Analytic expression for the mean time to absorption for a random walker on the Sierpinski gasket

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The exact analytic expression for the mean time to absorption (or mean walk length) for a particle performing a random walk on a finite Sierpinski gasket with a trap at one vertex is found to be $T^{(n)}=[3^n 5^{n+1}+4(5^n)-3^n]/(3^{n+1}+1)$ where n denotes the generation index of the gasket, and the mean is over a set of starting points of the walk distributed uniformly over all the other sites of the gasket. In terms of the number N_n of sites on the gasket and the spectral dimension $\tilde{d}$ of the gasket, the precise asymptotic behavior for large N_n is $T^{(n)}\rightarrow 1/3(2N_n)^{2/\tilde{d}}\sim N^{1.464}$. This serves as a partial check on our result, as it is (a) intermediate between the known results $T\sim N^2$ $(d=1)$ and $T\sim N \ln N$ $(d=2)$ for random walks on d-dimensional Euclidean lattices and (b) consistent with the known result for the asymptotic behavior of the mean number of distinct sites visited in a random walk on a fractal lattice.

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I. INTRODUCTION

Random walks and diffusion on fractal structures, both regular and disordered, occur in a truly diverse variety of physical situations ranging from transport in amorphous and porous media to heterogeneous catalysis and other chemical reactions on substrates. The literature is now quite extensive, and includes several reviews [1–7]. While asymptotic behaviors and scaling relations are known in considerable detail, it is always useful to have an exact, closed-form solution on a finite structure in order to understand more quantitatively the approach to asymptopia. We concern ourselves here with such a solution on a particular fractal, the Sierpinski gasket [8]. Though deterministic, this structure possesses relevant fractal characteristics such as ramification and lacunarity to a degree that makes it a very useful model for physical applications. It therefore appears to be worthwhile to analyze the random walk problem on the gasket [9–12] in closer detail. In particular, we focus on an unbiased random walk on the fractal in the presence of a trap (a perfect absorber) at a vertex, and ask for the mean walk length before absorption (equivalently, the mean survival time). This involves a double average: the mean length of the walk from a given origin must first be computed, and then averaged over a uniform distribution of the origin over all the sites of the gasket other than the trap site. It turns out that a remarkably simple analytic expression can be obtained for this quantity, by using the exact scaling of random walks on the gasket and a resummation procedure akin to real-space renormalization.

II. FORMULATION OF THE PROBLEM

We index the stages of the hierarchical construction of the Sierpinski gasket by the generation number n, with $n=0$ corresponding to the primary equilateral triangle. There are $N_n=3/2(3^n+1)$ sites on the nth generation gasket. These will be labeled sequentially from the top to the bottom by the site index i. The trap is taken to be located at the apex, i.e., at site 1. The left-hand corner site of the bottom row corresponds to $i=N_n-2^n$, while the right-hand corner site is of course $i=N_n$ itself. It is convenient to refer to these special points as sites L and R, respectively. Figure 1 shows the gasket corresponding to $n=3$, for which $N_3=42$. We note that the triangles with vertices at sites (2, 3, 5), (4, 6, 13), (7, 8, 12), etc., are lacunary regions of the gasket.

We consider an unbiased Markovian random walk of a particle on the gasket, starting from any site other than the trap: at each time step (taken to be unity), the particle jumps with equal probabilities to any of its nearest-neighbor sites. This probability is therefore 1/4 for all sites $i\geqslant 2$, except for the bottom vertices L and R, for which this probability is 1/2. The Markov chain representing such a random walk is ergodic in the absence of the trap, i.e., the particle will visit all sites with probability 1, no matter which site it starts from; and in the presence of the trap, it is sure to be absorbed there.

