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Nonequilibrium steady states of phonon-fermion systems

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Model kinetic equations are used to investigate a nonequilibrium steady state of a phonon-fermion system in the presence of external scatterers. The general relation between the thermodynamic values ensuring a steady state of a fermion gas in the prepense of macroscopic scatterers is found. The effect due to phonon drag is investigated.

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A recently proposed approach [1] based on the extention of the Bhatnagar-Gross-Krook (hereafter, BGK) approximation can be effectively used for the investigation of nonequilibrium states of quasiparticle mixtures. In this paper, we apply this method to study the effect of phonon drag [2] on the formation of a nonequilibrium steady state of a fermion system in the presence of macroscopic scatterers. The problem under consideration concerns a variety of physical applications such as the phonon-^{3}He quasiparticle system of ^{3}He-^{4}He mixtures [3–7] and the phonon-electron system of metals [6,8,9]. Another problem of central interest to the present paper is the effects due to the limitation of the quasiparticle mean free paths caused by external scatterers (porous media, Vycor glass, aerogel, impurities, etc.). This arises, in particular, from recently discovered nonlinearities in the transport coefficients of some superconductors [10–13] and unconventional behavior of quantum fluids in aerogel [14–17]. Both effects can be explained by the above-mentioned limitations due to the quasiparticle-scatterer collisions [18].

The existing studies of the nonequilibrium steady states of phonon-fermion systems are mainly based on the methods of nonequilibrium thermodynamics (see [8] and references therein). The variational approach of Kohler [19] constitutes the main tool of these methods. The calculations performed in a variational scheme show results which are, however, of qualitative character. Besides, the output of this method is restricted to the input: a choice of so-called trial functions. This choice is the matter of physical intuition rather than the subject of the exact analysis.

Some qualitative evaluations of the phonon contribution to the kinetic coefficient of metals were done using the gas dynamic methods [2,20,21]. Gurevich [21] and Sondheimer [22] both pointed out the necessity of the consideration of the coupled-Boltzmann equations for electrons and phonons. However, these authors did not present the exact treatment of the problem. Hanna and Sondheimer [23] obtained the approximate solution of the above-mentioned coupled-kinetic equations in the framework of the variational scheme. Baylin [24,25] treated the same problem with the inclusion of *U*-processes.

It should also be noted that most of the above-mentioned results were obtained either for classical Boltzmann or for degenerate Fermi gas. To date, the influence of phonon drag on the size effect in a gas of quasiparticles was not studied in the intermediate temperature region. However, this investigation is actual for the Fermi gas of ^{3}He quasiparticles dissolved in ^{4}He because of comparatively low Fermi temperature $T_F \sim 0.3$ K of the former.

In spite of its importance for the above-mentioned applications, a systematic analysis of the phonon contribution to the size effect in a fermion gas is still absent. The presence of external scatterers makes the quasiparticle system unclosed. If the distance between scatterers is of the order of the quasiparticle mean free path, the gas dynamics methods are inapplicable. The same holds true for the more precise Chapman-Enskog and Hilbert expansions [26], at least for its lowest orders used in practice. Violation of the momentum conservation law in the quasiparticle-scatterer collisions produces the additional difficulties in the formulation of an appropriate kinetic model. This point was discussed in [1], in detail.

In the present paper, we calculate the phonon contribution to the steady-state formation in a fermion system. In order to take into account the nonequilibrium of the phonon system, we find the solution of the coupled-Boltzmann equations in the spirit of the exact BGK approach. First of all, we focus on the effects that are due to the normal phonon scattering. This problem is considered in the most general form. We do not restrict ourselves to any particular system. For definiteness, we consider that the acoustic phonons are responsible for the main contribution to the kinetic coefficients. We also restrict ourselves to the consideration of the *N*-processes in quasiparticle collisions. The above two conditions are typical for a variety of physical realizations [2,6].

The model kinetic equation derived in [1] reads

$$\left[\left(E-\sum_{i=1}^{2}|e_i\rangle\langle e_i|\right)\hat{\nu}_{11}\left(E-\sum_{i=1}^{2}|e_i\rangle\langle e_i|\right) + (E-|e_1\rangle\langle e_1|)\hat{\nu}_{12}(E-|e_1\rangle\langle e_1|)+\hat{\nu}_L\right]|g\rangle=|V\rangle, \qquad (1)$$

where $\hat{\nu}_L=\mathrm{diag}\{-\nu_{1L},-\nu_{2L}\}$, $\hat{\nu}_{11}=\mathrm{diag}\{-\nu_{11},-\nu_{22}\}$, and $\hat{\nu}_{12}=\mathrm{diag}\{-\nu_{12},-\nu_{21}\}$; ν_{11}, ν_{22}, ν_{12}, and ν_{21} are the constant particle-particle, phonon-phonon, particle-

