Mean electromotive force in turbulent shear flow

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We consider the mean electromotive force in turbulent shear flow taking into account the stretching of turbulent magnetic field lines by the mean flow. The mean flow can change the properties of magnetohydrodynamics-turbulence in such a way that turbulent motions become suitable for the dynamo action. The contribution of shear to the mean electromotive force cannot be described in terms of the alpha effect. The instability of the mean field arises if shear is sufficiently strong. The growth rate of instability depends on the length scale of the mean field being higher for the field with a smaller length scale. The considered mechanism may be responsible for the generation of large-scale magnetic fields in various astrophysical bodies (galaxies, accretion discs, jets, etc.).

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I. INTRODUCTION

It is widely believed that, under certain conditions, turbulent fluid motions may amplify a large-scale magnetic field from a weak seed field (see, e.g., [1,2]). The generally accepted point of view is that turbulent motions showing a lack of the reflection symmetry are suitable for this process [3]. In rotating fluid, the Coriolis force may break the reflection symmetry of turbulent motions in such a way that the resulting mean electromotive force has a component proportional to the mean magnetic field (the alpha effect). The alpha effect is often attributed to the generation of large-scale magnetic fields in various astrophysical bodies (see, e.g., [4–6]).

However, possible mechanisms of the generation of large-scale magnetic fields in turbulence obviously cannot be represented by the alpha effect alone. Recently, a mechanism has been proposed by Yoshizawa, Yokoi, and Kato [7–9]) who considered the transport properties of inhomogeneous turbulence by making use of a two-scale direct-interaction approximation. These authors argued that the induction equation for the mean magnetic field should be supplemented by a source term proportional to the product of cross helicity and mean vorticity. This term plays the role of an effective turbulent battery that may produce a large-scale electric current and, hence, the mean magnetic field. In the proposed mechanism, the mean field may be induced, for example, by a large-scale rotational motion in the presence of the cross correlation between the small-scale velocity and magnetic field. This mechanism has also been considered by Blackman [10], who argued that the source term should be represented also in the mean induction equation for sheared rotators.

Apart from the alpha effect and the cross-helicity effect, qualitatively different mechanisms of the generation of the mean field may operate in the presence of shear. Shear changes the intrinsic properties of magnetohydrodynamics turbulence, stretching turbulent magnetic field lines, and due to this, the behavior of the mean field in shear flow differs from that predicted by simplified models taking account of the alpha effect alone. By making use of a two-scale approximation, it has been argued by Urpin [11,12] that additional terms appear in the mean electromotive force that are proportional to the production of spatial derivatives of the mean velocity $\vec{V}$ and the mean magnetic field $\vec{B}$. Note that despite the fact that additional terms are proportional to $\partial B_i/\partial x_j$, the effect of shear cannot generally be expressed in terms of anisotropic magnetic diffusivity because the latter enters the mean electromotive force only in products with the components of $\vec{\nabla}\times\vec{B}$. The additional terms change the type of equation governing the mean magnetic field and may result in a turbulent dynamo mechanism. Note that the presence of shear-induced terms in the mean electromotive force first has been shown by Hoyng [13] who considered the particular case of differential rotation with $\Omega=\Omega(z)$.

The shear-driven dynamo may effectively operate even in the simplest case of a plane Couette flow [12]. The proposed mechanism differs qualitatively from the conventional alpha dynamo. For instance, the shear-driven dynamo may generate only the field that is inhomogeneous in the direction of flow. Also, the considered mechanism does not require a large-scale inhomogeneity of turbulence that is absolutely necessary for the alpha dynamo because the pseudoscalar α may be formed from the axial vector of angular velocity only as a scalar product with some polar vector. In contrast to the alpha dynamo, the shear-driven mechanism allows the generation of two-dimensional fields [12].

In the present paper, we consider in detail the shear-driven mean electromotive force in a plane Couette flow. As a tool for our consideration, we use a simple kinematic model neglecting the influence of a generated magnetic field on turbulence that depends very much on assumptions regarding the origin of turbulence. This allows one to concentrate upon the main qualitative features of a generation mechanism. We generalize for the case of an arbitrary plane flow the nonlocal mean-field approach proposed by Rüdiger and Urpin [14] for the simplest case of a Couette flow with linear shear. This approach allows us to consider the mean-field electrodynamics without attributing a two-scale approximation, which is hardly fulfilled in real conditions. We show that even homogeneous turbulence with mirror symmetry can lead to a generation of the mean magnetic field.