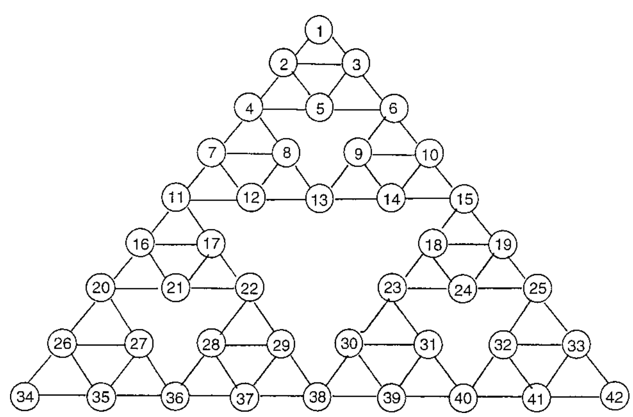

FIG. 1. The generation $n=3$ Sierpinski gasket $(N_n=42)$.

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These statements remain true even when $n \to \infty$ (as discussed briefly in the final section), though the mean time to absorption diverges in the infinite system.

The probability of survival (without absorption) of the particle after t time steps starting from any initial site i satisfies the discrete version of the backward Kolmogorov equation. As a consequence of the linearity of this equation, the moments of the survival time until absorption satisfy a system of inhomogeneous, linear, simultaneous equations. Let $T_{i,q}^{(n)}$ (where $q=0,1,\ldots,$) denote the qth moment of the survival time (the time to trapping) for a walk originating at site i on the nth generation gasket. Thus, $T_{i,q}^{(n)}$ is also the qth moment of the walk length for a random walk starting at i. By definition, $T_{1,q}^{(n)}=0$. For $2 \leqslant i \leqslant N_n$ we have, recalling that the time step has been set equal to unity,

$$-\Delta_{ij} T_{j,q+1}^{(n)} = (q+1) T_{i,q}^{(n)}, \tag{1}$$

where a summation over the repeated index j is implied, and Δ_{ij} stands for the discrete Laplacian

$$\Delta_{ij} = \frac{1}{\nu_i}\,\delta_{\langle ij\rangle} - \delta_{ij}\,. \tag{2}$$

Here, ν_i is the coordination number of site i, δ_{ij} is the Kronecker delta, and $\langle ij\rangle$ indicates that j is a nearest neighbor of i. The set of equations satisfied by the first moments or mean walk lengths $T_{i,1}^{(n)}$ is obtained by setting $q=0$ in Eq. (1). The quantity $T_{i,0}^{(n)}$ is the zeroth moment of the distribution of the time of first passage to the trap from the origin i, and is equal to unity since absorption at the trap is a sure event for every starting point i. As we shall be concerned throughout with just the set of first moments $T_{i,1}^{(n)}$, we drop the index corresponding to q henceforth and write $T_i^{(n)}$ for this quantity. We therefore have

$$-\Delta_{ij} T_j^{(n)} = 1. \tag{3}$$

We seek the mean walk length $T^{(n)}$, which is the average of $T_i^{(n)}$ over starting sites i distributed uniformly over all sites of the gasket other than the trap site $i=1$. This is given by

$$T^{(n)} = \frac{1}{(N_n-1)} \sum_{i=2}^{N_n} T_i^{(n)} = \frac{1}{(N_n-1)} \sum_{i=2}^{N_n} \sum_{j=2}^{N_n} (-\Delta^{-1})_{ij}\,, \tag{4}$$

where use has been made of Eq. (3) in writing the second equality. We note that Δ is a nonsingular matrix: each row sum is zero from the third row onwards, but the row sum is $-1/4$ for the first two rows, owing to the presence of the trap at site 1. Although Eq. (4) appears to be a rather compact expression, it must be borne in mind that Δ^{-1} is a matrix of order $(N_n-1)\times(N_n-1)$, and that N_n increases exponentially with n. Moreover, as $T^{(n)}$ involves the sum over all the elements of Δ^{-1}, and these are not preserved under a similarity transformation, we cannot, on the face of it, expect to re-express the required quantity in terms of invariants such as the trace of the matrix and its powers. This is why it is remarkable that a relatively simple closed form expression for $T^{(n)}$ exists, for general n. As mentioned earlier, the circumstance that makes this possible is the scaling of random walks together with a resummation procedure (amounting to an exact real-space renormalization) that is enabled by the regular hierarchical structure of the gasket.

