Non-Gaussian equilibrium distributions arising from the Langevin equation

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We study the Langevin equation of a point particle driven by random noise, modeled as a two-state Markov process. The corresponding master equation differs from the Fokker-Planck equation. In equilibrium, the velocity of the particle is distributed according to a binomial power law. We discuss transient (i.e., nonequilibrium) behavior, and the consequences of non-Markovian noise statistics.

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I. INTRODUCTION

The manifestation of non-Gaussian distributions in physical systems is becoming an interesting subject of research. Clear evidences of such violations have been found in many phenomena related to diffusion processes [1–3]. Recently, investigations are focusing on dynamical systems that relax to an equilibrium [4–9].

From a microscopic point of view, the success of Gaussian statistics rests on the possibility of modeling the total action of complex dynamics, in terms of a large sum of independent random variables. The central limit theorem ensures that the statistics of this sum converge to the Gaussian law, provided that their variances are defined [10]. In order to overcome this statistical hypothesis, some authors pursue a methodological approach based on the description of macroscopic quantities starting from microscopic dynamics (e.g., see [7,11–13]), hence attempting to provide a dynamical foundation of statistical physics. This methodology has difficulties due to taking into account many degrees of freedom, but the increasing power of computer machines encourages investigations in this direction (e.g., see [6,13,14]).

The one-dimensional Langevin equation provides us a simple example of this kind of description [7,12,15,16]. A point particle, of velocity $V(t)$, is immersed in a thermal bath, and the exchange action between the particle and the thermal bath is summarized by a viscous friction $-\gamma V$ and random collisions dw,

$$dV(t) = -\gamma V(t)dt + dw(t). \tag{1}$$

Usually one assumes that $dw(t)$ be an infinitesimal increment of a Wiener (or Brownian) process, having zero mean and variance per unit of time equal to $2D$. With this assumption, Eq. (1) defines a so-called Ornstein-Uhlenbeck process. The equilibrium distribution solution to the related Fokker-Planck equation is Gaussian, with zero mean and variance σ_γ^2 determined by

$$\sigma_\gamma^2 = D/\gamma. \tag{2}$$

Hence the Gaussian statistics have been recovered from the dynamical equation (1).

However the Gaussian noise $dw(t)$ is a mathematical abstraction. In fact, the finite increment $\Delta w = \int_0^{\Delta t} dw(t)$ represents, in the interval of time Δt, the total Gaussian displacement due to an infinite number of collisions [17]. When this number is large, then the Wiener noise is a good approximation of the physical reality; if not, that sort of modelization might fail (see also [4,7]). From another point of view, that assumption could be physically unplausible, because Δw could be of arbitrarily large intensity, although with a very small probability. On this basis, and following the framework of Refs. [4], [5], in this paper will be shown some results concerning the application of a more realistic noise to the Langevin equation.

II. LANGEVIN EQUATION WITH DICHOTOMOUS MARKOV NOISE

Let us imagine that random and independent events ξ_k affecting the motion of $V(t)$, happen according to a temporal sequence $\{\ldots, t_k, t_{k+1}, \ldots\}$. A simple choice for the resulting random force would be a two-state (or dichotomous) noise, defined by $\xi(t) = \Sigma_k \xi_k \chi(t; t_k, t_{k+1})$, where $\chi(t; t_k, t_{k+1}) = 1$ if $t \in [t_k, t_{k+1}]$, zero otherwise. Here ξ_k is a random variable assuming the values $-W$ or $+W$, with equal probability, corresponding to the two states of the noise. Its modulus represents the strength of a fluctuating force. The dynamics of the random force $\xi(t)$ appears like an alternation of values $\pm W$: $\xi(t)$ waits in a state, determined by the value of ξ_k, for a time interval $\tau = t_{k+1} - t_k$; at the end of which a change can occur, according to the value of ξ_{k+1}. This noise has zero mean and variance $\langle \xi^2 \rangle = W^2$.

