory ("quiver").

Substitution of the expressions for the electron velocity (21) and trajectory (22) in Eq. (12) results in

$$\mathcal{H}_{q_i}(z,\omega)=\frac{\zeta_q}{4\mathcal{P}_q}\frac{1}{v_{zi}}\tilde{\mathcal{V}}_\perp^w\cdot\tilde{\mathcal{E}}^*_{q\perp}(\overline{x_i},\overline{y_i})e^{+j[\int_0^z\theta_{q_i}(z',\omega)dz']}, \tag{23}$$

where we define

$$\zeta_q\equiv\begin{cases}1 & \text{for } TE \text{ modes,}\\ 1-\dfrac{k_{\perp q}^2}{k_{eq}k_w} & \text{for } TM \text{ modes,}\end{cases} \tag{24}$$

and

$$\theta_{q_i}(z,\omega)\equiv\frac{\omega}{v_{zi}(z)}-(k_{zq}+k_w) \tag{25}$$

is the detuning parameter of the ith electron at position z.

When the effect of electromagnetic radiation on electron motion is low, resulting in negligible amplification of the excited radiation (low-gain regime), it is assumed that all electrons in the beam move at a constant (averaged over wiggler period) axial velocity $v_{zi}(z)=v_{z0}$ and that they maintain their initial detuning parameter θ_q along the wiggler. Consequently, the solution of Eq. (12) at the exit of a wiggler of length L_w is found to be

$$\mathcal{H}_{q_i}(L_w,\omega)=\mathcal{A}_{q_i}\,\mathrm{sinc}(\tfrac{1}{2}\theta_qL_w)e^{j1/2\theta_qL_w} \tag{26}$$

where $\mathcal{A}_{q_i}\equiv(\zeta_q/4\mathcal{P}_q)(L_w/v_{z0})\tilde{\mathcal{V}}_\perp^w\cdot\tilde{\mathcal{E}}^*_{q\perp}(\overline{x_i},\overline{y_i})$ and $\mathrm{sinc}(x)\equiv\sin(x)/x$.

According to Eq. (17), the spectral density of the radiation power emitted by the stream of electrons passing through the FEL undulator is

$$\frac{dP_q(L_w)}{d\omega}=\frac{1}{2\pi}\frac{1}{T}[e\overline{|\mathcal{A}_{q_i}|^2}\mathcal{I}(0)+\overline{\mathcal{A}_{q_i}\mathcal{A}^*_{q_{i'}}}|\mathcal{I}(\omega)|^2]\mathcal{P}_q\mathrm{sinc}^2(\tfrac{1}{2}\theta_qL_w). \tag{27}$$

The first term in Eq. (27) is the spontaneous emission spectrum

$$\frac{dP_q^{\rm sp}(L_w)}{df}=\tau_{\rm sp}P_q^{\rm sp}(L_w)\mathrm{sinc}^2(\tfrac{1}{2}\theta_qL_w), \tag{28}$$

where $P_q^{\rm sp}(L_w)=1/(T\tau_{\rm sp})e\mathcal{I}(0)\overline{|\mathcal{A}_{q_i}|^2}\mathcal{P}_q$ is the total spontaneous emission power carried by the transverse mode q, and $\tau_{\rm sp}=|L_w/v_{z0}-L_w/v_g|$ is the *slippage* time. The second term in Eq. (27) corresponds to the spectral density of the super-radiant power. Assuming that $\overline{\mathcal{A}_{q_i}\mathcal{A}^*_{q_{i'}}}\simeq\overline{|\mathcal{A}_{q_i}|^2}$, it can be written as

$$\frac{dP_q^{\rm sr}(L_w)}{df}=\frac{|\mathcal{I}(f)|^2}{e\mathcal{I}(0)}\frac{dP_q^{\rm sp}(L_w)}{df}=\frac{1}{e\mathcal{I}(0)}\tau_{\rm sp}P_q^{\rm sp}(L_w)|\mathcal{I}(f)|^2\mathrm{sinc}^2(\tfrac{1}{2}\theta_qL_w). \tag{29}$$