II. THE MEAN TURBULENT ELECTROMOTIVE FORCE

Decompose magnetic-field $\vec{\mathcal{B}}$ and velocity $\vec{u}$ into the mean and fluctuating parts, $\vec{\mathcal{B}}=\vec{B}+\vec{b}$ and $\vec{u}=\vec{V}+\vec{v}$, where $\vec{B}$ and $\vec{V}$ are the mean field and velocity, respectively. The mean induction equation reads,

$$\frac{\partial \vec{B}}{\partial t}=\vec{\nabla}\times(\vec{V}\times\vec{B})+\eta\Delta\vec{B}+\vec{\nabla}\times\vec{\mathcal{E}}, \tag{2.1}$$

where η is the magnetic viscosity and $\vec{\mathcal{E}}=\langle\vec{v}\times\vec{b}\rangle$ is the mean electromotive force; $\langle\ \dots\ \rangle$ denotes ensemble averaging. We consider mean electromotive force $\vec{\mathcal{E}}$ in a quasilinear approximation. In this approximation, mean quantities are governed by equations including nonlinear effects in fluctuating terms, while the linearized equation is used for the fluctuating quantities [3]. A quasilinear approximation is sufficiently accurate, for example, to describe an ensemble of turbulent motions with relatively high frequencies and small amplitudes when the Strouhal number $S=v\tau/\ell$ is small; here, τ and ℓ are the correlation time and the lengthscale of turbulence, respectively; v is the turbulent velocity.

We assume that the magnetic Reynolds number is very large for turbulent motions, thus we may neglect the dissipative term in the induction equation. In what follows, we will need only the sign of the dissipative terms to properly choose the integration path when calculating Fourier integrals. Then, the linearized induction equation for the fluctuating magnetic field reads

$$\frac{\partial \vec{b}}{\partial t}=\vec{\nabla}\times(\vec{V}\times\vec{b})+\vec{\nabla}\times(\vec{v}\times\vec{B}). \tag{2.2}$$

Consider a flow with the mean velocity given by $\vec{V}=V(x)\vec{e}_y$ where x, y, and z are the Cartesian coordinates; $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ are the unit vectors. Substituting this expression into Eq. (2.2), we have

$$\frac{\partial \vec{b}}{\partial t}+V(x)\frac{\partial \vec{b}}{\partial y}-\vec{e}_y b_x V'(x)=\vec{A}, \tag{2.3}$$

where

$$\vec{A}=(\vec{B}\cdot\vec{\nabla})\vec{v}-(\vec{v}\cdot\vec{\nabla})\vec{B}, \tag{2.4}$$

and $V'(x)=dV/dx$. The fluid is assumed to be incompressible.

Equation (2.3) may be solved by making use of a partial Fourier transformation. Since coefficients in Eq. (2.3) do not depend on y and z, we make initial transformations in these coordinates and obtain the equation for the quantity

$$\hat{\vec{b}}_1(x,k_y,k_z,t)=\frac{1}{(2\pi)^2}\int dy\,dz\,e^{ik_y y+ik_z z}\vec{b}(\vec{r},t). \tag{2.5}$$

This equation reads

$$\frac{\partial \hat{\vec{b}}_1}{\partial t}-ik_yV(x)\hat{\vec{b}}_1-\vec{e}_yV'(x)\hat{b}_{1x}=\hat{\vec{A}}_1, \tag{2.6}$$

where $\hat{\vec{A}}_1$ is the corresponding Fourier amplitude of $\vec{A}$. Then, $\hat{\vec{b}}_1(x,k_y,k_z,t)$ may be represented as

$$\hat{\vec{b}}_1(x,k_y,k_z,t)=e^{ik_yV(x)t}\hat{\vec{b}}_2(x,k_y,k_z,t). \tag{2.7}$$

The equation for $\hat{\vec{b}}_2(x,k_y,k_z,t)$ does not contain the advective term,

$$\frac{\partial \hat{\vec{b}}_2}{\partial t}-\vec{e}_yV'(x)\hat{b}_{2x}=e^{-ik_yV(x)t}\hat{\vec{A}}_1. \tag{2.8}$$

This equation may easily be solved by Fourier transformation in t. The solution is not the complete Fourier transform of $\vec{b}(\vec{r},t)$, but depends on the x coordinate and reads

$$\hat{\vec{b}}(\omega,\vec{k}_\perp,x)=-\frac{i}{\omega}\hat{\vec{A}}(\omega,\vec{k}_\perp,x)-\frac{V'(x)}{\omega^2}\hat{A}_x(\omega,\vec{k}_\perp,x)\vec{e}_y, \tag{2.9}$$

where the Fourier transformation of $\hat{\vec{A}}(\omega,\vec{k}_\perp,x)$ is given by

$$\hat{\vec{A}}(\omega,\vec{k}_\perp,x)=\frac{1}{(2\pi)^3}\int dy\,dz\,dt\,e^{i\vec{k}_\perp\vec{r}-i(\omega+k_yV)t}\vec{A}(\vec{r},t), \tag{2.10}$$

where $\vec{k}_\perp=k_y\vec{e}_y+k_z\vec{e}_z$. Note that, in fact, there is no singularity in Eq. (2.9) because the neglected dissipative terms would result in small negative imaginary corrections to ω, thus, we would have $\omega-i0$, instead of ω in singular terms; ± 0 denotes a positive (or negative) contribution proportional to η.