TABLE I. Mean walk length $T_i^{(3)}$ from the site i on the $n=3$ Sierpinski gasket with a trap at the apex site $i=1$.

i	2,3	4,6	5	7,10	8,9	11,15	12,14
$T_i^{(3)}$	80	130	106	168	162	200	176
i	13	16–19	20, 25	21, 24	22, 23	26–33	34–42
$T_i^{(3)}$	170	226	240	234	240	248	250

III. NUMERICAL RESULTS

Before proceeding to the derivation of the general formula for $T^{(n)}$, it is instructive to note the numerical values obtained by direct calculation for the first few generations n. Wherever possible, we drop the superscript in $T_i^{(n)}$, to keep the notation uncluttered.

The case $n=0$ is trivial: we have $N_0=3$, and it is easy to see that $T_2=T_3=2$, so that $T^{(0)}=2$. The first generation that is representative of the structure is $n=1$, for which $N_1=6$. Taking advantage of the obvious symmetries, we find $T_2=T_3=8$, $T_4=T_6=10$ and $T_5=10$, so that $T^{(1)}=46/5$. Similarly, for $n=2$, the solutions are $T_2=T_3=26$, $T_4=T_6=40$, $T_5=34$, $T_7=\cdots=T_{10}=48$ and $T_{11}=\cdots=T_{15}=50$, yielding $T^{(2)}=608/14$. The values of $T_i^{(3)}$ and $T_i^{(4)}$ for the third- and fourth-generation gaskets ($N_3=42, N_4=123$) are given in Tables I and II. The values of $T_i^{(5)}$ ($N_5=366$) and $T_i^{(6)}$ ($N_6=1095$) have also been computed explicitly [13]. The mean walk lengths $T^{(n)}$ for every generation from $n=0$–6 are listed in Table III. These explicit numerical values serve as direct checks on the analytical formula to be derived below.

IV. CALCULATION OF THE MEAN WALK LENGTH

We first establish certain scaling and symmetry relations satisfied by the quantities $T_i^{(n)}$, and then use these to derive the general formula for $T^{(n)}$.

A. Time scaling on the gasket

The numerical values presented in the preceding section show that $T_4^{(1)}=10=5T_2^{(0)}$, $T_{11}^{(2)}=50=5T_4^{(1)}=5^2T_2^{(0)}$. Similarly, $T_4^{(2)}=40=5T_2^{(1)}$, $T_{13}^{(2)}=50=5T_5^{(1)}$, and so on. Doubling the chemical distance systematically increases the mean time to reach a given point for the first time by a factor of five: on a given structure (in a given generation), the mean time to hit any of the four points two lattice constants away from any site, and along the same directions as its four nearest-neighbor sites, is equal to five time steps. Exactly the same scale factor occurs in the case of the two corner sites L and R with coordination number two. This scaling is exact on the Sierpinski gasket, and is essentially the statement that the random walk dimension of the gasket [5] is $d_w=\ln 5/\ln 2$. It

TABLE II. Mean walk length $T_i^{(4)}$ from the site i on the $n=4$ Sierpinski gasket with a trap at the apex site $i=1$.

i	2, 3	4, 6	5	7, 10	8, 9	11, 15
$T_i^{(4)}$	242	400	322	528	504	650
i	12, 14	13	16–19	17, 18	20, 25	21, 24
$T_i^{(4)}$	554	530	760	754	840	792
i	22, 23	26, 33	27, 32	28, 31	29, 30	34, 42
$T_i^{(4)}$	810	914	890	848	842	1000
i	35, 41	36, 40	37, 39	38	43, 44	47, 49, 50, 52
$T_i^{(4)}$	922	880	856	850	1080	1130
i	48, 51	53, 56, 57–60	54, 55, 58, 59	61, 65, 66, 70	62, 64, 67, 69	63, 68
$T_i^{(4)}$	1106	1168	1162	1200	1176	1170
i	71–78	79, 81, 82, 84, 85, 87, 88, 90	80, 83, 86, 89	91–106	107–123	
$T_i^{(4)}$	1226	1240	1234	1248	1250	

