Iterative renormalization group for anomalous dimension in a nonlinear diffusion process

Tao Tu,[1,*] G. Cheng,[1] and Hua Sheng[2]

[1]Laboratory of Quantum Communication and Quantum Computation, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

[2]Department of Physics, University of Maryland, College Park, Maryland 20742

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We construct a classical successive method, the Picard method in integral equation theory, to make an iterative algorithm with the renormalization group (RG) approach to calculate the anomalous dimension in a nonlinear diffusion equation. We find our result improves than the original RG work because we begin with the εth RG solution, not the trivial fixed-point solution.

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Renormalization group (RG), and in particular, its quantum-field-theory implementation has provided us essential tools for the description of phase transitions and critical phenomena beyond mean-field theory [1–3]. Some years ago, it had been found that there are also important applications in nonequilibrium phenomena and asymptotic analysis [4–5]. In particular, applications to calculate the anomalous dimensions in the asymptotic behavior of the nonlinear partial differential equations have been discussed by Goldenfeld and his colleagues, in the case of Barenblatt's equation [6], modified porous medium equation [7], and turbulent-energy-balance equation [8], etc.

The equation discussed here is a one-dimensional nonlinear diffusion equation

$$\partial_t u = D \partial_x^2 u \tag{1}$$

with a discontinuous diffusion coefficient $D=1/2$ for $\partial_t u > 0$ and $D=(1+\varepsilon)/2$ for $\partial_t u < 0$. For the diffusion indicated in Eq. (1), there is a certain time-dependent radius $r_0(t)$ beyond which $\partial_t u < 0$ and behind which $\partial_t u > 0$. Thus, there are different diffusion coefficients in the two regions. This equation, hereafter referred to as Barenblatt's equation, describes the filtration of a compressible fluid through an elastic porous medium which is irreversibly deformable [9]. Here, we consider it with the initial condition

$$u(x,0) = \frac{1}{\sqrt{2\pi l^2}} \exp\left(-\frac{x^2}{2l^2}\right). \tag{2}$$

The formal solution to the Eq. (1) is

$$u(x,t) = \int dy\, G(x-y,t) u(y,0) + \frac{\varepsilon}{2}\int_0^t ds \int dy\, G(x-y,t-s)\Theta[-\partial_s u(y,s)]\partial_y^2 u(y,s), \tag{3}$$

where G is the Green's function

$$G(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \tag{4}$$

and Θ is the Heaviside function corresponding to the discontinuity of D.

The reason that this RG treatment is potentially useful is that the fixed point of the RG transformation is a self-similar asymptotic solution of the equation [4]. In Goldenfeld's original work [6], they construct a ε-perturbation theory for the equation, in the lowest order of ε Eq. (1) reduces to a linear diffusion equation such as $\partial_t u(x,t) - 1/2\partial_x^2 u(x,t) = 0$. The fixed point of the simple equation, i.e. the long-time behavior of the problem is $u \sim m_0/\sqrt{t}\exp(-x^2/2t)$. They start from this trivial fixed point solution to εth calculation and obtain the fixed-point solution of Eq. (1) is $u \sim A/(t^{1/2+\alpha})\exp(-x^2/2t)$ by the RG approach [6], where the anomalous dimension α appears naturally.

On the other hand, the numerical result shows that some time after the beginning of computation the following relation holds: $u \sim A/(t^{1/2+\alpha})\exp(-x^2/2t)$, i.e., the asymptotics for Eq. (1) rapidly converges to the above fixed-point solution [9]. But the value of the anomalous dimension α quan-

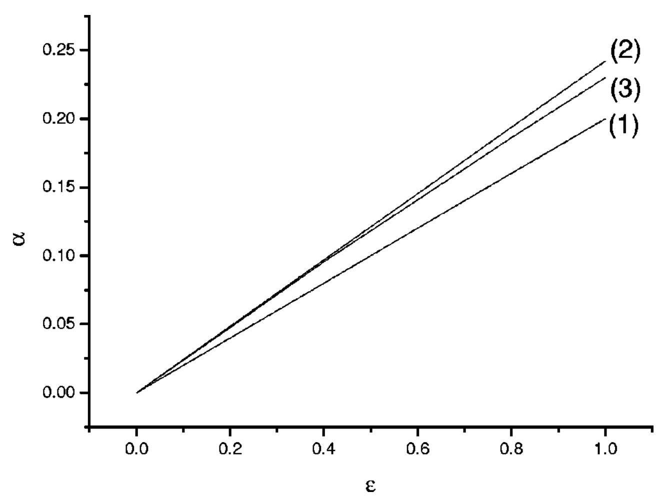

FIG. 1. The anomalous dimension α as a function of ε. The curve 1 is the numerical result from Barenblatt [9]; The curve 2 is the RG result given by Goldenfeld *et al.* [6]; The curve 3 is our RG result by *Picard*-like iteration.