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phonon, and phonon-particle collision frequencies, respectively. The momentum-dependent frequency ν_{1L} (ν_{2L}) describes the particle (phonon) collisions with external scatterers. Note that the frequencies ν_{12} and ν_{21} are not independent. The general relation

$$\nu_{12}\int f_0^{(1)\prime}d\Gamma_1=\nu_{21}\int f_0^{(2)\prime}d\Gamma_2,$$

obtained in [1], can be specified for the case of a phonon-fermion system to be written as

$$n_1\nu_{12}=1.37n_{ph}\nu_{21}, \tag{2}$$

where n_{ph} and n_1 are the number densities of phonons and fermions, respectively.

The vectors $|g\rangle$, $|e_k\rangle$ ($k=1,2$), and $|V\rangle\equiv|V_1,V_2\rangle$ introduced in Eq. (1) are defined by

$$|g\rangle=\left|\begin{matrix}g_1\\ g_2\end{matrix}\right\rangle,$$

$$|e_k\rangle=\frac{\sqrt{3}(\langle p_1|p_1\rangle_1\langle p_2|p_2\rangle_2)^{-(k-1)/2}}{\sqrt{\langle p_1|p_1\rangle_1+\langle p_2|p_2\rangle_2}}\times\left|\begin{matrix}\langle p_2|p_2\rangle_2^{k-1}p_1\\ (-\langle p_1|p_1\rangle_1)^{k-1}p_2\end{matrix}\right\rangle,$$

$$|V\rangle=\left|\begin{matrix}\dfrac{\partial\epsilon_1}{\partial p_1}\partial_z\left(\dfrac{\epsilon_1-\mu}{T}\right)\\ -\dfrac{c^2p_2}{T^2}\partial_zT\end{matrix}\right\rangle. \tag{3}$$

Here, p_1 (p_2) is the projection of the fermion (phonon) momentum $\vec{p}_1$ ($\vec{p}_2$) onto the direction of gradients, ϵ_1 and μ are the fermion energy and chemical potential, respectively, f_i are the distribution functions of fermions ($i=1$) and phonons ($i=2$), and $g_if_0^{(i)\prime}=f_i-f_0^{(i)}$ are the small corrections to the equilibrium distribution functions

$$f_0^{(1)}=\frac{1}{\exp\left(\dfrac{\epsilon_1-\mu}{T}\right)+1},\qquad f_0^{(2)}=\frac{1}{\exp\left(\dfrac{c|\vec{p}_2|}{T}\right)-1}.$$

The prime denotes differentiation with respect to the argument, T is the temperature, c is the sound velocity, the gradients are directed along the z axis, and the Boltzmann constant is set equal to one.

Note that $\langle e_l|e_k\rangle=\delta_k^l$, where the scalar products are defined by [1]

$$\langle\phi_1(\vec{p_1}),\varphi_1(\vec{p_2})|\phi_2(\vec{p_1}),\varphi_2(\vec{p_2})\rangle=\langle\phi_1(\vec{p_1}),\phi_2(\vec{p_1})\rangle_1+\langle\varphi_1(\vec{p_2}),\varphi_2(\vec{p_2})\rangle_2,$$

$$\langle h(\vec{p_i}),g(\vec{p_i})\rangle_i=-\int f_0^{(i)\prime}\left(\frac{\varepsilon_i}{T}\right)h^*(\vec{p_i})g(\vec{p_i})d\Gamma_i, \tag{4}$$

and $d\Gamma_i$ is the volume element of the momentum phase space of fermions ($i=1$) and phonons ($i=2$).

Coupled integral Eqs. (1) can be reduced to the set of linear equations. We multiply Eq. (1) by the bra vectors $\langle e_l|$ ($l=1,2$) and solve a set of the obtained equations in the moments $\langle e_l|g\rangle$. After some straightforward algebra, one finds

$$\langle p_l|g_l\rangle_l=-\frac{\langle p_m(\nu_{mL}+\hat{\nu}_l)\nu_m^{-1}|p_m\rangle_m}{\langle p_m|p_m\rangle_m}\langle p_l\nu_l^{-1}|V_l\rangle_l-\hat{\nu}_l\frac{\langle p_l\nu_l^{-1}|p_l\rangle_l}{\langle p_l|p_l\rangle_l}\langle p_m\nu_m^{-1}|V_m\rangle_m,\quad l\neq m, \tag{5}$$

where $\nu_l=\nu_{ll}+\nu_{lm}+\nu_{lL}$ and

$$\hat{\nu}_l=\frac{\langle p_l|p_l\rangle_l(\nu_{12}\langle p_2|p_2\rangle_2+\nu_{21}\langle p_1|p_1\rangle_1)}{(\langle p_1|p_1\rangle_1+\langle p_2|p_2\rangle_2)^2}.$$