is in fact a special case of, and follows rigorously from, a more general scaling relation for the Laplace transforms of first passage time densities on the Sierpinski gasket that can be established using renormalization arguments [12]. In the present context, it has two consequences: (i) a direct scaling of mean walk lengths from specific sites that are related by symmetry in gaskets of different generations; and (ii) any relationship among the $\{T_i^{(n)}\}$ that preserves a symmetry as one goes from one generation to another can also be shown to obey a scaling rule.

B. Decimation procedure

The scaling referred to above enables us to re-express partial sums of the $T_i^{(n)}$ in a manner that helps "decimate" sets of sites level by level in the complete sum.

Consider first the two bottom corner sites L and R. Clearly, $T_L^{(n)}=T_R^{(n)}$, by an obvious symmetry. The distance from L or R to the trap doubles in each generation while preserving the symmetry of the site. Therefore, one must have $T_L^{(n)}=5T_L^{(n-1)}=5^nT_L^{(0)}$. This is indeed borne out by direct calculation, which yields the values $T_2^{(0)}=2$, $T_4^{(1)}=10$, $T_{11}^{(2)}=50$, $T_{34}^{(3)}=250$, $T_{107}^{(4)}=1250, \ldots$.

The next, crucial, step is based on the "triangular symmetry" of the gasket (and the symmetry of the trap's location). Consider the gasket in any given generation n. Let (i_1,j_1,k_1) label the three sites demarcating any one of the *smallest* (or "size 1") lacunary triangles in the gasket, and (I_1,J_1,K_1) the three vertices of the triangle containing (i_1,j_1,k_1) as its central lacunary region: for example, referring to Fig. 1, the sites (2, 3, 5) and (1, 4, 6), respectively; or (16, 17, 21) and (11, 20, 22), respectively; and so on. Writing down Eq. (3) for the sites (i_1,j_1,k_1), it is easy to see that the sum of the mean times for this set is directly related to the corresponding sum for the set (I_1,J_1,K_1) according to

$$T_{i_1}^{(n)}+T_{j_1}^{(n)}+T_{k_1}^{(n)}=T_{I_1}^{(n)}+T_{J_1}^{(n)}+T_{K_1}^{(n)}+6. \tag{5}$$

Now let (i_2,j_2,k_2) denote the vertices of a lacunary region of size 2, such as (4, 6, 13) or (20, 22, 36); and (I_2,J_2,K_2) the corresponding triangle whose central lacunary region is delimited by (i_2,j_2,k_2), such as (1, 11, 15) or (11, 34, 38). It then follows from the scaling derived above that

$$T_{i_2}^{(n)}+T_{j_2}^{(n)}+T_{k_2}^{(n)}=T_{I_2}^{(n)}+T_{J_2}^{(n)}+T_{K_2}^{(n)}+(6\times 5), \tag{6}$$

a relation that can also be verified by direct computation after elimination of all the intermediate T_i involved. Moving up the hierarchy, if (i_r, j_r, k_r) are the sites demarcating a lacunary triangle of size r in ascending order of size, starting from the smallest at size 1, and (I_r, J_r, K_r) the vertex labels of the triangle with (i_r, j_r, k_r) as its central lacunary region, then

$$T_{i_r}^{(n)}+T_{j_r}^{(n)}+T_{k_r}^{(n)}=T_{I_r}^{(n)}+T_{J_r}^{(n)}+T_{K_r}^{(n)}+(6\times 5^{r-1}). \tag{7}$$

TABLE III. Mean walk length $T^{(n)}$ on the Sierpinski gasket at generation n, in the presence of a trap at the apex site.

