Choice of dynamical variables for global reconstruction of model equations from time series

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The success of modeling from an experimental time series is determined to a significant extent by the choice of dynamical variables. We propose a method for preliminary investigation of a time series whose purpose is to find out whether a global dynamical model with smooth functions can be constructed for the chosen variables. The method consists in the estimation of single valuedness and continuity of relations between dynamical variables and variables to enter left-hand sides of model equations. The method is explained with numerical examples. Its efficiency is demonstrated by modeling a real nonlinear electric circuit.

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I. INTRODUCTION

Dynamical modeling implies the specification of a state vector $\mathbf{x}=(x_1,x_2,\ldots,x_D)\in\mathrm{R}^D$, where x_k are dynamical variables and D is a model dimension, and of an evolution operator which provides unique prediction of future states starting from an initial one. A model which describes the behavior of an object in a broad region of the phase space $(x_1,x_2,\ldots,x_D)$ is called global. A relevant approach to obtaining such a model is the reconstruction of equations from a time series, i.e., from a discrete sequence of experimental data $\{\boldsymbol{\eta}(t_i)\}$, where $t_i=i\Delta t$, $i=1,2,\ldots,N$, Δt is a sampling interval. Different methods of constructing ordinary differential equations (ODE's) [1–6], discrete maps [7–9], and delay differential equations (DDE's) [10,11] have already been suggested. Such phenomenological models have shown their efficiency for short-term prediction [7], estimation of some characteristics of an observed process (such as fractal dimensions [3] and Lyapunov exponents [2,12]), and signal classification [13].

In general, a procedure of constructing model equations $\mathbf{y}(t)=\mathbf{f}[\mathbf{x}(t)]$ from a time series $\{\boldsymbol{\eta}(t_i)\}$[1] is as follows. First, a time series of state vectors $\{\mathbf{x}(t_i)\}$ is formed from the original series $\{\boldsymbol{\eta}(t_i)\}$.[2] Second, a series $\{\mathbf{y}(t_i)\}$ is obtained from $\{\mathbf{x}(t_i)\}$ according to the chosen model type; either by numerical differentiation of the series $\{\mathbf{x}(t_i)\}$ {for ODE's $d\mathbf{x}/dt=\mathbf{f}[\mathbf{x}(t)]$}, or just by the shift of the series $\{\mathbf{x}(t_i)\}$ along the time axis {for discrete maps $\mathbf{x}(t_{i+1})=\mathbf{f}[\mathbf{x}(t_i)]$}. Third, the forms of the approximating functions f_k (components of the vector-valued function $\mathbf{f}$) are specified and their coefficients are calculated. The latter is often performed via the least-squares routine [5,8,9], i.e., so as to minimize the values

$$\sum_{i=1}^{N}[y_k(t_i)-f_k(\mathbf{x}(t_i))]^2=min,\quad k=1,\ldots,D.$$

But we should note that the least-squares routine gives sufficiently accurate estimates only if the noise level is low (say, less than 5%). An advanced routine has recently been suggested [14] which utilizes a more general form of the maximum likelihood principle and provides more accurate estimates of the coefficients. That routine should be always preferred when the number of coefficients to be calculated is quite small and the noise level is high. The most important and difficult steps of the described procedure are the choice of the dynamical variables x_k and the specification of the forms of the functions f_k. An inappropriate choice of the variables can make approximation of the dependencies $y_k(\mathbf{x})$ with smooth functions extremely problematic (see, e.g., [15]) or even make these dependencies nonunique.

In this paper we propose a method of estimating the suitability and "convenience" of the chosen variables x_k for constructing a *global* dynamical model. It is based on testing the series $\{\mathbf{y}(t_i)\}$ and $\{\mathbf{x}(t_i)\}$ for single valuedness and continuity of each dependency $y_k(\mathbf{x})$ in the entire region of an observed motion. A somewhat similar idea is considered in [10] where the value of a delay time (for reconstruction of DDE's) is selected so as to minimize the "filling factor." The latter characteristic is convenient and easy to calculate but, in general, it achieves a minimum even if the dependency is not single valued. Our method employs, to a certain extent, the idea of the δ-ε method which has been suggested in [16] for detecting determinism in an observed process. However, our research addresses a different problem, namely, the problem of the global reconstruction of model equations.