The momentum change $\xi(t)dt$ is going to substitute for the Wiener noise $dw(t)$ of Eq. (1), so that it becomes

$$dV(t) = -\gamma V(t)dt + \xi(t)dt. \tag{3}$$

Indeed, $\int \xi(t)dt$ represents a random walk with varying steps. Hence, in order to distinguish it from the dichotomous $\xi(t)$ (i.e., its temporal derivative), $\xi(t)dt$ will be named *random walk noise*. The replacement of the Wiener noise is in agreement with what has been discussed at the end of the Introduction (see [4,5] for details). In fact, fluctuations of the Wiener process are "fast," i.e., of order $\sqrt{\Delta t}$, whereas those of the random walk are "slow," i.e., of order Δt.

This model is complete when the statistics of the stochastic series of events $\{\ldots, t_k, t_{k+1}, \ldots\}$ is given. The Poisson process is the simplest choice, where the waiting time interval τ is a random variable exponentially distributed,

$$\psi(\tau)=\lambda e^{-\lambda\tau}, \tag{4}$$

$1/\lambda$ being the mean waiting time. An alternative description is given in terms of a process of single independent events that can occur, with a rate λ, in small time steps of fixed length. This represents a Markov process, which allows us to write the corresponding master equation in a simple form. If the waiting time interval τ is not exponentially distributed, then the process will not be Markovian, and another variable between the temporal steps, representing a memory state, will be needed.

III. MASTER EQUATION AND EQUILIBRIUM SOLUTION

We are interested in finding the density distribution function related to the process $V(t)$ defined by Eq. (3). For that purpose two density distribution functions have to be defined [4,5], one for each state of the system: $p_1(v,t)dv=\mathbf{P}\{v<V(t)\leqslant v+dv, s=+W\}$ represents the probability that $V(t)\in(v,v+dv]$ and the state be $s=+W$; $p_2(v,t)dv=\mathbf{P}\{v<V(t)\leqslant v+dv, s=-W\}$ has the same probabilistic meaning, but it refers to the state $s=-W$.

With these definitions the master equation for the process established by Eq. (3), with Poissonian events according to Eq. (4), is (see [4,5] for more general cases)

$$\partial_t p_1(v,t)+\partial_v[(W-\gamma v)p_1(v,t)] = -\frac{\lambda}{2}[p_1(v,t)-p_2(v,t)], \tag{5}$$

$$\partial_t p_2(v,t)-\partial_v[(W+\gamma v)p_2(v,t)] = +\frac{\lambda}{2}[p_1(v,t)-p_2(v,t)].$$

This equation states the balance between the two states of the noise. The left-hand side represents the operator of the dynamical evolution. The right-hand side specifies, e.g., in the first equation, the amount of state change per unit time. Events occur with a rate λ; therefore, $-(\lambda/2)p_1$ is the leakage of processes escaping from the state $+W$ and going into $-W$; whereas $+(\lambda/2)p_2$ is the gain due to the processes coming from the state $-W$. The second equation descends from the first, by replacing $+W$ with $-W$, and p_1 with p_2.

The partial differential equation (5) has to be taken in conjunction with the initial and the normalizing condition,

$$p_1(v,0)=p_{01}(v), \quad p_2(v,0)=p_{02}(v),$$

$$\int [p_1(v,t)+p_2(v,t)]dv=1. \tag{6}$$

Hence the solution of Eq. (5) gives us the complete description of the process of Eq. (3). We are seeking for the distribution function regardless of the state of the system: $p(v,t)=p_1(v,t)+p_2(v,t)$. From Eq. (5), by addition and subtraction, and removing p_1-p_2, we obtain

$$\partial_v[(W^2-\gamma^2v^2)\partial_v p]-\partial_t^2 p+2\gamma v\,\partial_v\partial_t p-\gamma\partial_v[(2\gamma-\lambda)vp] +(\gamma-\lambda)\partial_t p=0. \tag{7}$$

This is a linear hyperbolic equation of the second order, departing from the Fokker-Planck equation in the presence of the second-order derivatives $\partial_t^2 p$ and $\partial_v\partial_t p$. In the absence of friction (i.e., $\gamma=0$), it becomes a telegrapher's equation driven by the two-state noise alone [18].