VI. RADIATION FROM A SINGLE BUNCH

We consider an electron beam pulse having a temporal Gaussian current shape

$$i(t)=\frac{I_0}{\sqrt{2\pi}}e^{-t^2/2T^2}, \tag{30}$$

where T is the temporal "standard deviation" of the pulse. The Fourier transform of the current distribution is given by a Gaussian function in the frequency domain

$$\mathcal{I}(f)=I_0Te^{-1/2(2\pi Tf)^2}. \tag{31}$$

For $f=0$, the Fourier transform results in $\mathcal{I}(0)=I_0T$. The power spectral density of the FEL spontaneous emission is given by Eq. (28), where the total spontaneous emission power is $P_q^{\rm sp}(L_w)=1/\tau_{\rm sp}eI_0\overline{|\mathcal{A}_{q_i}|^2}\mathcal{P}_q$. In a FEL, utilizing a magnetostatic planar wiggler, the transverse wiggling amplitude is $\mathcal{V}_\perp^w=a_wc/\gamma$ (where $a_w=eB_w/mck_w$ is the wiggler parameter) and the total power of the spontaneous emission is given by

TABLE I. The operational parameters of millimeter wave free-electron maser.

Accelerator	
Electron beam energy:	$E_k = 1.375\,\mathrm{MeV}$
Electron beam current:	$I_0 = 1\,\mathrm{A}$
Wiggler	
Magnetic induction:	$B_w = 2000\,\mathrm{G}$
Period:	$\lambda_w = 4.444\,\mathrm{cm}$
Number of periods:	$N_w = 20$
Waveguide	
Rectangular waveguide:	$1.01\,\mathrm{cm} \times 0.9005\,\mathrm{cm}$
Mode:	TE_{01}

$$P_q^{\mathrm{sp}}(L_w) = \frac{1}{8}\,\frac{eI_0}{\tau_{\mathrm{sp}}}\left(\frac{a_w}{\gamma\beta_{z_0}}\right)^2 \frac{\xi_q^2 Z_q}{A_{em_q}} L_w^2 \tag{32}$$

where Z_q is the mode impedance and

$$A_{em_q} = \frac{\int\int |\tilde{\mathcal{E}}_q(x,y)|^2 dx\,dy}{|\tilde{\mathcal{E}}_q(0,0)|^2}$$

is its effective area. The spectrum of the super-radiant power radiated from the electron bunch is

$$\frac{dP_q^{\mathrm{sr}}(L_w)}{df} = \overline{k}\tau_{\mathrm{sp}} P_q^{\mathrm{sp}}(L_w) e^{-(2\pi T f)^2} \mathrm{sinc}^2(\tfrac{1}{2}\theta_q L_w), \tag{33}$$

where $\overline{k} = I_0 T/e$ is the expected number of electrons in the pulse.

We shall investigate spontaneous and super-radiant emissions radiated when an electron pulse passes through a wiggler of a FEL having operational parameters as given in Table I. Figure 2 shows that the beam and waveguide dispersion curves intercept at two points corresponding to the upper- and lower-synchronism frequencies—100 and 29 GHz, respectively. In such FEL schemes, the electron bunch is emitted when a photocathode is illuminated by a pulsed UV laser radiation [23–28]. Utilizing state-of-art femptosecond UV laser system, enables the generation of ultra-short e-beam bunches with duration of less than a period of the FEL radiation at millimeter wavelength and thus demonstrate super-radiant emission at this regime.

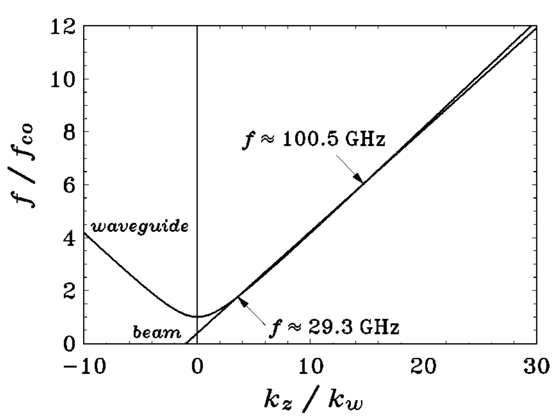

FIG. 2. FEL dispersion curves.