According to our approach, relative variations of a variable y_k inside small volumes ΔV in the space of the selected variables $x_1,x_2,\ldots,x_D$ are found. Then, one finds out how these variations behave when $\Delta V\to 0$. We note that the main role is played by the local characteristics that is different from [15,16] where the averaged (integral) estimates are

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[1]Its length N and the dimension of its vectors are limited by the conditions of the experiment.

[2]Coordinates of a vector $\mathbf{x}$ can be obtained by using the methods of successive derivatives [3,5,6], time delays [1,4,7–9], integration [6], or weighted summation [4]. Besides, they may just coincide with the observables. The length of the series $\{\mathbf{x}(t_i)\}$ can be less than N, but the difference is usually small. To avoid additional notations, we assume the length of $\{\mathbf{x}(t_i)\}$ to be equal to N.

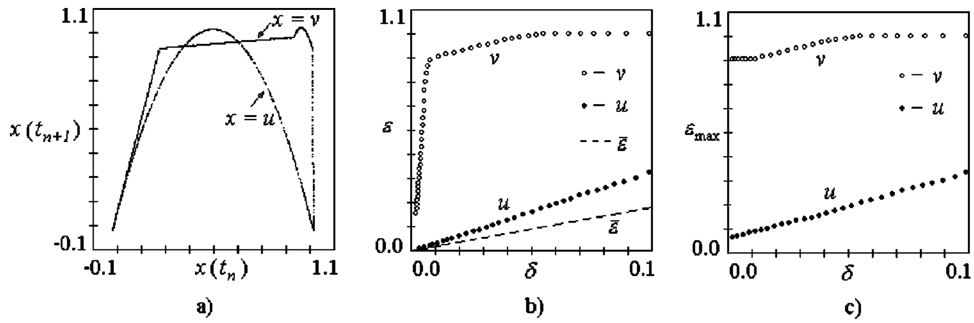

FIG. 1. (a) Maps for time series of the variables u and v formed from a chaotic solution of the system (1). There is a region of steep slope of the graph for v (see its right-hand side). (b) The graphs $\varepsilon_{max}(\delta)$ for observables v (white circles) and u (filled circles) when noise is absent. The presence of the region of steep slope for the observable v [Fig. 1(a)] reveals itself in appearing a ''breakpoint'' at $\delta \approx 0.005$. The graphs $\overline{\varepsilon}(\delta)$ coincide for both variables and are shown with the dashed line. (c) The graphs $\varepsilon_{max}(\delta)$ when noise is present. The graph indicates nonsingle valuedness for the variable v, while for the variable u it is situated just a little higher than in Fig. 1(b).

used. Namely, we suggest to select the dynamical variables so as to provide for each of the model dependencies $y_k(\mathbf{x})$ minimal local variation and its tending to zero for $\Delta V \to 0$. The latter is evidence for single valuedness, continuity, and the absence of very steep ''slopes'' of a dependency. Otherwise, sufficiently accurate approximations of those dependencies with the usually employed smooth functions (e.g., with standard polynomials) become too difficult or even impossible.

The proposed method is described in Sec. II. It is illustrated with examples of the reconstruction of difference and differential equations from their ''clean'' and noisy numerical solutions in Sec. III. Its efficiency is demonstrated by modeling a real nonlinear electric circuit in Sec. IV. Limitations and prospects of the approach are discussed in Sec. V.

II. DESCRIPTION OF THE METHOD

Let us consider the following problem setup: $\{\boldsymbol{\eta}(t_i)\}$ is an observable time series, the type of model equations is selected, and time series $\{\mathbf{x}(t_i)\}$ and $\{\mathbf{y}(t_i)\}$ are formed. It is necessary to assess single valuedness and continuity of the dependencies $y_k(\mathbf{x})$ (for $k=1,2,\ldots,D$) and to find a criterion for selecting a set of variables which is the most suitable for the construction of a global model.

If a dependency $y(\mathbf{x})$ is single valued and continuous in a domain V, then the difference $|y(\mathbf{x})-y(\mathbf{x}_0)|$ tends to zero when $\|\mathbf{x}-\mathbf{x}_0\| \to 0$ for each $\mathbf{x}_0 \in V$. In practice, violation of this condition may be viewed as a sign of nonsingle-valuedness or discontinuity of the dependency $y(\mathbf{x})$. Since the observable time series has a finite length, the above-mentioned limit, strictly speaking, cannot be found. However, it is possible to trace a tendency of the variation of the value $|y(t_i)-y(t_j)|$ when the vectors $\mathbf{x}(t_i)$ and $\mathbf{x}(t_j)$ are made closer and closer, down to a certain *finite* distance. For a sufficiently large amount of data N, high accuracy of measurements, and low noise level, this distance can be made sufficiently small for each region of the observed motion.