By neglecting the terms with the temporal derivative in Eq. (7), the equation for the equilibrium density distribution $p_{\text{eq}}(v)$ reads

$$(W^2-\gamma^2v^2)p'_{\text{eq}}(v)=\gamma^2(2-\lambda/\gamma)v\,p_{\text{eq}}(v), \tag{8}$$

the explicit solution of which is

$$p_{\text{eq}}(v)=\frac{\gamma}{W\sqrt{\pi}}\frac{\Gamma\left[\frac{1}{2}\left(\frac{\lambda}{\gamma}+1\right)\right]}{\Gamma(\lambda/(2\gamma))}\left(1-\frac{\gamma^2}{W^2}v^2\right)^{\lambda/(2\gamma)-1}, \tag{9}$$

where $\Gamma(\cdot)$ is Euler's gamma function [19]. The distribution is confined to $[-W/\gamma, W/\gamma]$, because outside this interval, the damping $-\gamma V$ becomes so large that it overcomes the acceleration provided by the dichotomous force.

The exponent $\lambda/(2\gamma)-1$ determines three kinds of behavior,

(i) $\lambda/(2\gamma)>1$: the convexity is negative and the density distribution vanishes on the boundaries of the definition interval.

(ii) $\lambda/(2\gamma)<1$: the convexity is positive ($\cup$ shaped) and the density distribution diverges when approaching the end points of the support, but its integrability is always satisfied.

(iii) $\lambda/(2\gamma)=1$: the function is constant, i.e., the distribution is uniform.

The three characteristic behaviors can be recognized in Fig. 1. In order to compare shapes, the rescaled version $\tilde{p}(z)=(W/\gamma)\,p_{\text{eq}}(zW/\gamma)$ of Eq. (9) is plotted for different values of γ, provided that λ is kept fixed.

The variance (or mean quadratic velocity) of the equilibrium distribution (9) is

$$\lim_{t\to\infty}\langle V(t)^2\rangle=\int v^2 p_{\text{eq}}(v)dv=\frac{W^2}{\gamma(\lambda+\gamma)}. \tag{10}$$

When the viscosity γ is much larger than the rate of events λ, we have $\langle V^2\rangle=W^2/\gamma^2$. This corresponds to what intuition suggests, namely, the distribution collects at the endpoint values $\pm W/\gamma$. When $\gamma\ll\lambda$, the variance becomes $\langle V^2\rangle=W^2/(\lambda\gamma)$, belonging to a Gaussian distribution. In fact, in the weak viscosity limit [4,5], by neglecting the terms of order γ^2 in Eq. (8) and preserving all those of the first, that is,

$$f'(v)\approx-(\lambda\gamma/W^2)v\,f(v), \tag{11}$$

the Gaussian density distribution results,

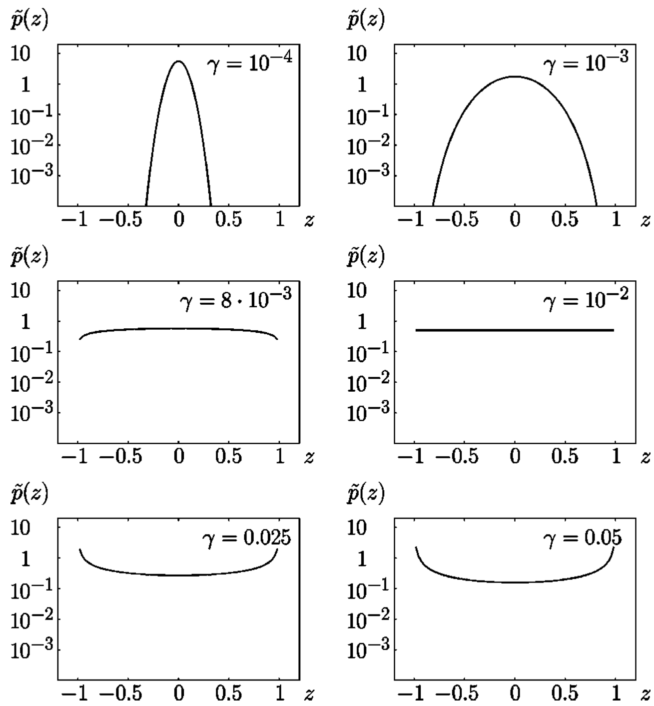

FIG. 1. Plots of Eq. (9) for different values of γ. The abscissa refers to the rescaled velocity z, the ordinate to the rescaled distribution $\tilde{p}(z)$ in logarithmic scale. Parameters: $W=1$, $\lambda=0.02$.

$$f(v)\approx(2\pi\sigma_\gamma^2)^{-1/2}\exp[-v^2/(2\sigma_\gamma^2)],\quad \sigma_\gamma^2=W^2/(\lambda\gamma), \tag{12}$$

at least for values of v not too close to the end points. Similar considerations are valid for the high-rate limit $\lambda\to\infty$ of the Poisson process.