n	N_n	$T^{(n)}$
0	3	2
1	6	46/5
2	15	608/14
3	42	8674/41
4	123	127 772/122
5	366	1 904 566/365
6	1095	28 507 448/1094

C. Formula for the mean walk length

The foregoing suggests how the mean walk length $T^{(n)}$ on the Sierpinski gasket may be computed for arbitrary n. This is done by suitably regrouping the terms in the sum $\sum_{i=2}^{N_n}T_i^{(n)}$, and systematically and repeatedly using Eq. (7) as one moves upwards through triangles of increasing size. It is evident that this amounts to the sort of decimation process familiar in applications of real-space renormalization. Some combinatorics, involving the enumeration of the number of lacunary triangles of each size in the gasket, is required. The final result is that $\Sigma_{i=2}^{N_n} T_i^{(n)}$ can be expressed entirely in terms of known numerical factors and the combination $(T_1^{(n)}+T_L^{(n)}+T_R^{(n)})$. Since $T_1^{(n)}\equiv 0$, while $T_L^{(n)}=T_R^{(n)}=2\times 5^n$, this leads directly to the desired expression for $T^{(n)}$. As an illustration of the procedure, consider the gasket of generation $n=4$, with $N_n=123$. Dropping for a moment the generation superscript for brevity, and with $L=107$, $R=123$, we obtain (after carrying out the procedure described above)

$$\sum_{i=2}^{123} T_i = 2(T_1+T_L+T_R)+(3^3\times 6\times 5^0)+3\{(T_1+T_L+T_R) + (3^2\times 6\times 5^1)+3[(T_1+T_L+T_R)+(3^1\times 6\times 5^2) + 3((T_1+T_L+T_R)+(3^0\times 6\times 5^3))]\}. \tag{8}$$

Thus, $\Sigma_i T_i$ has been recast in terms of the sum $(T_1+T_L+T_R)$ of the mean walk lengths from the three primary vertex sites. Note that 3^3 (the factor multiplying 6×5^0 in the expression above) is the number of lacunary regions (triangles) of size 1 on the $n=4$ gasket, 3^2 (the factor multiplying 6×5^1) is the number of lacunary triangles of size 2, and so on. Using the fact that $T_L=2\times 5^4$ in this example, we get on simplification the numerical value $T^{(4)}=127\,772/122$.

We may now carry out a similar procedure for the case of general n. The analog of Eq. (8) yields

$$\sum_{i=2}^{N_n} T_i^{(n)} = \left(1+\sum_{m=0}^{n-1} 3^m\right)(T_1^{(n)}+T_L^{(n)}+T_R^{(n)}) + (3^{n-1}\times 6)\sum_{m=0}^{n-1} 5^m. \tag{9}$$

Carrying out the summations involved, we arrive at the following result for the mean walk length, or equivalently, the mean time to absorption at the trap:

$$T^{(n)} = \frac{3^n 5^{n+1}+4(5^n)-3^n}{3^{n+1}+1}. \tag{10}$$

The values one obtains from this formula for $0\leqslant n\leqslant 6$ are in complete agreement with the ones listed in Tables I–III, which are based on the direct computation of each $T_i^{(n)}$, thus providing detailed numerical checks on our result.

The corresponding expressions for $T^{(n)}$ when traps are present at two vertices of the primary triangle, or at all three vertices, can readily be obtained by using the techniques described in the foregoing.

V. EXPRESSION IN TERMS OF FRACTAL DIMENSIONS

Further insight into the origin of the specific numbers that occur in the exact formula of Eq. (10) is obtained by re-expressing the latter in term of the system size (or number of sites) N_n, and the fractal dimensions associated with random walks on the gasket [1–7].

