Numerical solution of the mode-coupling equations for the Kardar-Parisi-Zhang equation in one dimension

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We have studied the Kardar-Parisi-Zhang equation in the strong coupling regime in the mode-coupling approximation. We solved numerically in dimension $d=1$ for the correlation function at wave vector $\mathbf{k}$. At large times t we found the predicted stretched exponential decay consistent with our previous saddle point analysis [Phys. Rev. E **63**, 057103 (2001)], but we also observed that the decay to zero occurred in an unexpected oscillatory way. We have compared the results from mode coupling for the scaling functions with the recent exact results from Prähofer and Spohn (e-print cond-mat/0101200) for $d=1$ who also find an oscillatory decay to zero.

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The Kardar-Parisi-Zhang (KPZ) [1] equation is one of the most important phenomenological equations in physics. It is essentially a nonlinear Langevin equation and was proposed in 1986 as a coarse grained description of a growing interface. It is the simplest generalization of the diffusion equation which includes a relevant nonlinear term, and probably as a consequence of this the KPZ equation also arises in connection with many other important physical problems (the Burgers equation for one-dimensional turbulence [2], directed polymers in a random medium [3–5], etc.).

The KPZ equation in the context of a growing interface describes it by a single valued height function $h(\mathbf{x},t)$ on a d-dimensional substrate $\mathbf{x}\in \mathrm{Re}^d$ is

$$\partial_t h(\mathbf{x},t)=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+\eta(\mathbf{x},t). \tag{1}$$

The first term on the right of Eq. (1) represents the forces which tend to smooth the interface, the second describes the nonlinear growth locally normal to the surface, and the last is a noise term which mimics the stochastic nature of the growth process [6], usually chosen to be Gaussian, with zero mean and second moment $\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle =2D\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$. The equal time interface profile is usually described in terms of the roughness: $w=\sqrt{\langle h^2(\mathbf{x},t)\rangle-\langle h(\mathbf{x},t)\rangle^2}$ which for a system of size L behaves like $L^\chi f(t/L^z)$, where $f(x)\rightarrow$const as $x\rightarrow\infty$ and $f(x)\sim x^{\chi/z}$ as $x\rightarrow 0$, so that w grows with time like $t^{\chi/z}$ until it saturates to L^χ when $t\sim L^z$. χ and z are the roughness and dynamic exponent, respectively.

Above two dimensions, there are two distinct types of solutions to the KPZ equation. In the weak coupling regime ($\lambda<\lambda_c$) the nonlinear term is irrelevant and the behavior is governed by the Gaussian ($\lambda=0$) fixed point and $z=2$. The strong coupling regime ($\lambda>\lambda_c$), where the nonlinearity is relevant (and $\lambda_c=0$ for all $d\leqslant 2$), is characterized by exponents which are not known exactly in general dimension d. From the Galilean invariance [2] [invariance of Eq. (1) under an infinitesimal tilting of the surface] one can derive the relation $\chi+z=2$, which leaves just one independent exponent. For the special case when $d=1$, the existence of a fluctuation-dissipation theorem gives the exact results $\chi=1/2$, $z=3/2$.

Almost by definition there is no small parameter for a systematic perturbative treatment of the strong coupling regime. One is forced either into numerical studies of the KPZ equation, which are naturally difficult for dimension d greater than 2, or into *ad hoc* approximations. The best known of these is the so-called mode-coupling approximation [7–9], in which in the diagrammatic expansion for the correlation and response function only diagrams which do not renormalize the three point vertex λ are retained. One of the purposes of this paper is to investigate the accuracy of the mode-coupling approximation by comparing it with the recently obtained exact solution of Prähofer and Spohn [10] for $d=1$. The mode-coupling approximation is gratifyingly close to the exact solution, which encourages one to believe in the utility of the mode-coupling approach in higher dimensions where no exact solution is known or likely to be found.

In a recent paper we studied the long time properties of the KPZ equation within the mode-coupling approximation, and we predicted a stretched exponential decay for the correlation function at long times. In this paper we have found numerically the solution of the mode-coupling equations in $d=1$ which confirms the results of the previous asymptotic analysis but which also reveals that the correlation functions decay to zero in an oscillatory manner—a fact which was not revealed by our previous asymptotic analysis.