Then, the solution for a fluctuating magnetic field is

$$\vec{b}(\vec{r},t)=\int_{-\infty}^{+\infty}\frac{d\omega\,d\vec{k}_\perp}{i\omega}e^{i(\omega+k_yV)t-i\vec{k}_\perp\vec{r}}\left(\hat{\vec{A}}-\frac{iV'}{\omega}\hat{A}_x\vec{e}_y\right). \tag{2.11}$$

The result looks like we use a Doppler-shifted Fourier transform with the frequency $\omega+k_yV(x)$.

For the sake of simplicity, we consider the case of locally isotropic and homogeneous turbulence with the correlation tensor given by

$$\langle\hat{v}_i(\omega,\vec{k})\hat{v}_j(\omega',\vec{k}')\rangle=\frac{1}{3}v^2(\omega,\vec{k})(\delta_{ij}-k_ik_j/k^2)\times\delta(\vec{k}+\vec{k}')\,\delta(\omega+\omega'), \tag{2.12}$$

where $v^2(\vec{k},\omega)$ is the spectral function (see, e.g., [15]). Of course, shear may affect the correlation properties and, probably, turbulence is not isotropic and homogeneous in real conditions, however, we will neglect this effect.

Substituting the complete Fourier transformation of $\vec{v}(\vec{r},t)$ and the expression (2.11) for $\vec{b}(\vec{r},t)$ into the definition of $\vec{\mathcal{E}}$, we obtain the integral representation for $\vec{\mathcal{E}}$, which does not assume a two-scale approximation. Then, after averaging Eq. (2.12), the expression for $\vec{\mathcal{E}}$ transforms into

$$\vec{\mathcal{E}}=\int \frac{d\omega' d\vec{k}' d\omega\, d\vec{k}_\perp}{3(2\pi)^3\omega} dy' dz' dt' v^2(\omega',\vec{k}') e^{i(\omega+\omega')(t-t')} \times e^{i(\vec{k}_\perp+\vec{k}')(\vec{r}'-\vec{r})+ik_y V(x)(t-t')}\left(i\vec{E}_1-\frac{V'}{\omega}\vec{e}_y\times\vec{E}_2\right), \tag{2.13}$$

where

$$\vec{E}_1=\vec{\nabla}'\times\vec{B}-\vec{e}_i\varepsilon_{ijk}\frac{k_j' k_m'}{k'^2}\frac{\partial B_k}{\partial x_m'}, \tag{2.14}$$

$$\vec{E}_2=\vec{\nabla}' B_x-\vec{k}'\frac{k_m'}{k'^2}\frac{\partial B_x}{\partial x_m'}-i(\vec{k}'\cdot\vec{B})\left(\vec{e}_x-\vec{k}'\frac{k_x'}{k'^2}\right), \tag{2.15}$$

$\vec{\nabla}'=(\partial/\partial x,\partial/\partial y',\partial/\partial z')$, $\vec{B}$ in the expressions (2.14) and (2.15) is a function of $\vec{r}'=(x,y',z')$ and t'; summation is over repeated indexes.

Introducing the wave-vector $\vec{q}=\vec{k}_\perp+\vec{k}_\perp'$ instead of $\vec{k}_\perp$, the integrals over $d\omega$ and $d\vec{k}_\perp$ in Eq. (2.15) may be transformed as in

$$\int \frac{d\omega\, d\vec{k}_\perp}{\omega^\nu} e^{i(\omega+\omega')(t-t')-i(\vec{k}+\vec{k}')(\vec{r}-\vec{r}')+ik_y V(t-t')} = e^{ik_y' V(t'-t)}\int d\vec{q}\, e^{i\vec{q}[\vec{r}'-\vec{r}+\vec{V}(t-t')]}\int \frac{d\omega\, e^{i\omega(t-t')}}{(\omega-\omega')^\nu}. \tag{2.16}$$

Integrals over $d\omega$ may be reduced to the known integrals [16],

$$\int_{-\infty}^{+\infty}\frac{e^{-ipx}dx}{(ix+\beta)^\nu}=\frac{2\pi}{\Gamma(\nu)}(-p)^{\nu-1}e^{\beta p} \quad \text{if} \quad p<0,$$

and 0 if $p>0$; $\Gamma(\nu)$ is the gamma-function; Re $\nu>0$, Re $\beta>0$. Substituting the value of these integrals into equation (2.16), we obtain

$$\vec{\mathcal{E}}=\int \frac{d\omega' d\vec{k}'}{3(2\pi)^2} v^2(\omega',\vec{k}')\int_{-\infty}^{t} dy' dz' dt' e^{i(\omega'-k_y' V)(t-t')} \times\int d\vec{q} e^{i\vec{q}[\vec{r}'-\vec{r}+\vec{V}(t-t')]}[(t-t')V'\vec{e}_y\times\vec{E}_2-\vec{E}_1]. \tag{2.17}$$