As is well known, the fractal dimension of the gasket (in $d=2$) is $d_f=\ln 3/\ln 2\approx 1.584$, while its random walk dimension (as already mentioned) is $d_w=\ln 5/\ln 2\approx 2.321$. The spectral (or "fraction") dimension of the gasket is $\tilde{d}=2d_f/d_w=\ln 9/\ln 5\approx 1.365$. It is $\tilde{d}$ that controls the behavior of random walks on a fractal at a deep level. For instance, on a fractal of infinite extent (and in the absence of traps), $\tilde{d}\leqslant 2$ implies persistence, i.e., sure return to any starting point; while $\tilde{d}>2$ implies transience (or a return probability <1). Since $\tilde{d}<2$ for the Sierpinski gasket, a random walk on it remains persistent even when $N_n\to\infty$. However, such a walk is null recurrent, i.e., the mean time to return to any site (or mean first passage time from any site to any other site) is infinite on the infinite gasket. In the present context, this means that $T^{(n)}$, which is just the mean first passage time to reach the trap at site 1 (averaged over all possible starting points of the walk), becomes unbounded as $N_n\to\infty$. The interesting question is the precise form of this divergence.

Since the total number of sites on the nth generation gaket is $N_n=3/2(3^n+1)$, we have $3^n=2/3N_n-1$, while 5^n can be written as $(2/3N_n-1)^{2/\tilde{d}}$. Hence, Eq. (10) becomes

$$T^{(n)} = \frac{(2N_n-3)}{(N_n-1)}\left[\frac{1}{6}(2N_n-3)^{2/\tilde{d}}+\frac{2}{5}(2N_n-3)^{2/\tilde{d}-1}-\frac{1}{6}\right], \tag{11}$$

an expression in which the N_n dependence of $T^{(n)}$ is explicit. Therefore, as $N_n\to\infty$, we have the leading asymptotic behavior

$$T^{(n)} \to \tfrac{1}{3}(2N_n)^{2/\tilde{d}} = \tfrac{1}{3}(2N_n)^{d_\omega/d_f}. \tag{12}$$

Montroll [14] has obtained rigorous results for the mean walk length of a random walk on regular lattices of N sites in the presence of traps. For large N, the leading behavior of this quantity is $\sim N^2$ in $d=1$, while it is $\sim N\ln N$ in $d=2$. One may therefore anticipate that on the Sierpinski gasket, which has a fractal dimension d_f lying between 1 and 2, and more pertinently, a spectral dimension $\tilde{d}$ lying between 1 and 2, the mean walk length $T^{(n)}$ would have a dependence on N_n that is intermediate between these two dependences. The result in Eq. (12), which shows that $T^{(n)}\sim N_n^{1.464}$, bears this out.

The result is also consistent with a known relation that is, in a sense, complementary to it. Let S_t denote the mean number of distinct sites visited in an t-step random walk on a trap-free fractal lattice. Then, as $t\to\infty$, the leading asymptotic behavior of S_t is given by $S_t\sim t^{\tilde{d}/2}$ provided $\tilde{d}<2$ (it is $\sim t$ when $\tilde{d}\geqslant 2$). As $\tilde{d}<2$ on the gasket, we have $t\sim S_t^{2/\tilde{d}}$. In the present context, N_n plays the role of S_t, as the walker eventually visits all sites with probability one from any starting point; and the mean walk length $T^{(n)}$ is roughly like the number t of steps of the walk. One may therefore expect that $T^{(n)} \sim N_n^{2/\tilde{d}}$, as found above. Of course our result establishes this relationship rigorously, and also gives the exact coefficient of proportionality.

Finally, it would be interesting to extend the technique developed here to the case of a Sierpinski gasket embedded in an arbitrary number $d > 2$ of Euclidean dimensions. The leading asymptotic behavior of $T^{(n)}$ as a function of N_n can be written down right away for such higher-dimensional analogs of the gasket. The spectral dimension of such a structure is $\tilde{d} = 2\ln(d+1)/\ln(d+3)$, which tends to 2 from below as $d \to \infty$. Therefore a random walk on the gasket in any $d \geqslant 2$ remains persistent and null recurrent. For the problem at hand (the mean walk length or survival time before absorption at the trap), we have the leading asymptotic behavior $T^{(n)} \sim (N_n)^{\ln(d+3)/\ln(d+1)}$.

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