The method of testing the selected variables consists of the following. Let us assume (without any loss of generality) that the difference between the maximal and minimal values for each of the variables $x_1, \ldots, x_D$, and y is equal to unity.[3] In other words, the investigated set of vectors $\{\mathbf{x}(t_i)\}$ is contained inside a hypercube $V \in \mathrm{R}^D$ with the side of the unit length. Let us partition V into identical hypercubic boxes of the size δ, select all boxes containing at least two vectors, and denote them as $s_1, s_2, \ldots, s_M$. We call the difference between maximal and minimal values of y inside a box s_k a ''local variation'' ε_k: $\varepsilon_k = \max_{\mathbf{x}\in s_k} y(\mathbf{x}) - \min_{\mathbf{x}\in s_k} y(\mathbf{x})$. The maximal local variation $\varepsilon_{max} = \max_{1\le k\le M} \varepsilon_k$ and its graph $\varepsilon_{max}(\delta)$ are used as the main characteristics of the investigated dependency. The suitability of the considered variables $\mathbf{x}$ and y for global modeling is assessed with the aid of the following ideas.

(1) If a dependency $y(\mathbf{x})$ is single valued and continuous, the value of ε_{max} is sufficiently small for small δ and tends to zero when $\delta \to 0$. It is not hard to show that a graph $\varepsilon_{max}(\delta)$ is a straight line for sufficiently small δ.

(2) If a single valued and continuous dependency has a region of very steep slope (a ''jump''), then ε_{max} remains rather big for sufficiently small δ, since that region is contained inside one box. However, the further decrease of δ leads to the decrease of ε_{max} because the region of a jump becomes divided into several boxes. The graph $\varepsilon_{max}(\delta)$ exhibits a ''breakpoint'' at the value of δ equal to the size of the region of steep slope [e.g., Fig. 1(b), white circles]. In such a case, the dependency $y(\mathbf{x})$ is also difficult to approximate with a smooth function. Therefore, dynamical variables should be selected so that the graph $\varepsilon_{max}(\delta)$ tends to the origin gradually, without breakpoints.

(3) In practice, an achievable value of δ is bounded from below because of the finite amount of data N. For example, if vectors $\mathbf{x}(t_i)$ are distributed uniformly in the hypercube V, this boundary can be estimated as $N^{-1/D}$. If N is very small, there are no sufficiently close vectors $\mathbf{x}(t_i)$ and $\mathbf{x}(t_j)$ in many

[3]This condition may be easily provided by an appropriate normalization of the variables.

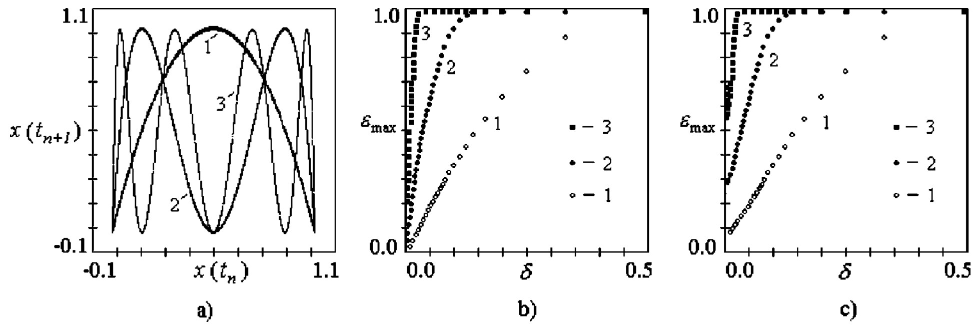

FIG. 2. (a) Maps for time series formed from a chaotic solution of the system (1) by recording every iteration (the graph 1), every second iteration (the graph 2), and every third iteration (the graph 3). (b) The graphs $\varepsilon_{max}(\delta)$. A steeper graph corresponds to a dependency which oscillates more intensively. (c) Noise is present and the graphs are situated higher than in (b). A dependency, which oscillates more intensively, is distorted more significantly (and the corresponding graph is situated higher).