As a rule, $p_{\rm eq}(v)$ departs from the Gaussian that one would obtain if a Wiener noise were used; although the random walk noise is an approximation to the Wiener noise, which is valid for times greater than the correlation time scale T_ξ [16], with diffusion coefficient $D=\langle\xi^2\rangle T_\xi$. The correlation time is defined as the integral of the correlation function, $T_\xi=\int_0^\infty\Phi_\xi(t)dt$, provided that the integral converges [20]. Sometimes T_ξ is referred to as the characteristic time scale, i.e., the time needed for the establishment of the statistical equilibrium. In the case of the two-state statistics with Poissonian intervals, the correlation function is exponential [21],

$$\Phi_\xi(t)=\langle\xi(t)\xi(0)\rangle/\langle\xi^2\rangle=e^{-\lambda|t|}, \tag{13}$$

so that $T_\xi=\lambda^{-1}$ and the diffusion coefficient is found to be

$$D=W^2/\lambda. \tag{14}$$

Nevertheless the presence of viscosity, in the Langevin equation, breaks this similarity, leading to different equilibrium distributions. Only in the limit of weak viscosity (or in the high-rate Poisson process) does the distribution of Eq. (9) approximate the Gaussian of the Ornstein-Uhlenbeck process, with the variance of Eq. (12) retrieved by Eqs. (2) and (14).

IV. NONEQUILIBRIUM TIME SCALE

This section is devoted to estimating the time required for the establishment of the equilibrium. The variance $\langle V^2(t)\rangle$ and the correlation function $\Phi_V(t-t')=\langle V(t)V(t')\rangle_{\rm eq}/\langle V^2\rangle$ allows us this kind of analysis without solving the time-dependent master equation (7). They can be directly calculated from the solution of Eq. (3), with the initial conditions $V(t_0)=0$ and $t_0=0$: $V(t)=\int_0^t e^{-\gamma(t-\tau)}\xi(\tau)d\tau$. By multiplying it for two distinct times t and t', taking the average $\langle\cdot\rangle$ over many realizations of the process, and inserting the correlation function, the covariance results to be

$$\langle V(t)V(t')\rangle=e^{-\gamma(t+t')}\langle\xi^2\rangle\int_0^t d\tau\int_0^{t'}d\tau' e^{\gamma(\tau+\tau')}\times\Phi_\xi(\tau-\tau'). \tag{15}$$

This is a general expression having validity for any kind of noise. By using the explicit expression of Eq. (13) in Eq. (15), the covariance results,

$$\langle V(t)V(t')\rangle=\frac{\langle\xi^2\rangle}{\gamma(\gamma^2-\lambda^2)}\{\gamma e^{-\lambda|t-t'|}-\lambda e^{-\gamma|t-t'|}+e^{-\gamma(t+t')}[\gamma(1-e^{(\gamma-\lambda)t}-e^{(\gamma-\lambda)t'})+\lambda]\}. \tag{16}$$

The first line contains the stationary addend; the second represents the transient, corresponding to the nonequilibrium properties of the covariance. The duration of this transient is approximately of the order of $1/\lambda$ and $1/\gamma$. Note that this expression is well defined only if λ differs from γ. The case $\lambda=\gamma$ can be obtained by taking the limit $\lambda\to\gamma$.

For large values of both t and t', the transient is negligible and one finds the covariance function when the process is in the steady-state,

$$\langle V(t)V(t')\rangle_{\rm eq}=\frac{W^2[\gamma e^{-\lambda|t-t'|}-\lambda e^{-\gamma|t-t'|}]}{\gamma(\gamma+\lambda)(\gamma-\lambda)}, \tag{17}$$

which is a function of $|t-t'|$.