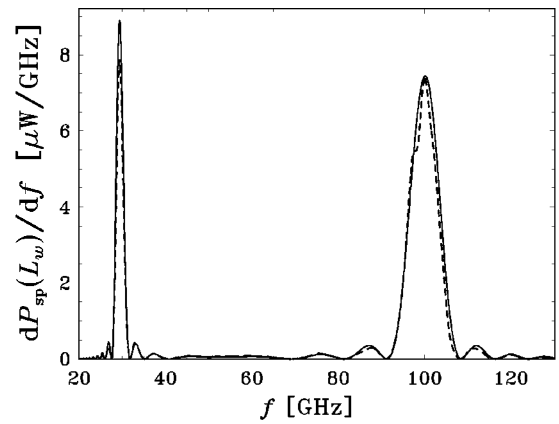

FIG. 3. Power spectral density of spontaneous emission. Analytical calculations (solid line) and numerical simulation (dashed line).

Analytical and numerical calculations of the resulting spontaneous emission spectrum are drawn in Fig. 3. The dashed line curves results from a particle simulation WB3D code, which is based on a three-dimensional, space-frequency model, utilizing an expansion of the total electromagnetic field (radiation and space-charge waves) in terms of transverse eigenmodes of the waveguide. Since shot noise is proportional to the particle charge [see Eq. (18)], the spontaneous emission spectrum obtained from N particles simulation is $\overline{k}/N$ times that of the spontaneous emission resulting from $\overline{k}$ expected electrons in the e-beam pulse.

In the following, we shall focus our attention on radiation near the upper-synchronism frequency $f_0 = 100\,\mathrm{GHz}$ and calculate the power spectrum of the radiation emitted when a pulse of electrons of temporal length T passes through the wiggler. Figure 4 shows a curve of the spectral distribution of radiation energy $dW_q^{\mathrm{sr}}(L_w)/df = T[dP_q^{\mathrm{sr}}(L_w)/df]$ emitted when the bunch period T is smaller than the temporal period $1/f_0$ of the signal. For $Tf_0 = 0.1$, the super-radiant emission power is observed to be much higher than the power of spon-

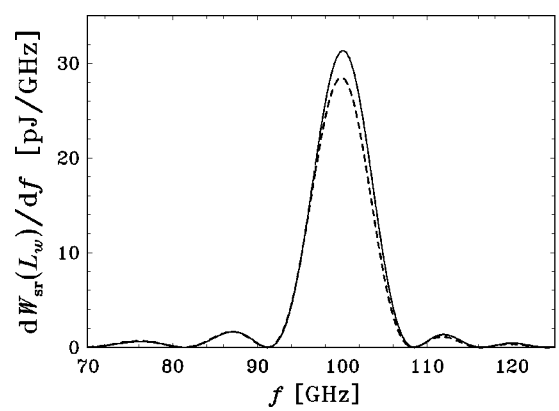

FIG. 4. Energy flux spectral density of super-radiant emission from a short $T = 1$ pS electron bunch ($Tf_0 = 0.1$). Analytical calculations (solid line) and numerical simulation (dashed line).

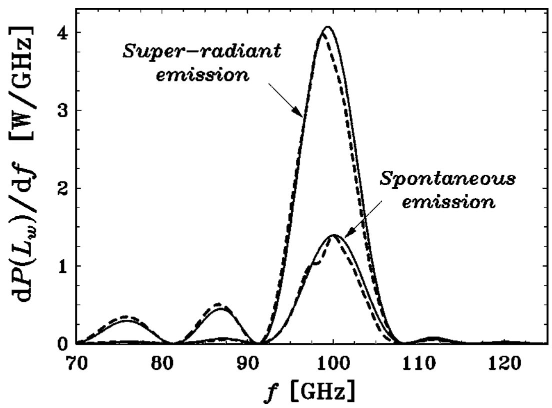

FIG. 5. Power spectral density of spontaneous and super-radiant emissions from $T=3$ pS electron bunch ($Tf_0=0.3$). Analytical calculations (solid line) and numerical simulation (dashed line).