regions. Hence, our approach is not applicable since there is no possibility of investigating the local properties of the dependency $y(\mathbf{x})$.[4]

(4) In a physical experiment there are unavoidable measurement errors (e.g., determined by the number of bits of an ADC) and noise (the influence of numerous factors that could not be described in a deterministic way). Let us denote their common effect on $\mathbf{x}$ and y as σ_{noise}. When δ becomes less than σ_{noise}, the value of ε_{max} no longer decreases even if there is a certain law relating y to $\mathbf{x}$, e.g., if the value of σ_{noise} is more than the size of a region of steep slope, a graph $\varepsilon_{max}(\delta)$ indicates nonsingle-valuedness of the investigated dependency [Fig. 1(c), white circles]. So, the considered variables are assumed inappropriate for global modeling according to our criterion.

As an *additional* characteristic, the value of an average local variation $\overline{\varepsilon}$ may be employed: $\overline{\varepsilon}=1/M\sum_{k=1}^{M}\varepsilon_k$. If $\overline{\varepsilon}\to 0$ for $\delta\to 0$ and the slope of the graph $\overline{\varepsilon}(\delta)$ is small, it could sometimes point to a "gradual" (in average) dependency $y(\mathbf{x})$ which can be better (under equal other conditions) approximated with a smooth function. We illustrate further that the value $\overline{\varepsilon}$ does not contain by itself all the information on the properties of $y(\mathbf{x})$ which is necessary for global modeling, e.g., if $y(\mathbf{x})$ has a localized region of nonsingle-valuedness or discontinuity, it could affect the average value $\overline{\varepsilon}$ only slightly and a graph $\overline{\varepsilon}(\delta)$ would look like a graph for a smooth single-valued dependency [Fig. 3(b), the dashed line].

III. DEMONSTRATIVE NUMERICAL EXAMPLES

A. Reconstruction of difference equations

Let us illustrate the described ideas with an example of the reconstruction of difference equations from a chaotic time series generated by the quadratic map

$$u_{n+1}=ru_n(1-u_n^2) \tag{1}$$

at $r=4.0$ for two situations. First, an observable is $\eta(t_i)=u_i$, where u_i are the successive iterations of the system (1). Second, $\eta(t_i)=v_i$, where the variable v is related to u in a one-to-one way—via a piecewise-linear function h:

$$v=h(u)=\begin{cases}5u, & 0\leqslant u\leqslant 0.18,\\ 0.9+(u-0.18)/8.2, & 0.18<u\leqslant 1,\end{cases} \tag{2}$$

which can be interpreted as a transformation of the signal by a measurement device. Analyzing time series of these two variables (their length is $N=10^4$) with the aid of the proposed method, we assess the possibility of constructing a global model in the form of a one-dimensional map $x(t_{i+1})=f[x(t_i)]$, the dynamical variable x coinciding with the observable η. For all cases, models are constructed according to the procedure described briefly in Sec. I.

In the first situation, constructing a global model is not a hard problem. It is sufficient to employ a polynomial of the second order as a function f. The model obtained provides one-step prediction practically with a machine precision. In the second situation, modeling is rather difficult, e.g., using a polynomial of the 11th (!) order allows for reducing a relative root-mean-squared one-step prediction error only to 30%.

We apply the proposed method to test a dependency of $x(t_{i+1})$ on $x(t_i)$ both for an observable u and for an observable v [see Fig. 1(a)]. Both graphs $\varepsilon_{max}(\delta)$ [Fig. 1(b)] indicate single valuedness and continuity. But in the first case ε_{max} tends to zero "gradually" when $\delta\to 0$, while in the second case the graph $\varepsilon_{max}(\delta)$ exhibits a breakpoint at small δ. The breakpoint reflects the presence of a region of very steep slope of the investigated dependency [see Fig. 1(a), the region $x(t_i)\approx 1$]. The graphs $\overline{\varepsilon}(\delta)$ practically coincide for both variables [Fig. 1(b), the dashed line].