Mode-coupling equations are coupled equations for the correlations and response function. The correlation and response function are defined in Fourier space by

$$C(\mathbf{k},\omega)=\langle h(\mathbf{k},\omega)h^*(\mathbf{k},\omega)\rangle,$$

$$G(\mathbf{k},\omega)=\langle \partial h(\mathbf{k},\omega)/\partial\eta(\mathbf{k},\omega)\rangle,$$

where $\langle\cdot\rangle$ indicate an average over η. In the mode-coupling approximation, the correlation and response functions are the solutions of two coupled equations,

$$G^{-1}(\mathbf{k},\omega)=G_0^{-1}(\mathbf{k},\omega)+\lambda^2\int\frac{d\Omega}{2\pi}\int\frac{d^dq}{(2\pi)^d}[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]\times[\mathbf{q}\cdot\mathbf{k}]G(\mathbf{k}-\mathbf{q},\omega-\Omega)C(\mathbf{q},\Omega), \quad (2)$$

$$C(\mathbf{k},\omega)=C_0(\mathbf{k},\omega)+\frac{\lambda^2}{2}|G(\mathbf{k},\omega)|^2\int\frac{d\Omega}{2\pi}\int\frac{d^dq}{(2\pi)^d}\times[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]^2C(\mathbf{k}-\mathbf{q},\omega-\Omega)C(\mathbf{q},\Omega), \quad (3)$$

where $G_0(\mathbf{k},\omega)=(\nu k^2-i\omega)^{-1}$ is the bare response function, and $C_0(\mathbf{k},\omega)=2D|G(\mathbf{k},\omega)|^2$. In the strong coupling limit, $G(\mathbf{k},\omega)$ and $C(\mathbf{k},\omega)$ take the following scaling forms

$$G(\mathbf{k},\omega)=k^{-z}g(x),$$

$$C(\mathbf{k},\omega)=k^{-(2\chi+d+z)}n(x),$$

with $x=\omega/k^z$. In $d=1$, the mode-coupling equations simplify, due to the existence of a fluctuation dissipation theorem which relates the correlation function to the response function. In t and k space, the fluctuation dissipation theorem can be written as

$$G(k,t)=\frac{\nu k^2}{D}\Theta(t)C(k,t). \quad (4)$$

(We use the same notation G and C in t space and ω space and indicate which one we mean by the arguments.) We choose $\nu=1$ and $D=1$ in what follows. The mode-coupling equations are then reduced to one single equation that in terms of the response function in k and t space $G(tk^z)=G(k,t)$ reads

$$G(\tau)=1-\lambda^2\int_0^\tau d\sigma\int_0^\sigma ds\,\nu(s)G(\sigma-s), \quad (5)$$

where τ is the scaling variable tk^z,

$$\nu(s)=\frac{1}{2\pi}\int_0^\infty dx G(|1/2+x|^z s)G(|1/2-x|^z s), \quad (6)$$

and $z=3/2$. A similar analysis to the one we have done can be found in Ref. [11], where similar results were found (compare Fig. 1 with Fig. 1 in Ref. [11]), but here we focus on the long time asymptotics. To make this comparison we set here and in what follows $\lambda=1$. In a previous study [12], we argued that an asymptotic solution $\hat{n}(\tau)=\hat{n}_\infty(\tau)$ for $\tau\to\infty$ for the scaling part of the correlation function in t and $\mathbf{k}$ space [$\tau=tk^z$, and $\hat{n}(\tau)$ is the Fourier transform of $n(x)$] is given by

$$\hat{n}_\infty(\tau)=A(B\tau)^{\gamma/z}e^{-|B\tau|^{(\alpha/z)}}, \quad (7)$$

with $\gamma=(d-1)/2$, $\alpha=1$, and

$$A=\frac{g(0)^{-2}4\Gamma(4z-4)}{P2^{(d-1)/2}\Gamma(2z-2)^2}, \quad (8)$$

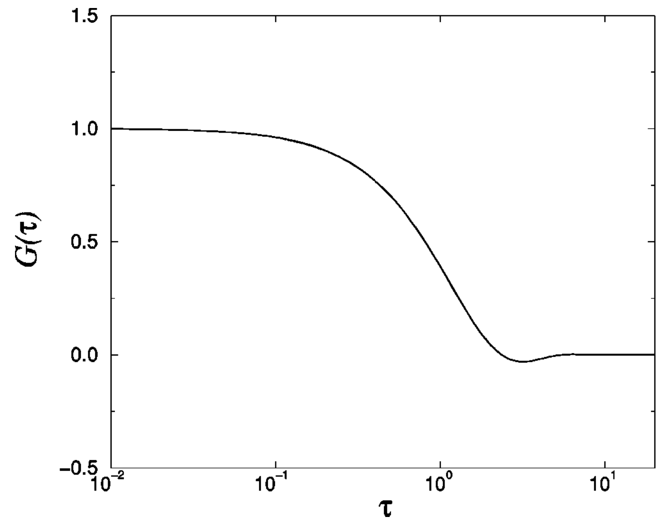

FIG. 1. Scaling function for the response function $G(\tau)$.

with $P=\lambda^2/(2^d\pi^{(d+3)/2})$.

In $d=1$, $G(\tau)\propto\hat{n}(\tau)$ due to the fluctuation dissipation theorem, so that we expect the same asymptotic expression for G. Our numerical analysis shows, much to our surprise, an oscillatory behavior for the correlation function superimposed on the stretched exponential decay.