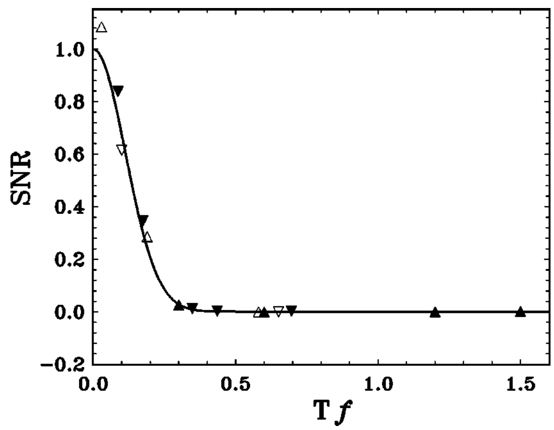

FIG. 6. Signal-to-noise ratio (SNR) for a Gaussian pulsed beam. The triangles on the graph correspond to numerical simulation results.

taneous emission (by a factor $\overline{k}=I_0T/e=6.25\times 10^6$—the number of electrons expected in the pulse). The power spectrum peaks at f_0, where the condition of phase matching occurs. The bandwidth of the main lobe of the energy spectral distribution function is approximately $\Delta f \approx 1/\tau_{\mathrm{sp}} \simeq f_0/N_w$. If the e-beam bunch is much shorter than a wavelength, the total energy of the super-radiant wave packet is given in terms of the spontaneous emission power by $\overline{k}TP_q^{\mathrm{sp}}(L_w)$.

The super-radiant power decreases as compared to the spontaneous emission power as the pulse duration T is increased. Figure 5 shows the power spectra of spontaneous and super-radiant emissions for an intermediate case, where the e-beam pulse duration is slightly smaller than the temporal period of the signal $Tf_0=0.3$ ($\overline{k}=1.875\times 10^7$ electrons in the pulse). For long e-beam pulses, the power of the super-radiant emission is reduced below the spontaneous (shot-noise) power and diminishes as $T\rightarrow\infty$. The *signal-to-noise ratio* (SNR) is the relation between the super-radiant and spontaneous emissions power spectra. Using Eq. (20) for the case of a Gaussian pulsed electron beam, the signal-to-noise ratio can be written as

$$\frac{dP_q^{\mathrm{sr}}(z)/df}{dP_q^{\mathrm{sp}}(z)/df}=\overline{k}e^{-(2\pi Tf)^2}. \tag{34}$$

The graph of the signal-to-noise ratio is shown in Fig. 6. The triangles correspond to the results of numerical simulations with particles numbers $N=100$ (empty symbols) and $N=1000$ (solid symbols) at synchronism frequencies $f_0=29$ GHz (triangles down) and $f_0=100$ GHz (triangles up).

We use the model to investigate the evolution of the total spontaneous emission power generated when a long e-beam pulse ($Tf_0=10$) is passing through a wiggler. Figure 7 shows the power growth along the wiggler as a function of the wiggling periods N_w. In the first few periods, the spontaneous radiation is excited from short noise in the electron beam and its power increases proportional to N_w^2 [see Eq. (32)]. Within this stage, the mutual interaction between the electromagnetic radiation and the electron beam is small, the power amplification is low (low-gain regime) and the power growth follows the analytical solution given in Eq. (32). An exponential growth of SASE is inspected later after passing a sufficient number of periods, revealing that the interaction enters to the high gain regime, until saturation occurs when arriving to the nonlinear regime of the FEL operation.

VII. PERIODIC BUNCHING

Assume a continuous electron beam with sinusoidal modulated current at frequency f_0 [39]

$$i(t)=I_0[1+m\cos(\omega_0 t)], \tag{35}$$

where I_0 is the average DC current and m is the modulation depth. In that case,

$$\lim_{T\rightarrow\infty}\tfrac{1}{T}\mathcal{I}(f=0)=\mathcal{I}_0$$

and

$$\lim_{T\rightarrow\infty}\tfrac{1}{T}|\mathcal{I}(f)|^2=I_0^2\left[\delta(f)+\frac{m^2}{4}\delta(f-f_0)+\frac{m^2}{4}\delta(f+f_0)\right].$$