Advantages of using one of these variables for global modeling are even more obvious if observed series are noisy. Let $\eta(t_i)$ be equal to $\eta(t_i)=u_i+\xi_i$ and $\eta(t_i)=v_i+\xi_i$, respectively [where $\{\xi_i\}$ is a sequence of independent random values distributed uniformly in the interval $(-0.005,0.005)$ that corresponds approximately to 1% of the signal level]. It

[4]There are no similar limitations in [16] because the purpose of that work is different: to find at least "traces of determinism." Therefore, it is sufficient to find at least some domains of single valuedness ("exceptional events") while large, but rarely populated, regions may be ignored.

is yet possible to obtain an efficient global model with a second-order polynomial from the series $\{u_i+\xi_i\}$ (relative one-step prediction error is comparatively small, about 3%). However, the series $\{v_i+\xi_i\}$ appears completely irrelevant for modeling. The graphs $\varepsilon_{max}(\delta)$ warn about such results [Fig. 1(c)]; the graph for u is situated just a little ''higher'' than in Fig. 1(b), while the graph for v indicates nonsingle-valuedness.

Another illustrative example is the estimation of the suitability of variables and the reconstruction of a model in the form $x(t_{i+1})=f[x(t_i)]$ from a series $\{\eta(t_i)\}=\{u_i\}$ for the following three cases: (1) $x(t_i)=\eta(t_i)$, (2) $x(t_i)=\eta(t_{2i})$, and (3) $x(t_i)=\eta(t_{3i})$. They correspond to recording the first, the second, and the third iteration of the logistic map (1), respectively. For a bigger number of the iteration, a dependency $x(t_{i+1})=f[x(t_i)]$ is more difficult to approximate [Fig. 2(a)] that reveals itself in a bigger slope of the graph $\varepsilon_{max}(\delta)$ [Fig. 2(b)]. Similarly to the previous example, the influence of noise is more dramatic for a more complicated dependency [Fig. 2(c)].

B. Reconstruction of ODE's

Let us consider the Rossler system

$$\dot{u}=-v-w,$$
$$\dot{v}=u+av, \tag{3}$$
$$\dot{w}=b+w(u-c),$$

at the parameters values $a=0.398$, $b=2.0$ and $c=4.0$ which correspond to a chaotic regime. We present preliminary estimates provided by the criterion $\varepsilon_{max}(\delta)$ and results of the reconstruction of model equations in the standard form [3]

$$\dot{x}_1=x_2,$$
$$\dot{x}_2=x_3, \tag{4}$$
$$\dot{x}_3=f(x_1,x_2,x_3),$$

where x_1 coincides with an observable η, its time series is formed from a time realization of the variable v. The values $v(t_i)$ are derived via numerical integration of the Eqs. (3) using the fourth-order Runge-Kutta routine with the step $\Delta t=0.01$ (the length of the series is $N=10^5$). The proposed method is applied for the investigation of a dependency $\dot{x}_3(x_1,x_2,x_3)$ in all cases. The values of the x_1 coordinate are formed from the time series $\{v(t_i)\}$ in different ways. Time series of x_2, x_3, and $\dot{x}_3$ are obtained via numerical differentiation of the series $\{x_1(t_i)\}$ with the aid of different techniques. Coefficients of model equations are calculated by using the linear least-squares technique (see Sec. I). The maximum likelihood principle [14] is difficult to employ here because the number of coefficients to be estimated is big (about 10). But the noise level is small or can be reduced, so both routines should give approximately the same results [14]. After estimating the coefficients, efficiency of the obtained model is assessed via comparison of the original and model phase portraits and calculation of the prediction time [7], that is the time interval on which the model provides sufficiently accurate forecast (namely, the relative prediction error remains less than a certain threshold value—we use the value of 0.05).

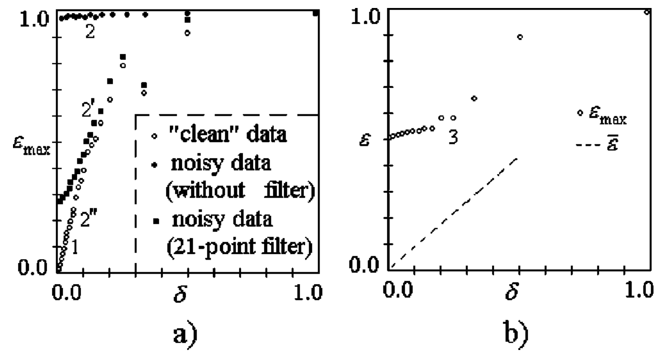

FIG. 3. Employing the graphs $\varepsilon(\delta)$ to assess different variants of the choice of variables for the Rossler system (3). (a) The graphs $\varepsilon_{max}(\delta)$ for $x_1=v$ in the following cases: (1) without noise—the graph 1 (white circles), (2) with noise, derivatives are calculated in different ways: without filtering—the graph 2 (filled circles), using a 21-point smoothing polynomial—the graph 2′ (filled squares), and using a 41-point smoothing polynomial—the graph 2″ (practically coincides with the graph 1). (b) The graph $\varepsilon_{max}(\delta)$ in the case 3: $x_1=v^2$, without noise—the graph 3 (white circles). The dependency is not single valued, but the graph $\overline{\varepsilon(\delta)}$ looks like the graph for a single-valued dependency (the dashed line).