The integral over $d\vec{q}$ yields one of the representations of the δ-function (see, e.g., [17]. Then, integrating over $dy'dz'$, we have for the mean electromotive force

$$\vec{\mathcal{E}}=-\tfrac{1}{3}\int d\omega' d\vec{k}' v^2(\omega',\vec{k}')\int_{-\infty}^{t} dt' e^{i(\omega'-k_y' V)(t-t')} \times[\vec{E}_1(\vec{r}',t')-(t-t')V'\vec{e}_y\times\vec{E}_2(\vec{r}',t')]\big|_{\vec{r}'=\vec{r}-\vec{V}(t-t')}. \tag{2.18}$$

Introducing $\omega=\omega'-k_y' V$ and assuming that $\omega>k_y' V$, we may expand the spectral function $v^2(\omega+k_y' V,\vec{k})$ in a power series of V. Restricting ourselves to the lowest order in V, we have

$$\vec{\mathcal{E}}=-\tfrac{1}{3}\int d\omega\, d\vec{k}' v^2(\omega,\vec{k}')\int_{-\infty}^{t} dt' e^{i\omega(t-t')}[\vec{E}_1(\vec{r}',t') -(t-t')V'\vec{e}_y\times\vec{E}_2(\vec{r}',t')]\big|_{\vec{r}'=\vec{r}-\vec{V}(t-t')}. \tag{2.19}$$

Since, according to our assumption, turbulence is locally isotropic and homogeneous, we may average $\vec{E}_1$ and $\vec{E}_2$ over directions of the vector $\vec{k}'$, then

$$\vec{E}_1(\vec{r}',t')=\frac{2}{3}\vec{\nabla}'\times\vec{B}(\vec{r}',t'), \quad \vec{E}_2(\vec{r}',t')=\frac{2}{3}\vec{\nabla}' B_x(\vec{r}',t'). \tag{2.20}$$

Finally, the expression for the mean electromotive force reads

$$\vec{\mathcal{E}}=-\int_{-\infty}^{t} dt' F(t-t')[\vec{\nabla}'\times\vec{B}(\vec{r}',t')-(t-t')V' \times(x)\vec{e}_y\times\vec{\nabla}' B_x(\vec{r}',t')]\big|_{\vec{r}'=\vec{r}-\vec{V}(x)(t-t')}, \tag{2.21}$$

where

$$F(t)=\tfrac{2}{9}\int d\omega\, d\vec{k}\, v^2(\omega,\vec{k})e^{i\omega t}. \tag{2.22}$$

Note that deriving this equation, we did not assume that the turbulent scale ℓ is small compared to that of the mean field. As a result, the mean electromotive force (2.22) is nonlocal. The expression for $\vec{\mathcal{E}}$ does not contain the component proportional to $\vec{B}$ that is typical for the alpha effect.

III. TURBULENT DISSIPATION IN THE INTEGRAL APPROACH

Consider, initially, the decay of the mean field in the case of a vanishing mean flow $V(x)=0$. Then, we have from Eq. (2.21)

$$\vec{\mathcal{E}} = -\int_{-\infty}^{t} dt' F(t-t') \vec{\nabla} \times \vec{B}(\vec{r},t')$$

$$= -\vec{\nabla} \times \int_{0}^{\infty} d\xi F(\xi) \vec{B}(\vec{r},t-\xi). \tag{3.1}$$

Dissipation turns out to be local but is determined by the previous evolution of the magnetic field. Substituting this electromotive force into the mean induction Eq. (2.1) and assuming that the magnetic field is parallel to the x axis, we obtain in a high-conductivity limit

$$\frac{\partial B_x}{\partial t} = \Delta \int_0^\infty d\xi F(\xi) B_x(\vec{r},t-\xi). \tag{3.2}$$

Since the coefficients of this equation do not depend on coordinates, we may consider the simplest solution corresponding to the plane wave

$$B_x(\vec{r},t) = f(t) e^{-i\vec{K}\cdot\vec{r}}, \tag{3.3}$$

where $\vec{K}$ is the wave vector. Substituting this expression into Eq. (3.2), we have

$$\frac{df}{dt} = -K^2 \int_0^\infty d\xi F(\xi) f(t-\xi). \tag{3.4}$$

This equation describes the turbulent (eddy) dissipation of the mean magnetic field. By analogy with the microscopic magnetic viscosity, we may define the quantity

$$\eta_T = \frac{1}{f(t)} \int_0^\infty d\xi F(\xi) f(t-\xi) \tag{3.5}$$

as a turbulent magnetic viscosity. Contrary to the standard definition (see, e.g., [3]), η_T in the integral approach is generally determined by the history of the mean field.