The other quantity of interest is the variance, which is obtained by using Eq. (16) supplied with the condition $t=t'$,

$$\langle V^2(t)\rangle=\frac{2W^2}{\gamma+\lambda}\left[\frac{1-e^{-2\gamma t}}{2\gamma}+\frac{e^{-(\gamma+\lambda)t}-e^{-2\gamma t}}{\lambda-\gamma}\right], \tag{18}$$

valid for $\gamma\neq\lambda$, and

$$\langle V^2(t)\rangle=\frac{W^2}{2\gamma^2}[1-e^{-2\gamma t}(1+2\gamma t)],$$

when $\lambda=\gamma$. Define $T_V^*=\max[1/(2\gamma),1/(\lambda+\gamma)]$, so that for $t\gg 4T^*$ the equilibrium is reached and the expression of Eq. (10) is recovered. For small t the variance grows as W^2t^2.

As mentioned above, the characteristic time scale for the relaxation of the process $V(t)$ to equilibrium may be esti-

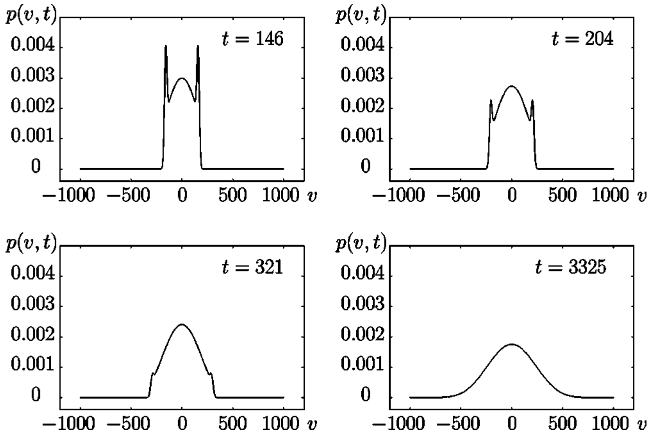

FIG. 2. Plots of $p(v,t)$ resulting from a numerical integration. Parameters: $\gamma=10^{-3}$, $\lambda=0.02$, $W=1$, $4T_V^*=2000$, $T_V=1050$. End points are at $v=-1000$ and $v=+1000$.

mated by using the correlation function $\Phi_V(t-t')$. Dividing Eq. (17) by Eq. (10) one obtains,

$$\Phi_V(t-t')=\frac{\gamma e^{-\lambda|t-t'|}-\lambda e^{-\gamma|t-t'|}}{\gamma-\lambda} \quad \text{if} \quad \gamma\neq\lambda, \qquad (19)$$

and

$$\Phi_V(t-t')=e^{-\gamma|t-t'|}(1+\gamma|t-t'|) \quad \text{if} \quad \gamma=\lambda.$$

The characteristic time scale T_V is established by the integral,

$$T_V=\int_0^\infty \Phi_V(\tau)d\tau=\frac{1}{\lambda}+\frac{1}{\gamma}=T_\xi+\frac{1}{\gamma}. \qquad (20)$$

All this means that both T_V^* and T_V depend on the two parameters characterizing the process: the viscosity and the rate of random events. Whereas in the Ornstein-Uhlenbeck process the time scale depends only on the viscosity

Numerical solution

In Figs. 2, 3, and 4 are plotted the results of a numerical integration of Eq. (5) with initial conditions $P_1(v,0)=\delta(v)/2$ and $P_2(v,0)=\delta(v)/2$. Each shows four samples of the distribution $p(v,t)$ at increasing times. The two Dirac δ peaks, not shown at the initial time, cause two peaks traveling towards the end points of the definition interval. These correspond to the density of the process that has not been subject to any transitions from the beginning. In the meantime their amplitudes decrease and the distribution takes its final equilibrium form in the middle. The middle of the distribution represents the paths that have been subject to transitions. For $t>4T^*$ or $t>T_V$ the equilibrium is almost reached, as the largest time plots show. Figure 2 shows the densities in the weak viscous case, a bell shaped distribution is obtained at the equilibrium. The uniform distribution case ($\lambda=2\gamma$) is shown in Fig. 3. In Fig. 4 ($t=117$) the two peaks