(1) $\{x_1(t_i)\}=\{v(t_i)\}$, noise is absent, derivatives are calculated using simple finite-difference formulas of the form $\dot{x}_1(t_i)=[x_1(t_i+\Delta t)-x_1(t_i-\Delta t)]/(2\Delta t)$. A graph $\varepsilon_{max}(\delta)$ [Fig. 3(a), white circles] indicates single valuedness and continuity of the dependency $\dot{x}_3(x_1,x_2,x_3)$ that just confirms a previously known result [3]. Quite an efficient model (4) is obtained using a polynomial of the second order as a function f. It provides an accurate forecast approximately $15T$ ahead, where T is a basic period containing about 600 points.

(2) $\{x_1(t_i)\}=\{v(t_i)+\xi_i\}$, where $\{\xi_i\}$ is a sequence of independent random values uniformly distributed in the interval $(-0.0005,0.0005)$ that corresponds approximately to 0.01% of the signal level. Without filtration of the series, reconstructed models have nothing in common with the original system. The failure is predicted by the graph $\varepsilon_{max}(\delta)$ [Fig. 3(a), filled circles] and occurs due to essential amplification of the noise during differentiation. The results become better if one employs a smoothing polynomial (Savitsky-Golay filter [17]) and a sufficient width of a window on the time series for its construction. In the case presented with filled squares in Fig. 3(a), a window consisting of 21 points is used that appears insufficient; the graph $\varepsilon_{max}(\delta)$ points to nonsingle valuedness. A reconstructed model (4) with a polynomial of the second order is essentially worse than in the case of ''clean'' data, it provides prediction time about $3T$. However, if a wider window (e.g., 41 points) is used, the graph $\varepsilon_{max}(\delta)$ practically coincides with the graph for the ''clean'' series [Fig. 3(a)]. A model becomes significantly more efficient and provides an accurate forecast $7T$ ahead.

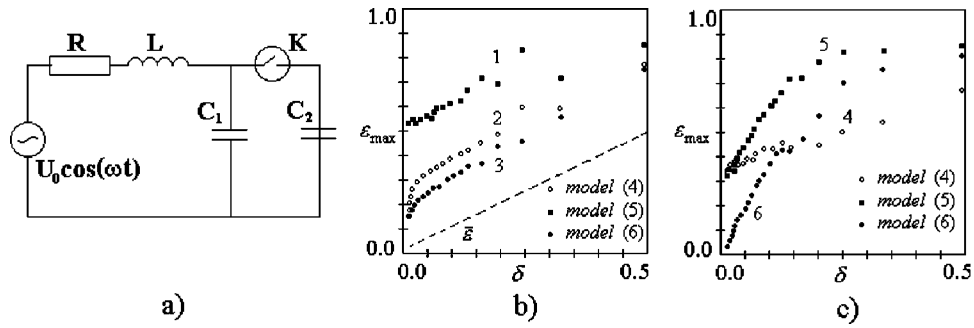

FIG. 4. (a) The scheme for the circuit with switched capacitors: $R=10\ \Omega$, $L=14$ mH, $C_1=0.1\ \mu$F, $C_2=4.4\ \mu$F, $U_0=2.3$ V, $\omega/(2\pi)=2.98$ kHz, $U_{thr}=-0.2$ V, and the sampling frequency is 250 kHz. (b) The graphs $\varepsilon_{max}(\delta)$ for different variants of the model structure (for the dynamical variable $x_1=I$): (1) for a dependency $\dot{x}_3(x_1,x_2,x_3)$ of a model (5)—filled squares, (2) for a dependency $\dot{x}_3(x_1,x_2,x_3)$ of a model (4)—white circles, (3) for a dependency $\dot{x}_2(x_1,x_2,\varphi)$ of a model (6)—filled circles. The graphs $\overline{\varepsilon(\delta)}$ are approximately the same for all examples (the graph for the case 1 is shown with the dashed line). (c) The graphs $\varepsilon_{max}(\delta)$ for different variants of the model structure (when the dynamical variable x_1 is an integral of the current I): (1) for a dependency $\dot{x}_3(x_1,x_2,x_3)$ of a model (4)—filled squares, (2) for a dependency $\dot{x}_3(x_1,x_2,x_3)$ of a model (5)—white circles, and (3) for a dependency $\dot{x}_2(x_1,x_2,\varphi)$ of a model (6)—filled circles.