The long time behavior can be revealed by plotting $|G(\tau)|$ as a function of $\tau^{2/3}$ in a linear-log scale (see Fig. 2), where both the presence of the oscillations and the overall stretched exponential decay of the envelope become apparent. We do not have any simple argument to explain the presence of such oscillations. However, we can show that they are perfectly consistent with the saddle point analysis performed in Ref. [12]. The same calculation can in fact be repeated with a complex exponential with the constant B a complex number. While the calculation leads to the same values of γ and α, it is not now possible to predict the amplitude constant A.

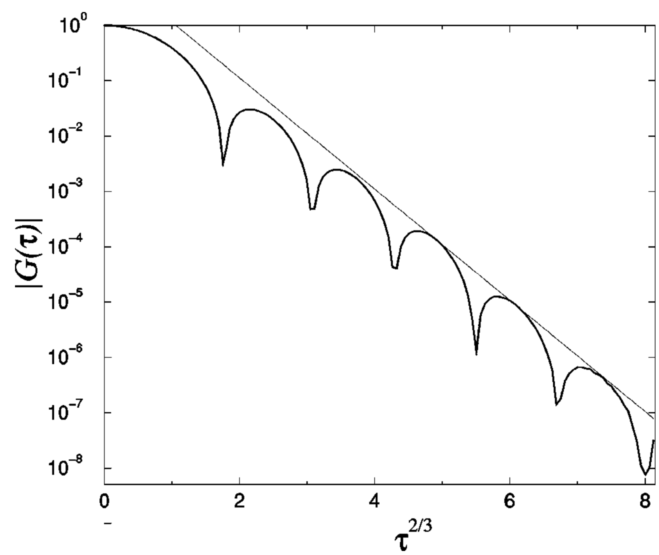

FIG. 2. Oscillation and stretched exponential behavior in the response function $G(\tau)$. The straight line indicates a line of slope -1 as predicted by Eq. (7).

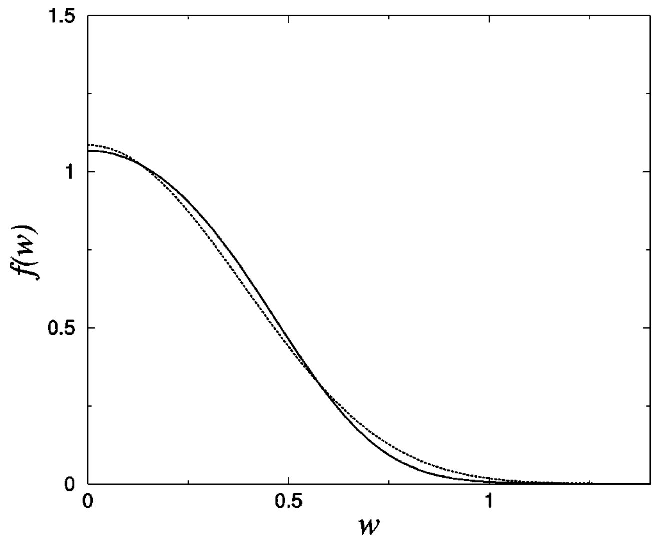

FIG. 3. Comparison with the results in Ref. [10]: the solid line is our result for $f(w)$, the dotted line is the same function from Ref. [10].

We next compare the results from mode coupling with the result for the scaling function in $d=1$ from the exact solution, which recently has became available [10]. The result in Ref. [10] is given in terms of a function $f(w)$ which is related to our $G(\tau)$ by

$$f(w)=\frac{1}{\pi}\int_0^\infty dy\,\cos(wy)G(y^z/4). \tag{9}$$

The results are shown in Fig. 3, and show a reasonable agreement between the mode-coupling approximation and the exact solution [13]. From the exact solution it is also possible to numerically calculate $G(\tau)$ and compare it with our approximate mode-coupling solution. The exact solution also displays the oscillatory behavior which we have discovered in the mode-coupling approximation [13].

Note that the function $f(w)[g''(w)/8$ in Ref. [10]] is also related to the truncated correlation function in real space

$$\tilde{C}(x,t)\equiv\int_{-\infty}^{\infty}\frac{dk}{\pi}[C(k,0)-\cos(kx)C(k,t)]=xF(t/x^{3/2}) \tag{10}$$

by $f(w)=\frac{1}{2}(d^2/dw^2)[wF(1/4w^{2/3})]$.

An earlier study of the accuracy of the mode-coupling approximation was undertaken by Frey *et al.* [11], who studied the magnitude of the corrections of higher order diagrams to the bare vertex. They found that such diagrams did produce substantial corrections. It is clear, however, that such contributions are relatively unimportant for the correlation function we have studied.

In summary, we have presented a numerical study of the mode-coupling equations for the strong coupling regime of the KPZ equation in the long time limit in $d=1$. We recovered the stretched exponential relaxation for the correlation function predicted previously in Ref. [12], but found a superimposed oscillation. Such oscillations are consistent with our previous analysis, even though we had not anticipated them. Finally, we compared the results from mode-coupling theory in $d=1$ with the exact solution from Ref. [10].

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