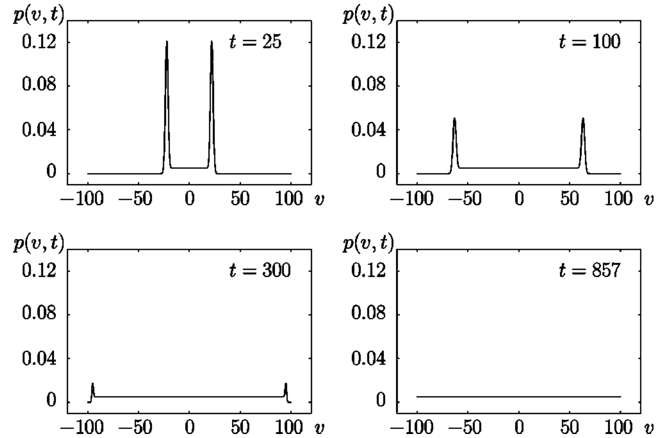

FIG. 3. Plots of $p(v,t)$ resulting from a numerical integration. Parameters: $\gamma=0.01$, $\lambda=0.02$, $W=1$, $4T_V^*=200$, $T_V=150$. End points are at $v=-100$ and $v=+100$.

at the end points represent the characteristic ∪ shaped distribution of the large friction case, and have no correspondence with the initial peaks.

We should comment on the limit of vanishing λ, in which all the paths of $V(t)$ are not subject to transitions and the final distribution is singular. The characteristic time scale T_V diverges according to the fact that the process is strongly correlated, nevertheless the singular equilibrium is almost reached after a time $4T_V^*$. Indeed, T_V is a correlation time and it lacks an interpretation as a statistical time scale due to the deterministic dynamics.

V. NON-MARKOVIAN PROCESSES

The Poisson statistics is used in the definition of the two-state process as a hypothesis that simplifies the writing of the master equation (5). For a non-Markovian dichotomous process, i.e., when the waiting time density $\psi(t)$ departs from exponential (4), it is required that one solve a master equation for a density distribution function with a memory state, characterized by integrals of a memory kernel. However, it is possible to write, at least in principle, the equilibrium equation in the following form:

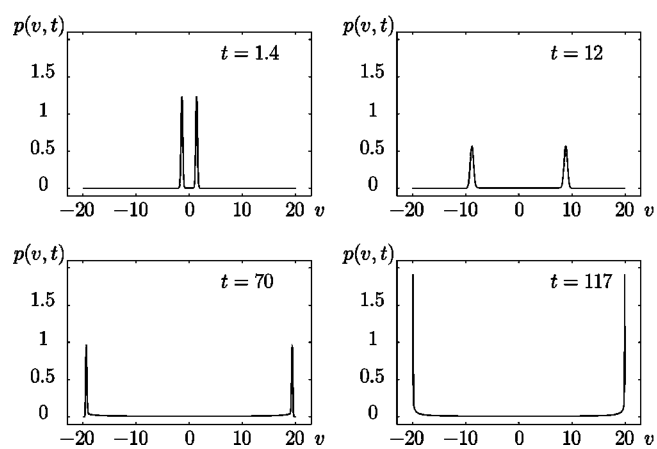

FIG. 4. Plots of $p(v,t)$ resulting from a numerical integration. Parameters: $\gamma=0.05$, $\lambda=0.02$, $W=1$, $4T_V^*=57.2$, $T_V=70$. End points are at $v=-20$ and $v=+20$.

$$\partial_v[(W-\gamma v)\hat{p}_1(v)] = -\lambda_1(v)\hat{p}_1(v) + \lambda_2(v)\hat{p}_2(v),$$
$$-\partial_v[(W+\gamma v)\hat{p}_2(v)] = +\lambda_1(v)\hat{p}_1(v) - \lambda_2(v)\hat{p}_2(v). \tag{21}$$

The integrals of the memory kernel are represented as products $\lambda_{1,2}(v)\hat{p}_{1,2}(v)$, so that the non-Markovian character of the memory state is translated into the dependence of the transition rate λ on v.

The equation for the total equilibrium density distribution $\hat{p}_{\mathrm{eq}}(v)=\hat{p}_1(v)+\hat{p}_2(v)$ is

$$(W^2-\gamma^2 v^2)\partial_v \hat{p}_{\mathrm{eq}}(v) = \{W[\lambda_1(v)-\lambda_2(v)] + \gamma[2\gamma-\lambda_1(v) - \lambda_2(v)]v\}\hat{p}_{\mathrm{eq}}(v), \tag{22}$$

the solution of which is

$$\hat{p}_{\mathrm{eq}}(v) = A\,\frac{\exp\left\{\int \lambda_1(v)/(W+\gamma v)\,dv\right\}}{W+\gamma v} \times \frac{\exp\left\{-\int \lambda_2(v)/(W-\gamma v)\,dv\right)}{W-\gamma v}, \tag{23}$$

where A is a normalization constant. The functions $\lambda_{1,2}(v)$ are unknown and have a hidden dependence on γ and W, but they help us to understand that the equilibrium solution (23) departs from Eq. (9). Nevertheless, in a framework different from that discussed here, $\lambda_{1,2}(v)$ may be computed or assigned, so Eq. (23) would furnish an explicit expression.

