Scalings of scalar structure functions in a velocity field with coherent vortical structures

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In planar turbulence modeled as an isotropic and homogeneous collection of two-dimensional noninteracting compact vortices, the structure functions $\mathcal{S}_p(r)$ of a statistically stationary passive scalar field have the following scaling behavior in the limit where the Péclet number $\mathrm{Pe}\rightarrow\infty$: $\mathcal{S}_p(r)\sim\mathrm{const}+\ln(r/L\,\mathrm{Pe}^{-1/3})$ for $L\,\mathrm{Pe}^{-1/3}\ll r\ll L$, $\mathcal{S}_p(r)\sim(r/L\,\mathrm{Pe}^{-1/3})^{6(1-D)}$ for $L\,\mathrm{Pe}^{-1/2}\ll r\ll L\,\mathrm{Pe}^{-1/3}$, where L is a large scale and D is the fractal codimension of the spiral scalar structures generated by the vortices $(1/2\leqslant D<2/3)$. Note that $L\,\mathrm{Pe}^{-1/2}$ is the scalar Taylor microscale that stems naturally from our analytical treatment of the advection-diffusion equation. The essential ingredients of our theory are the locality of interscale transfer and Lundgren's time average assumption. A phenomenological theory explicitly based only on these two ingredients reproduces our results and a generalization of this phenomenology to spatially smooth chaotic flows yields $(k\ln k)^{-1}$ generalized power spectra for the advected scalar fields.

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I. INTRODUCTION

The theory of turbulent passive scalars has received much attention recently [1–3]. The mixing of a scalar field θ in a velocity field $\mathbf{v}$ is governed by the advection-diffusion equation

$$\partial_t\theta(\mathbf{x},t)+\mathbf{v}(\mathbf{x},t)\cdot\nabla\theta(\mathbf{x},t)=\kappa\nabla^2\theta(\mathbf{x},t)+f(\mathbf{x},t), \quad (1.1)$$

where κ denotes the molecular diffusivity of the scalar θ and $f(\mathbf{x},t)$ is an external source (forcing) driving the scalar. The mixing of a scalar field θ is characterized by its structure functions $S_p\equiv\langle[\theta(\mathbf{x}+\mathbf{r})-\theta(\mathbf{x})]^p\rangle=\langle[\delta\theta(\mathbf{r})]^p\rangle$ for any number p. If we want to find S_p then we need models of the velocity field $\mathbf{v}$.

The model that has attracted much attention recently is the Kraichnan model [1] where the velocity field $\mathbf{v}$ is considered to be incompressible, statistically isotropic, white noise in time (δ correlated) and Gaussian. Furthermore, it has homogeneous increments with power law spatial correlations

$$\langle[v_i(\mathbf{r},t)-v_i(0,0)][v_j(\mathbf{r},t)-v_j(0,0)]\rangle =2\,\delta(t)r^h\left[(h+d-1)\,\delta_{ij}-h\frac{r_ir_j}{r^2}\right], \quad (1.2)$$

where the scaling exponent $h\in]0,2[$ and d is the dimension of space so that $i,j=1,2,\ldots,d$. The above tensorial structure of the velocity field is in conformity with incompressibility. The Kraichnan model also assumes a forcing $f(\mathbf{x},t)$ in Eq. (1.1) that is an independent Gaussian random field with zero mean. The forcing is white in time and its covariance is assumed to be a real, smooth, positive-definite function with rapid decay in space so that the forcing is homogeneous, isotropic and takes place on the integral scale L. The generic scaling behavior of the structure function $S_p\sim r^{\zeta_p}$, $r\ll L$, of passive scalars in the Kraichnan model was established in Refs. [1–4]. The scaling exponents of this formalism are of the form $\zeta_p=\zeta_p(d,h,p)$ where $h\in[0,2]$ is the Hölder exponent in Eq. (1.2). In the context of this model Balkovsky and Lebedev [5] and Chertkov [6] used the instantonic formalism in a d-dimensional space to find the scaling exponents for large p. It was also shown in the instantonic formalism that $\lim_{p\rightarrow\infty}\zeta_p\simeq d(2-h)^2/8h$ [5] which is independent of p. The scaling exponents were also calculated using other techniques in the limits $h\rightarrow0$ [2,7], $d\rightarrow\infty$ [8,3], and $p\rightarrow\infty$ [5], and a $2-h$ expansion of ζ_p was proposed in Ref. [4]. It was found that ζ_p does depend on p for small values of p in the Kraichnan model.