(3) $\{x_1(t_i)\}=\{v^2(t_i)\}$, noise is absent. The graph $\varepsilon_{max}(\delta)$ clearly indicates nonsingle valuedness of the dependency $\dot{x}_3(x_1,x_2,x_3)$ [Fig. 3(b), white circles]. An effective model (4) can not be obtained.[5] We note that the graph $\overline{\varepsilon(\delta)}$ [Fig. 3(b), the dashed line] looks similar to the graph for a single-valued continuous dependency, i.e., it does not allow for detecting the inappropriateness of the variables for global modeling.

IV. MODELING A NONLINEAR ELECTRIC CIRCUIT

The scheme of a nonlinear electric circuit (harmonically driven LCR circuit with switched capacitors) is shown in Fig. 4(a). The element K is an electronic key, a microscheme comprising dozens of transistors and other passive elements which is fed from a special source of dc potential. At small values of voltage U on the capacity C_1, the resistance of the key is very large and linear oscillations occur in the circuit LC_1R. When the voltage U achieves a threshold value U_{thr}, the resistance of the key decreases abruptly and the capacity C_2 becomes connected to the circuit. Back switching occurs approximately at the same value of U. As a result, the system is nonlinear and exhibits complex dynamics (in particular, chaotic oscillations) at big values of the driving amplitude [18,19].

Let us consider the effect of the choice of dynamical variables and model equations structure on the results of modeling. We employ a chaotic time realization of the current I through the resistor R as an observable time series $\{\eta(t_i)\}$. The data are recorded with the aid of a 12-bit ADC, the sampling interval is $\Delta t=4\ \mu$s, the driving period is $T=84\Delta t$, and the length of the series is $N=30\,000$. Six examples are considered below (three variants of the model ODEs structure for two different observables). The results of the application of the proposed method [Figs. 4(b) and 4(c)] and of the model construction are presented. Model ODE's are constructed and their efficiency is assessed exactly as in Sec. III B. The graphs in Fig. 4 are numbered corresponding to the numbers of the following examples.

(1) A popular model structure

$$\dot{x}_1=f_1(x_1,x_2,x_3),$$
$$\dot{x}_2=f_2(x_1,x_2,x_3), \qquad (5)$$
$$\dot{x}_3=f_3(x_1,x_2,x_3),$$

where $x_1(t_i)=\eta(t_i)$, $x_2(t_i)=\eta(t_i+\tau)$, and $x_3(t_i)=\eta(t_i+2\tau)$ are time delay coordinates, $\tau=21\Delta t$ is the first zero of the autocorrelation function. A smoothing polynomial is constructed for numerical differentiation. All three dependencies $\dot{x}_k(x_1,x_2,x_3)$ are analyzed. The value of ε_{max} does not tend to zero when δ decreases for all k. All graphs $\varepsilon_{max}(\delta)$ look similar, one of them is presented in Fig. 4(b) with filled squares (for $k=3$). It indicates the impossibility of constructing an efficient global model that is completely confirmed in practice.

(2) A standard model (4) with $x_1(t_i)=\eta(t_i)$. The dependency $\dot{x}_3(x_1,x_2,x_3)$ is tested. $\varepsilon_{max}(\delta)$ decreases when δ decreases [Fig. 4(b), white circles] that points to the possible single valuedness. But an efficient model, where right-hand sides are algebraic polynomials, cannot be obtained. Obviously, a polynomial is not appropriate for approximation of the dependency $\dot{x}_3(x_1,x_2,x_3)$. Another form of the approximating function is necessary here. Its choice is a difficult problem which is not a subject of the present paper.

(3) Following the recommendations on the reconstruction of nonautonomous systems [20,21], we construct a model in the form

[5]By analytic transformations of the system (3), it is not hard to show that the investigated dependency is not single valued.