Of some interest [4], is the behavior of $\hat{p}_{\mathrm{eq}}(v)$ at the end points of the support. As an example, we examine the consequences of different behaviors of $\lambda_2(v)$ in the vicinity of $\overline{v}=W/\gamma$. The following behaviors of $\hat{p}_{\mathrm{eq}}(\overline{v}-\delta x)$ are possible:

(i) $\hat{p}_{\mathrm{eq}}(v)$ cannot be normalized, if $\lambda_2(\overline{v}-\delta x)\sim k(\delta x)^{\alpha}$; i.e., this transition rate does not define a density equilibrium distribution.

(ii) It exists and diverges as $|\ln(\delta x)|^{-k/\gamma}/\delta x$, if $\lambda_2(\overline{v}-\delta x)\sim k/|\ln(\delta x)|$ and the condition $k/\gamma<1$ is satisfied.

(iii) It diverges as $\exp\{-k/[\gamma(1-\alpha)]|\ln(\delta x)|^{1-\alpha}\}/\delta x$, if $\lambda_2(\overline{v}-\delta x)\sim k|\ln(\delta x)|^{-\alpha}$ with $0<\alpha<1$.

(iv) If $\lambda_2(\overline{v}-\delta x)\sim\lambda_2/2$, $\hat{p}_{\mathrm{eq}}(\overline{v}-\delta x)$ can be divergent, convergent or infinitesimal, as has been shown in Sec. III.

(v) It vanishes as $\exp\{-k/[\gamma(\alpha+1)]|\ln(\delta x)|^{\alpha+1}\}/\delta x$, if $\lambda_2(\overline{v}-\delta x)\sim k|\ln(\delta x)|^{\alpha}$ with $\alpha>0$.

(vi) It vanishes as $\exp[-k/(\gamma\alpha)(\delta x)^{-\alpha}]/(\delta x)$, if $\lambda_2(\overline{v}-\delta x)\sim k(\delta x)^{-\alpha}$ with $\alpha>0$.

These possible behaviors show that non-Markovian statistics lead to forms of equilibrium more general than Markovian do. On the other hand, it is known that a non-Markovian dynamics can be turned into a Markovian under a coarse-graining procedure [16]. But as has been explained above and with Eq. (23), this correspondence could be not realizable, pointing out that the coarse-graining approximation should be used with care, when applied to a fluctuation-dissipation system jointly to a dichotomous noise.

VI. CONCLUSIONS

In the framework of former investigations [4,5], when the Langevin equation is driven by a two-state noise, the equilibrium velocity distribution departs from the Gaussian that one would obtain by using a Wiener noise. When the two-state process has Poisson statistics, an explicit equilibrium distribution can be calculated. As has been illustrated in the plots of Fig. 1 there is a variety of equilibrium distributions.

In this way one can understand that only a complete knowledge of the noise will gives us a complete understanding of the steady-state velocity distribution. As mentioned in Sec. III in a diffusive process the validity of the central limit theorem guarantees that for times greater than of the characteristic time scale, the statistics are Gaussian, regardless of the microscopic origin of the noise. But in a fluctuation-dissipation system, in which the noise drives a velocity process, this origin plays an important role and the equilibrium distribution of the velocity may be highly non-Gaussian.

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[19] C. Tsallis, S. V. F. Levy, A. M. C. Souza, and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995). The equilibrium distribution of Eq. (9) has the same mathematical expression as that appearing in the third page, first equation, obtained by using entropic methods. When $\lambda/(2\gamma)>1$, the correspondence is satisfied provided that the anomalous index of entropy q is equal to $1-2/(\lambda/\gamma-2)$. However, it is not clear whether this is anything more than an accidental resemblance.

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