The solution of Eq. (3.4) depends very much on the frequency dependence of the spectral function, $v^2(\omega,\vec{k})$. For the sake of simplicity, we choose the simplest representation of a spectral function (see, e.g., [15]),

$$v^2(\omega,\vec{k}) = W(\vec{k})(\omega^2 + 1/\tau^2)^{-1}, \tag{3.6}$$

where τ is the characteristic correlation time of turbulence. Substituting this expression into Eq. (3.4), we have

$$\frac{df}{dt} = -\frac{2}{9} K^2 \int W(\vec{k}) d\vec{k} \int_0^\infty d\xi f(t-\xi) \int_{-\infty}^{+\infty} \frac{d\omega\, e^{i\omega\xi}}{\omega^2 + 1/\tau^2}. \tag{3.7}$$

We can represent the frequency dependence of $v^2(\omega,\vec{k})$ in this equation as

$$\frac{1}{\omega^2 + 1/\tau^2} = \frac{\tau}{2} \left[\frac{1}{i\omega + 1/\tau} - \frac{1}{i\omega - 1/\tau} \right]. \tag{3.8}$$

Then, integrals over $d\omega$ may be reduced to the table integrals (see [16], p. 332). Substituting their value, we obtain

$$\frac{df}{dt} = -v_T^2 K^2 \int_0^\infty d\xi\, e^{-\xi/\tau} f(t-\xi), \tag{3.9}$$

where we denote

$$v_T^2 = \frac{2}{9} \pi\tau \int W(\vec{k}) d\vec{k}. \tag{3.10}$$

The solution of this equation may be represented in a simple exponential form,

$$f(t) = e^{-\gamma t}. \tag{3.11}$$

The exponential solution exists if $\gamma < 1/\tau$, otherwise the integral in Eq. (3.9) goes to infinity. The dispersion equation for γ reads

$$\gamma = v_T^2 K^2 \int_0^\infty e^{-(1/\tau - \gamma)\xi} d\xi, \tag{3.12}$$

or, calculating the integral,

$$\gamma^2 - \frac{1}{\tau}\gamma + v_T^2 K^2 = 0. \tag{3.13}$$

The roots of dispersion equation are

$$\gamma_1 = \frac{1}{2\tau} - \sqrt{\frac{1}{4\tau^2} - v_T^2 K^2}, \quad \gamma_2 = \frac{1}{2\tau} + \sqrt{\frac{1}{4\tau^2} - v_T^2 K^2}. \tag{3.14}$$

Both these roots satisfy the condition $\gamma_{1,2} < 1/\tau$, which is necessary for the existence of an exponential solution. Depending on the parameters, the decay of the mean magnetic field may be qualitatively different.

If $2\tau v_T K > 1$, then the roots (3.14) are complex conjugate, and decay is accompanied by oscillations of the field,

$$\gamma_{1,2} = \frac{1}{2\tau} \mp i\Omega, \tag{3.15}$$

where the frequency of oscillation is given by

$$\Omega = \sqrt{v_T^2 K^2 - \frac{1}{4\tau^2}}. \tag{3.16}$$

Note that in this case, the decay time scale does not depend on the wavelength of the mean field and is equal 2τ.

If $2\tau v_T K < 1$, both roots (3.14) are real and the field decays monotonous. However, the rate of decay may be substantially different for different modes. In the limiting case of a very large wavelength $\tau v_T K \ll 1$, we have

$$\gamma_1 \approx \tau v_T^2 K^2, \quad \gamma_2 \approx \frac{1}{\tau} - \tau v_T^2 K^2. \tag{3.17}$$

The first root describes the standard decay of the mean field caused by turbulent magnetic viscosity. The decay time scale decreases with decreasing of the wavelength as K^{-2}. Only this decaying mode exists in a two-scale approximation. The second root describes a qualitatively different behavior. In the limit $\tau v_T K \ll 1$, the mean magnetic field rapidly decays on a time scale $\sim\tau$, which depends weakly on the length scale of the mean field. This decaying mode only appears due to the influence of the previous history of the magnetic field on its present evolution and does not exist in a two-scale model.