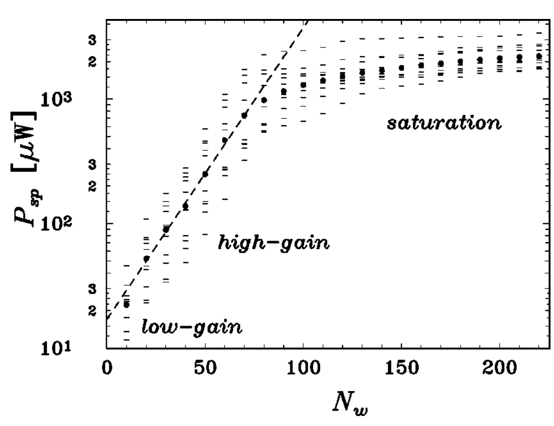

FIG. 7. Evolution of spontaneous emission power along the wiggler (statistical distribution is shown by dashes and the average by bullets).

Using these relations, the power spectral density of the spontaneous emission is found to satisfy Eq. (28), where the total spontaneous emission power is $P_q^{\mathrm{sp}}(L_w) = 1/\tau_{\mathrm{sp}} e I_0 \overline{|\mathcal{A}_{q_i}|^2} \mathcal{P}_q$. The power spectral density of the super-radiant emission is found from (29) to be

$$\frac{dP_q^{\mathrm{sr}}(L_w)}{df} = \frac{m^2}{4}\frac{I_0}{e}\tau_{\mathrm{sp}}P_q^{\mathrm{sp}}(L_w)\mathrm{sinc}^2\left(\frac{1}{2}\theta_q(f_0)L_w\right)\delta(f-f_0). \tag{36}$$

The total power of the super-radiant emission is given by

$$P_q^{\mathrm{sr}}(L_w) = \frac{m^2}{4}\frac{I_0}{e}\tau_{\mathrm{sp}}P_q^{\mathrm{sp}}(L_w)\mathrm{sinc}^2\left(\frac{1}{2}\theta_q(f_0)L_w\right). \tag{37}$$

Figure 8 shows the super-radiant power as a function of prebunching frequency f_0 for various modulation levels. A comparison is made with simulation results. Super-radiant power emitted by an infinite series of ultra-short bunches (impulses) is also shown. In this case, the current can be expanded in a Fourier series

$$i(t) = \sum_{n=-\infty}^{+\infty} I_0 T\delta(t-nT) = I_0\left[1 + 2\sum_{n=-\infty}^{+\infty}\cos\left(n\frac{2\pi}{T}t\right)\right]. \tag{38}$$

The resulting spectrum of super-radiant emission contains all harmonics of the prebunching frequency $f_0 = 1/T$ each having a sinusoidal current modulation with modulation index $m=2$. Figure 8 shows a curve of the super-radiant power emitted by a series of impulses as a function of the fundamental modulation frequency f_0. The discrepancy between analytical calculations and numerical simulations at high-modulation levels is due to stimulated emission effects that arise in the simulations, but not taken into account in the analytical calculations (where the effect of the radiation on electrons in not considered).

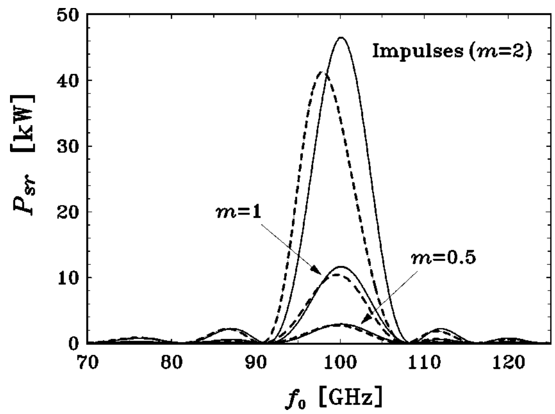

FIG. 8. Power spectral density of super-radiant emission from a sinusoidally modulated current and from an infinite series of ultra short bunches. Analytical calculations (solid lines) and numerical simulation (dashed lines).

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