Now we will obtain the condition ensuring the steady state of a closed phonon-fermion system. Because the particle current vanishes in this state, we have

$$\left\langle\frac{\partial\varepsilon_1}{\partial p_1}\middle|g_1\right\rangle_1=0. \tag{6}$$

Note that the analogous condition for the phonon gas does not hold because of the presence of temperature gradient producing the heat flux.

First, we consider condition (6) for a gas of noninteractive fermions in an external potential ϵ_{ex}. The explicit form of Eq. (6) can be obtained directly from Eq. (5) written for the case $l=1,m=2$. After a significant simplification, this reads

$$n_1T\left(\partial_z\left(\frac{\mu}{T}\right)+\frac{\langle\epsilon_1p_1\nu_1^{-1}|p_1\rangle_1}{\langle p_1\nu_1^{-1}|p_1\rangle_1}\frac{\partial_zT}{T^2}\right)+\frac{4}{3}\frac{E_{ph}}{1+\Phi(n_1,T)}\frac{\partial_zT}{T}=0, \tag{7}$$

where $E_{ph}=4\pi^5T/15(T/2\pi\hbar c)^3$ is the energy of the phonon gas per unit volume, and

$$\Phi(n_1,T)=\frac{\langle p_2\nu_{2L}\nu_2^{-1}|p_2\rangle_2}{\hat{\nu}_1\langle p_2\nu_2^{-1}|p_2\rangle_2}.$$

Relation (7) presents the desired condition ensuring the steady state of a fermion gas in the presence of external scatterers. According to Eq. (7), the temperature gradient is counterbalanced by the gradient of the chemical potential so that no particle flow occurs. Let us investigate the latter term in Eq. (7) describing the effect of phonon drag. As can be seen, the gradient of the chemical potential is larger than that observed in the absence of phonons. Thus, in a closed system, phonon drag leads to an effective increase in the gradient of particle-number density, compensating the phonon contribution to thermal diffusion of a fermion gas.

Let us note that the phonon-scatterer collisions reduce the above-described effect. This follows the fact that the value $\Phi(n_1,T)$ in Eq. (7) is positive for $\nu_{2L}\neq0$ and goes to zero for $\nu_{2L}=0$. Therefore, the second term in Eq. (7) exhibits the competition between the two processes. As the temperature increases, the cross phonon-fermion scattering gives rise to the above described effect due to phonon drag. The intensity of the phonon-scatterer collisions increases simultaneously. According to Eq. (7), this latter process reduces the drag effect. Because the two above processes are governed by different temperature dependences, it can cause nonmonotonic behavior of the phonon contribution to Eq. (7) versus temperature. A more detailed analysis of this effect is to be applied to the specific systems and will be reported elsewhere.

In the case $\nu_{2L}=0$, the contribution of phonons to the steady-state formation is maximal. In order to analyze this most interesting case in detail, we put $\nu_{2L}=0$, $\epsilon_1=\epsilon+\epsilon_{ex}$, and rewrite Eq. (7) in the form

$$\partial_z P+n_1\left(\frac{\langle \epsilon p_1 \nu_1^{-1}|p_1\rangle_1}{\langle p_1\nu_1^{-1}|p_1\rangle_1}-\frac{\langle \epsilon p_1|p_1\rangle_1}{\langle p_1|p_1\rangle_1}\right)\frac{\partial_z T}{T}+\frac{4}{3}E_{ph}\frac{\partial_z T}{T}+n_1X=0, \tag{8}$$

where $X=-\partial_z\epsilon_{ex}$ is the homogeneous external force, ϵ is the free fermion energy, and

$$P=\int f_0^{(1)}p_1\frac{\partial \epsilon_1}{\partial p_1}d\Gamma_1$$

is the fermion gas pressure defined conventionally.

According to Eq. (8), the presence of phonons and external scatterers makes the particle system unclosed and causes nonzero gradient of the fermion gas pressure. Note that in the absence of external scatterers ($\nu_{1L}=0$) and field ($X=0$), the second and fourth terms in Eq. (8) vanish and one finds the condition

$$\partial_z P+\frac{4}{3}E_{ph}\frac{\partial_z T}{T}=0 \tag{9}$$

obtained in [3].