IV. INSTABILITY OF THE MEAN FIELD

Consider instability of the mean field caused by shear. The shear-driven dynamo may generate even two-dimensional magnetic fields [12], therefore, we consider the mean field, which has only x and y components and does not depend on z. The x component of the mean induction Eq. (2.1) reads for a plane Couette flow

$$\frac{\partial B_x}{\partial t}+V(x)\frac{\partial B_x}{\partial y}=(\vec{\nabla}\times\vec{\mathcal{E}})_x. \tag{4.1}$$

Substituting the mean electromotive force from Eq. (2.21), we have

$$\frac{\partial B_x}{\partial t}+V(x)\frac{\partial B_x}{\partial y}=\int_0^\infty d\xi F(\xi)\Big[\vec{\Delta}' B_x(\vec{r}',t-\xi)-\xi V'(x)\frac{\partial^2}{\partial x'\partial y'}B_x(\vec{r}',t-\xi)\Big]_{\vec{r}'=\vec{r}-\vec{V}(x)\xi}. \tag{4.2}$$

The solution of this equation may be represented as

$$B_x=f(x)e^{\gamma t-iK_y y}, \tag{4.3}$$

where K_y is the wave vector in the y direction and γ is the growth rate. Then, $f(x)$ satisfies the equation

$$[\gamma-iK_yV(x)]f(x)=\int_0^\infty d\xi F(\xi)e^{-[\gamma-iK_yV(x)]\xi}\times\left[\frac{d^2f}{dx^2}-K_y^2f+i\xi V'(x)K_y\frac{df}{dx}\right] \tag{4.4}$$

or, after integrating over $d\xi$,

$$\frac{d^2f}{dx^2}-iK_yV'(x)\frac{\lambda_T}{\mu_T}\frac{df}{dx}-\left(K_y^2+\frac{\Gamma}{\mu_T}\right)f=0, \tag{4.5}$$

where

$$\mu_T=-\tfrac{2}{9}\int\frac{v^2(\omega,\vec{k})}{i\omega-\Gamma}d\omega\, d\vec{k},$$

$$\lambda_T=-\tfrac{2}{9}\int\frac{v^2(\omega,\vec{k})}{[i\omega-\Gamma]^2}d\omega\, d\vec{k}, \tag{4.6}$$

and $\Gamma=\Gamma(x)=\gamma-iK_yV(x)$. The coefficient μ_T represents magnetic viscosity in a turbulent shear flow, the coefficient λ_T describes the turbulent kinetic process that may be responsible for the generation of the mean field. Taking into account that for a steady-state turbulence $v^2(\omega,\vec{k})=v^2(-\omega,\vec{k})$, we may represent the kinetic coefficients as

$$\mu_T=\tfrac{2}{9}\Gamma\int\frac{v^2(\omega,\vec{k})}{\omega^2+\Gamma^2}d\omega\, d\vec{k},$$

$$\lambda_T=\tfrac{2}{9}\int\frac{(\omega^2-\Gamma^2)v^2(\omega,\vec{k})}{(\omega^2+\Gamma^2)^2}d\omega\, d\vec{k}. \tag{4.7}$$

Note that in our nonlocal model, all turbulent kinetic coefficients depend on the rate of a mean process that is the principle difference to any local theory such as a two-scale approximation. The kinetic coefficients are generally complex since γ is complex.

It is convenient to represent $f(x)$ as

$$f(x)=\psi(x)\exp\left(\frac{i}{2}K_y\int V'(x')\frac{\lambda_T}{\mu_T}dx'\right). \tag{4.8}$$

Then, the equation for $\psi(x)$ reads

$$\frac{d^2\psi}{dx^2}-\left[K_y^2+\frac{\Gamma}{\mu_T}-\frac{1}{4}K_y^2V'^2(x)\frac{\lambda_T^2}{\mu_T^2}-\frac{i}{2}K_y\frac{d}{dx}\left(V'(x)\frac{\lambda_T}{\mu_T}\right)\right]\psi=0. \tag{4.9}$$

The coefficients μ_T and λ_T depend generally on x, and these dependences are determined by the spectral function. As an example, consider again turbulence with the spectral function (3.6). Then, the turbulent kinetic coefficients are

$$\mu_T=\frac{\tau v_T^2}{1+\Gamma\tau},\quad \lambda_T=-\frac{\tau^2v_T^2}{(1+\Gamma\tau)^2}. \tag{4.10}$$

In the most interesting for applications case $\Gamma\tau\ll1$, we have

$$\mu_T\approx\tau v_T^2,\quad \lambda_T\approx-\tau^2v_T^2=-\tau\mu_T, \tag{4.11}$$

and the coefficients turn out to be independent of x. For this model of turbulence, Eq. (4.9) may be rewritten as

$$\frac{d^2\psi}{dx^2}-\left[\frac{\Gamma}{\mu_T}-K_y^2\left(\frac{1}{4}V'^2\tau^2-1\right)+\frac{i}{2}K_y\tau V''\right]\psi=0. \tag{4.12}$$

To illustrate the behavior of the mean field under different conditions, consider some particular solutions of this equation for shear flows between two planes, $x=0$ and $x=d$.