*Author to whom all correspondence should be addressed. Electronic address: tutao@mail.ustc.edu.cn

titatively differs from the εth RG calculation (in Fig. 1, one can see the difference between them explicitly) because the initial step of their RG method is the trivial fixed-point solution of linear diffusion equation. Therefore, the Heaviside function is kept in its zeroth-order approximation $\Theta[-\partial_s u_0(y,s)]$ in the εth RG calculation and the effect of the Heaviside function on the final fixed-point solution is neglected. It is clear that we can improve the result of RG by beginning with εth RG because the initial step is closer to the true solution. This has a better chance of converging to the true fixed-point solution, which we will see in the following content.

To begin with the εth RG fixed-point solution, not the trivial fixed-point solution and include the effect of the Heaviside function in a certain approach, we can construct a classical successive method, the so-called *Picard* method in integral equation theory [10], to make an iterative approach possible [11].

Our general strategy in this paper can be summarized with the following steps: First, we put the Heaviside function in its zeroth-order approximation

$$u^{(1)}(x,t)=\int dy\, G(x-y,t)u(y,0)+\frac{\varepsilon}{2}\int_0^t ds\int dy\, G(x-y,t-s)\Theta[-\partial_s u_0(y,s)]\partial_y^2 u^{(1)}(y,s). \quad (5)$$

From it, we obtain the solution $u^{(1)}$ and put it into the Heaviside function again to solve the equation secondly

$$u^{(2)}(x,\ t)=\int dy\, G(x-y,t)u(y,0)=\frac{\varepsilon}{2}\int_0^t ds\int dy\, G(x-y,t-s)\Theta[-\partial_s u^{(1)}(y,s)]\partial_y^2 u^{(2)}(y,s) \quad (6)$$

and the iterative process can be continued to any nth step

$$u^{(n)}(x,t)=\int dy\, G(x-y,t)u(y,\,0)+\frac{\varepsilon}{2}\int_0^t ds\int dy\, G(x-y,t-s)\Theta[-\partial_s u^{(n-1)}(y,s)]\partial_y^2 u^{(n)}(y,s). \quad (7)$$

Using this trick, we can handle this problem step by step to include the effect of the Heaviside function.

In the first step, we start with

$$u_0(x,t)=\frac{Q_0}{\sqrt{2\pi(t+l^2)}}\exp\left[-\frac{x^2}{2(t+l^2)}\right], \quad (8)$$

$$u^{(1)}(x,t)=\int dy\, G(x-y,t)u(y,0)+\frac{\varepsilon}{2}\int_0^t ds\int dy\, G(x-y,t-s)\Theta[-\partial_s u_0(y,s)]\partial_y^2 u^{(1)}(y,s). \quad (9)$$

We posit a naive ε expansion of u

$$u^{(1)}(x,t)=u_0^{(1)}(x,t)+\varepsilon u_1^{(1)}(x,t)+\cdots \quad (10)$$

The εth term can be calculated straightforwardly. As anticipated, $u_1^{(1)}$ diverges as $t\to\infty$. We find that

$$u^{(1)}(x,t)=\frac{Q_0}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\left[1-\frac{\varepsilon}{\sqrt{2\pi e}}\ln\left(\frac{t}{l^2}\right)\right]+\text{nonsingular terms}+O(\varepsilon^2). \quad (11)$$

To deal with the divergence, we use the RG approach introduced by Goldenfeld *et al.* [5]. Hence, we obtain

$$u_R^{(1)}(x,\ t)=\frac{A}{t^{1/2+\alpha(1)}}\exp\left(-\frac{x^2}{2t}\right) \quad (12)$$

with the anomalous dimension

$$\alpha^{(1)}=\frac{\varepsilon}{\sqrt{2\pi e}}+O(\varepsilon^2), \quad (13)$$

where the subscript R denotes the renormalized quantity. We notice that it is just the lowest RG result which has been obtained by Goldenfeld *et al.* before [6]. Obviously, we would find that in their RG approach, the Heaviside function is put in its zeroth-order approximation.