In the Knudsen regime ($\nu_{1L}\gg\nu_{11}+\nu_{12}$), the fermion-scatterer collisions contribute the most to the relaxation of a fermion gas to the steady state. Further, we restrict ourselves to the Lorentz approximation [2] for the particle-scatterer collision frequency ν_{1L} considering that it is proportional to the modulus of the particle momentum $p\equiv|\vec{p}_1|$. In this approximation, the Knudsen limit of condition (8) reads

$$\partial_z P+\frac{6F_{\frac{1}{2}}F_0-5F_{-\frac{1}{2}}F_1}{3F_{-\frac{1}{2}}F_0}n_1\partial_z T+\frac{4}{3}E_{ph}\frac{\partial_z T}{T}+n_1X=0, \tag{10}$$

where

$$F_m=\int_0^\infty \frac{x^{(1/2)+m}}{1+\exp\left(x-\dfrac{\mu}{T}\right)}dx$$

is the Fermi function.

For a classical Boltzmann gas, the Fermi function goes over into the gamma function $F_m=e^{\mu/T}\Gamma(m+3/2)$ and Eq. (10) reduces to the result generalizing the relation obtained in [1]

$$\partial_z(n_1\sqrt{T})+\frac{4}{3}\frac{E_{ph}}{\sqrt{T}}\frac{\partial_z T}{T}+\frac{n_1X}{\sqrt{T}}=0. \tag{11}$$

Note that in the absence of the external field, results (7) and (10) just yield the relation between temperature and the fermion number density gradients, ensuring the steady state in the presence of phonons. The experimental situation appropriate to the case is realized in the osmotic pressure measurement setup [4,5]. This device consists of two chambers connected with a superleak filled with the Vycor glass. The ^{3}He-^{4}He mixture placed in the chambers can overflow through the superleak to ensure the steady state. In this steady state, the temperature gradient is counterbalanced with the gradient of the ^{3}He number density according to relation (8) with $X=0$. Regrettably, the effect of collisions of quasiparticles with macroparticles of the Vycor glass [27] was neglected in the analysis of the experimental results [5]. Moreover, it is supposed that the ^{3}He quasiparticles cannot penetrate through the superleak either. In fact, this condition cannot be provided even in the smallest pores of Vycor of about 5 nm in diameter [27]. As can be easily observed from Eq. (8), the presence of macroscatterers changes the relation between gradients ensuring the steady state of the mixture. Therefore, the commonly used hydrodynamic limit (9) of the exact result (7) is not appropriate to the case. This limit rather describes the first stage of the steady-state formation due to the quick overflow of the superfluid component encountering no friction with a superleak. The true steady state is formed as a result of the relatively slow diffusion of the ^{3}He quasiparticles to the cold chamber. This process is subject to both effects of the collisions of ^{3}He quasiparticles with pore walls and phonon drag. The use of formulas (7) and (10), taking into account these effects, can lead to a better understanding of the experimental situation, described above.

Let us stress that the above-described method is not restricted to the consideration of a free fermion gas. It can be directly applied to the gas of quasiparticles. In order to illustrate this point, we write down the low-temperature limit ($T\ll T_F$) of condition (6) for a nonparabolic fermion band described by the energy-momentum relation of the form $H=H(\epsilon)$. For the sake of simplicity, we neglect the phonon-scatterer collisions and consider that $\nu_{1L}\sim p^\alpha\gg\nu_{12}$ [28]. Then the relation between gradients of temperature and chemical potential ensuring the steady state reads

$$\partial_z\mu=\left[\frac{\pi^2}{6H'_\epsilon}\frac{T}{T_F}\left((3-\alpha)\frac{T_F}{x_0}+2T_F\frac{H''_{\epsilon^2}}{H'_\epsilon}\right)+\left(1+\frac{T^2}{x_0}\frac{\pi^2}{6(H'_\epsilon)^4}(2x_0(H''_{\epsilon^2})^2-\alpha H''_{\epsilon^2}H'_\epsilon)\right)\frac{4}{3}\frac{E_{ph}}{n_1T}\right]\partial_zT, \tag{12}$$

where $x_0=H^{-1}(T_F)$. For a free fermion gas $H\equiv\epsilon$, $\alpha=0$ and the first term in Eq. (12) reduces to the well-known result $\pi^2/2\,(T/T_F)$ of the semiclassical theory [29]. Note that the fermion contribution to Eq. (12) is sensitive to momentum dependence of the fermion-scatterer collision frequency. Generally, such a dependence with $\alpha>3$ can cause the inversion of the sign of the above contribution. However, this is not the case observed in practice [28].

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