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = f(x_1, x_2, \varphi), \tag{6}$$

where $x_1(t_i) = \eta(t_i)$ and φ is the phase of driving. The dependency $\dot{x}_2(x_1, x_2, \varphi)$ is tested. The time series of the phase φ is obtained as $\varphi(t_i) = \omega t_i (\mathrm{mod}\, 2\pi)$, the angular frequency ω is assumed to be known. The graph $\varepsilon_{max}(\delta)$ (Fig. 4(b), filled circles) shows that the dependency is, possibly, single valued. However, an efficient model with harmonic driving and polynomial approximation cannot be obtained. Again, one needs to select a special form of the function *f*.

(4) A standard model (4) with $x_1(t_i) = \int_{t_1}^{t_i} \eta(t) dt$. This variable makes physical sense, it is the summed charge on the capacities C_1 and C_2. The time series $\{x_1(t_i)\}$ is obtained via the numerical integration of the measured time series of the current I (e.g., using the method of trapeziums). The value of ε_{max} for the dependency $\dot{x}_3(x_1, x_2, x_3)$ does not decrease when δ decreases [Fig. 4(c), white circles] and remains large. An effective model cannot be constructed.

(5) A model (5) with $x_1(t_i) = \int_{t_1}^{t_i} \eta(t) dt$ and delayed coordinates $x_3(t_i) = x_1(t_i + \tau)$ and $x_2(t_i) = x_1(t_i + 2\tau)$, where τ is again the first zero of the ACF. All three dependencies $\dot{x}_k(x_1, x_2, x_3)$ are tested. The graphs $\varepsilon_{max}(\delta)$ do not tend to the origin when δ decreases in all three cases. One of them (for $k=1$) is shown in Fig. 4(c) with filled squares. An effective model cannot be constructed.

(6) A model (6) with $x_1(t_i) = \int_{t_1}^{t_i} \eta(t) dt$. A graph $\varepsilon_{max}(\delta)$ shows that the dependency $\dot{x}_2(x_1, x_2, \varphi)$ is single valued and, moreover, varies "gradually" [Fig. 4(c), filled circles]. A reconstructed model (6) with an additive harmonic driving and a bivariate polynomial of the 11th order demonstrates a chaotic attractor qualitatively similar to the experimental one and provides an accurate forecast 5T ahead [22].

It is significant that an optimistic estimate according to the criterion $\varepsilon_{max}(\delta)$ and good results of the global reconstruction are achieved only in the last (the sixth) case. The graphs $\overline{\varepsilon}(\delta)$ are, however, practically the same for all abovementioned choices of variables [one of them is shown in Fig. 4(b) with the dashed line]. This fact confirms that the average characteristic $\overline{\varepsilon}$ does not, in general, allow to assess the suitability of variables for global modeling.

V. CONCLUSIONS

While performing global reconstruction of a dynamical model from a time series, two very important steps are the selection of dynamical variables and the specification of the forms of functions which approximate dependencies to enter model equations. If the choice of the variables is unsuccessful, these dependencies can appear too difficult for approximation or even nonunique.

The proposed method of testing time series $\{\mathbf{x}(t_i)\}$ and $\{\mathbf{y}(t_i)\}$ derived from observable data allows to estimate whether the dependencies $y_k(\mathbf{x})$ are single valued, continuous and do not have the regions of steep "slope." Hence, the method indicates whether the selected variables are appropriate for the construction of a global dynamical model. It is based on the analysis of the local properties of the investigated dependencies, which are much more relevant for the global reconstruction than averaged characteristics. The latter can, in general, be exploited only as additional information about the selected variables. Efficiency of the proposed method is shown with different numerical and experimental examples.

The method can be applied to the reconstruction of delay differential equations and partial differential equations, since in these cases one of the steps of a modeling procedure is also an approximation of some dependencies from experimental data [10,23]. As was illustrated by the examples of Sec. IV, single valuedness of a dependency does not yet guarantee obtaining an effective global model. A single-valued dependency can appear too difficult for global approximation (especially with standard polynomials), e.g., if it oscillates intensively. However, a local approach [7,8] might appear efficient for the latter case. Some recommendations on selecting the forms of approximating functions are given in [24].

In conclusion, we should note that the proposed method states only the results of the choice of variables, but it does not say in what way the set of variables should be changed in the case of failure. It may appear necessary to add new variables, exclude or transform some of the selected variables, and so on. However, this is, obviously, the theme of a different discussion.

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