Next, in the second step, we can put Eq. (12) into the Heaviside function, now we have

$$u^{(2)}(x,t)=\int dy\, G(x-y,t)u(y,\,0)+\frac{\varepsilon}{2}\int_0^t ds\int dy\, G(x-y,t-s)\Theta[-\partial_s u_R^{(1)}(y,s)]\partial_y^2 u^{(2)}(y,s). \quad (15)$$

The calculation differs slightly from the above with

$$u^{(2)}(x,t)=\frac{Q_0}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\left[1-\frac{\varepsilon\sqrt{1+2\alpha^{(1)}}}{\sqrt{2\pi\exp(1+2\alpha^{(1)})}}\ln\left(\frac{t}{l^2}\right)\right]+\text{nonsingular terms}+O(\varepsilon^2). \quad (16)$$

After the RG argument, the form of the solution remains unchanged

$$u_R^{(2)}(x,t)=\frac{A}{t^{1/2+\alpha^{(2)}}}\exp\left(-\frac{x^2}{2t}\right), \quad (17)$$

where the anomalous dimension changes into

$$\alpha^{(2)}=\frac{\varepsilon\sqrt{1+2\alpha^{(1)}}}{\sqrt{2\pi\exp(1+2\alpha^{(1)})}}. \quad (18)$$

Using the principle of mathematical induction, it is easy to prove that

$$\alpha^{(n)}=\frac{\varepsilon\sqrt{1+2\alpha^{(n-1)}}}{\sqrt{2\pi\exp(1+2\alpha^{(n-1)})}}. \quad (19)$$

Suppose that it is true for $n=m-1$, that is to say

$$\alpha^{(m-1)}=\frac{\varepsilon\sqrt{1+2\alpha^{(m-2)}}}{\sqrt{2\pi\exp(1+2\alpha^{(m-2)})}}. \tag{20}$$

Now, we substitute the $m-1$th solution

$$u_R^{(m-1)}(x,t)=\frac{A}{t^{1/2+\alpha^{(m-1)}}}\exp\left(-\frac{x^2}{2t}\right) \tag{21}$$

into the mth equation

$$u^{(m)}(x,t)=\int dy\,G(x-y,t)u(y,0)+\frac{\varepsilon}{2}\int_0^t ds\int dy\,G(x-y,t-s)\Theta[-\partial_s u_R^{(m-1)}(y,s)]\partial_y^2 u^{(m)}(y,s). \tag{22}$$

The same approach as above is repeated here and yields

$$u^{(m)}(x,t)=\frac{Q_0}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\left[1-\frac{\varepsilon\sqrt{1+2\alpha^{(m-1)}}}{\sqrt{2\pi\exp(1+2\alpha^{(m-1)})}}\ln\left(\frac{t}{l^2}\right)\right]+\text{nonsingular terms}+O(\varepsilon^2). \tag{23}$$

With RG, we can have

$$u_R^{(m)}(x,t)=\frac{A}{t^{1/2+\alpha^{(m)}}}\exp\left(-\frac{x^2}{2t}\right) \tag{24}$$

with

$$\alpha^{(m)}=\frac{\varepsilon\sqrt{1+2\alpha^{(m-1)}}}{\sqrt{2\pi\exp(1+2\alpha^{(m-1)})}}. \tag{25}$$

We have proved that it is true for $m=1$ and $m=2$. Thus, it is true for any m.

If the iterative process is convergent, the limit exists. Writing

$$\lim_{n\to\infty}\alpha^{(m)}=\alpha^{(\infty)}, \tag{26}$$

then one has

$$\alpha^{(\infty)}=\frac{\varepsilon\sqrt{1+2\alpha^{(\infty)}}}{\sqrt{2\pi\exp(1+2\alpha^{(\infty)})}}. \tag{27}$$

Two questions are to be considered. First, does the sequence converge and second, if it does converge, does it converge to the solution of the integral equation? We will discuss the convergence, the error made in replacing the final result $u_R(x,t)$ by the nth approximation $u_R^{(n)}(x,t)$ and the solution uniqueness in a separate paper.

The above equation (27) can be solved numerically, and we can compare our result with Both Goldenfeld's [6] and Barenblatt's [9] in the same picture as Fig. 1. It is obvious that our result improves than the lowest RG approach which has been done by Goldenfeld *et al.* [6].

In summary, we use a *Picard*-like iterative process to improve the calculation of RG for the anomalous dimension in Barenblatt's equation. We illustrate our proposal with the Barenbaltt's equation, however, it may be useful in studying the more interesting physical situations [12], such as the generalized porous medium equation

$$\partial_t u=D\Delta_d u^{n+1}, \tag{28}$$

which models a variety of nonequilibrium phenomena in, *inter alia*, fluid dynamics, plasma physics, and gas dynamics, depending on the value of n [13].

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