(1) *Turbulent drift waves*. Consider initially the case of a relatively weak shear $V'\tau<1$, when one cannot expect instability of the mean field. Assume that the velocity profile at $d\geqslant x\geqslant 0$ is given by

$$V(x)=D[d^2/4-(x-d/2)^2], \qquad (4.13)$$

where D is constant. Then, $V''(x)=-2D$. If $2\tau v_T>d$, then Eq. (4.12) transforms into

$$\frac{d^2\psi}{dx^2}-\left[\frac{\gamma}{\mu_T}+K_y^2+\frac{i}{2}K_y\tau V''\right]\psi=0. \qquad (4.14)$$

For the sake of simplicity, we assume B_x to be vanishing both at the top and bottom boundaries of a shear flow, $\psi=0$ at $x=0$ and $x=d$. Then, the solution of Eq. (4.14) is

$$\psi=\sin(n\pi x/d), \qquad (4.15)$$

where n is integer. This solution yields the dispersion equation

$$\gamma=iK_y\tau^2v_T^2D-\tau v_T^2Q^2, \qquad (4.16)$$

where $Q^2=K_y^2+(n\pi/d)^2$. If $K_y\tau D>Q^2$, the dispersion equation describes oscillatory drift waves that decay slowly because of the turbulent dissipation. The frequency of these waves is

$$\omega_d=4K_y\tau^2v_T^2V_{max}/d^2, \qquad (4.17)$$

where $V_{max}=Dd^2/4$ is the maximum velocity in the flow (4.13). Perturbations of the mean field represented by the drift waves move in the positive y direction with the drift velocity

$$V_d=4\tau^2v_T^2V_{max}/d^2. \qquad (4.18)$$

The drift velocity depends very much on the parameters of turbulence and, under considered conditions, may be larger than advection of the magnetic field lines.

(2) *Dynamo waves in WKB approximation*. For some perturbations, Eq. (4.12) may be solved by making use of the WKB approximation, which is well justified if ψ has many knots in the x direction (see, e.g., [18]). Note, however, that this approximation gives also qualitatively correct results even if the number of knots is ~ 1. In the WKB approximation, we may represent $\psi(x)$ as

$$\psi(x)=\sin\left(\int q(x')dx'+C\right), \qquad (4.19)$$

where $q(x)$ is the wave vector in the x direction and C is constant. This solution strictly applies if $qx\gg 1$. Then, in the lowest order in $1/qx$, we have

$$q^2(x)=\left[-\frac{\Gamma}{\mu_T}+K_y^2\left(\frac{1}{4}V'^2\tau^2-1\right)-\frac{i}{2}K_y\tau V''\right]. \qquad (4.20)$$

To obtain the solution of Eq. (4.12), we need the corresponding boundary conditions. Again, assuming that the field B_x is vanishing at $x=0$ and $x=d$, we obtain the dispersion equation

$$\int_0^d\left[-\frac{\Gamma}{\mu_T}+K_y^2\left(\frac{1}{4}V'^2\tau^2-1\right)-\frac{i}{2}K_y\tau V''\right]^{1/2}dx=n\pi, \qquad (4.21)$$

where n is integer. We may represent the dispersion equation by making use of the theorem on average,

$$\gamma=iK_y\left(V(x_0)-\frac{1}{2}\tau^3v_T^2V''(x_0)\right)+\frac{1}{4}\tau^3v_T^2K_y^2V'^2(x_0)-\tau v_T^2Q^2, \qquad (4.22)$$

where $d>x_0>0$. The instability arises if Re $\gamma>0$ or

$$\frac{1}{4}\tau^2K_y^2V'^2(x_0)>Q^2. \qquad (4.23)$$

Since $Q^2>K_y^2$, the necessary condition for instability is

$$\tau V'>2. \qquad (4.24)$$

Therefore, the mean field may be unstable only if shear is sufficiently strong. Note that the particular shape of the velocity profile is not important for instability. The growth time is of the order $(\tau^3v_T^2K_y^2V'^2)^{-1}$, and perturbations with a shorter wavelength grow faster.

(3) *Dynamo instability for the linear velocity profile*. In this case, the analytic solution of Eq. (4.12) may be obtained. Assume that $V(x)$ is given by

$$V(x)=a\tilde{x}, \quad \tilde{x}=x/d. \qquad (4.25)$$

Then, Eq. (4.12) transforms into

$$\frac{d^2\psi}{d\tilde{x}^2}-(g-ip\tilde{x})\psi=0, \qquad (4.26)$$

where

$$g=\frac{\gamma d^2}{\tau v_T^2}-K_y^2d^2\left(\frac{a^2\tau^2}{4d^2}-1\right), \quad p=\frac{aK_yd^2}{\tau v_T^2}. \qquad (4.27)$$

Introducing the coordinate,

$$\zeta=(i/p)^{-1/3}\tilde{x}+(i/p)^{2/3}g,$$

Eq. (4.26) may be transformed into the Airy equation,

$$\frac{d^2\psi}{d\zeta^2}-\zeta\psi=0. \qquad (4.28)$$

The solution of this equation may be expressed in terms of the Airy functions of the first and second types [19],

$$\psi(\zeta)=a_1\,\mathrm{Ai}(\zeta)+a_2\,\mathrm{Bi}(\zeta), \tag{4.29}$$

where a_1 and a_2 are constant. From the boundary conditions, we obtain the dispersion equation for eigenvalues

$$\mathrm{Bi}(\tilde{\omega})\mathrm{Ai}(\tilde{\omega}+i^{-1/3}p^{1/3})=\mathrm{Ai}(\tilde{\omega})\mathrm{Bi}(\tilde{\omega}+i^{-1/3}p^{1/3}), \tag{4.30}$$

where $\tilde{\omega}=i^{2/3}g/p^{2/3}$. We consider the case $\tilde{\omega}\gg 1$, which corresponds to $aK_y\tau > d/\tau v_T$. In this case, we may use the asymptotic behavior of the Airy functions [19]

$$\mathrm{Ai}(\tilde{\omega})\approx\frac{\tilde{\omega}^{-1/4}}{2\sqrt{\pi}}e^{-2\tilde{\omega}^{3/2}/3}, \quad \mathrm{Bi}(\tilde{\omega})\approx\frac{\tilde{\omega}^{-1/4}}{\sqrt{\pi}}e^{2\tilde{\omega}^{3/2}/3}. \tag{4.31}$$

Then, the dispersion Eq. (4.30) simplifies

$$\tilde{\omega}^{-1/4}(\tilde{\omega}+i^{-1/3}p^{1/3})^{-1/4}\{1-e^{4/3[\tilde{\omega}^{3/2}-(\tilde{\omega}+i^{-1/3}p^{1/3})^{3/2}]}\}=0, \tag{4.32}$$

or

$$\frac{4}{3}[\tilde{\omega}^{3/2}-(\tilde{\omega}+i^{-1/3}p^{1/3})^{3/2}]=2\pi n i, \tag{4.33}$$

where n is integer. If $\tilde{\omega}>p^{1/3}$ then we have

$$p^{1/3}\tilde{\omega}^{1/2}/i^{1/3}\approx -i\pi n, \tag{4.34}$$

or

$$\gamma\approx\tau v_T^2\left(\frac{a^2K_y^2\tau^2}{4d^2}-K^2\right), \tag{4.35}$$

where $K^2=K_y^2+(n\pi/d)^2$. The first term on the right-hand side of Eq. (4.35) describes the destabilizing effect of shear, and the second term is associated with turbulent dissipation. Obviously, this dispersion equation may correspond to unstable dynamo modes if $a^2\tau^2/4d^2>1$ or, in other words, if

$$\tau V'>2 \tag{4.36}$$

in the agreement with Eq. (4.24) (see, also, [14]). If shear is sufficiently strong, the instability may seize a wide range of wavelength. The growth time decreases with decreasing of the wavelength reaching the value $\sim 1/\tau V'^2$ for waves with $K_y\sim 1/\tau v_T$.

V. CONCLUSION

We have considered the turbulent dynamo action in shear flow. The principal result is that, in the presence of shear, even turbulent motions showing mirror symmetry become suitable for the generation of the mean magnetic field. The mean field amplification is caused by additional terms that appear in the mean electromotive force and are proportional to the production of spatial derivatives of the magnetic field and shear stresses. These terms are nonvanishing even if turbulence is isotropic and homogeneous. The considered generation mechanism is qualitatively different from the conventional alpha dynamo, which apart from the lack of mirror symmetry of turbulence, also requires a large-scale stratification of turbulence.

In the present paper, we did not use the two-scale model that assumes the length scale of turbulence is much smaller than that of mean quantities, but generalized the nonlocal approach proposed by Rüdiger and Urpin [14]. As a result, we obtain the integral equation governing the mean magnetic field. For a sufficiently strong shear, this equation has the solution that corresponds to unstable mean-field waves. The instability arises only for waves that are nonuniform in the direction of flow. The growth rate of unstable waves depends on the wavelength being larger for waves with a shorter wavelength.

The mechanism considered may play an important role in the generation of magnetic fields in various astrophysical bodies where hydrodynamic flows are characterized by a strong shear (accretion discs, galaxies, jets, etc.).

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