Our work lies in the opposite extreme of the Kraichnan model. We work in the regime where we have a persistent vortical velocity field frozen in time in two dimensions. The important differences between this model and the Kraichnan model are in the structure of the velocity field infinitely correlated in time in this model but delta correlated in time in the Kraichnan model; and vortical in space in this model, but Gaussian in the Kraichnan model. In this model, the velocity field is that of spatially distributed noninteracting two dimensional (2D) vortices with compact structure. We consider the spatial distribution of vortices to be dilute in that they are far from each other and, therefore, maintain their structure and spatial position for an indefinite period. We also consider this distribution to be homogeneous and isotropic and the velocity field to be incompressible, that is $\nabla\cdot\mathbf{v}=0$. The model of the velocity field considered here is an artificial model of planar homogeneous turbulence where the emphasis is on the coherent vortex aspect of the flow. In order of presentation, the first aim of this model is to demonstrate that in the case of the unforced scalar [$f=0$ in Eq. (1.1)] we can quantify the statistics of the turbulent scalar field in terms of the scalar's spiral geometry generated by the coherent vortical structures in the flow (Sec. III and Sec. IV). The second aim is to derive the Batchelor k^{-1} power spectrum and all the corresponding

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structure functions for the scalar field in the case where the scalar is forced and in a statistically steady state (Sec. V). Such a spectrum has been recently observed by Jullien *et al.*, [9] in a 2D turbulent flow with well defined, albeit short lived, coherent vortical structures.

In the next section we discuss the scenario of a decaying scalar field in an isolated vortex. In Sec. III, we define structure functions and calculate the spectrum of higher order correlation functions for a single spiral created by a single vortex and decaying by the action of molecular diffusion. In Sec. IV we generalize our analysis to many noninteracting vortices and calculate the scalings of the structure functions of the decaying scalar field. In Sec. V we calculate the generalized power spectra and the corresponding structure functions of statistically stationary scalar spirals by applying the time-average operation approach of Lundgren [10]. In Sec. VI we discuss the phenomenology behind the k^{-1} scalings of the generalized power spectra. Section VII contains conclusions, discussion and the obtainment of the $(k\ln k)^{-1}$ scalings of the generalized power spectra in smooth chaotic flows.

II. PASSIVE SCALAR IN A PLANAR VORTEX

The advection of a decaying passive scalar field by a single planar vortex has been studied by Flohr and Vassilicos [11]. We use the formulation used in Ref. [11], namely,

$$\partial_t\theta+\Omega(r)\partial_\phi\theta=\kappa\nabla^2\theta, \tag{2.1}$$

where $\partial_t=\partial/\partial t$ and $\partial_\phi=\partial/\partial\phi$ and $\Omega(r)=\Omega_0(r/L)^{-s}$ and L is the maximum distance of the scalar interface from the center of the vortex. This equation describes the advection and diffusion of a scalar field θ in the azimuthal plane $\mathbf{r}=(r,\phi)$ by a steady vortex with azimuthal velocity component $u_\phi(r)=r\Omega(r)=L\Omega_0(r/L)^{1-s}$ and vanishing radial and axial velocity components. Direct numerical simulations and experiments in the laboratory have demonstrated the existence and importance of coherent vortices in two-dimensional turbulence and in two-component turbulence in stably-stratified flow with and without rotation of the reference frame [12–15]. Note that axial velocity fields of the form $u_\phi(r)=L\Omega_0(r/L)^{1-s}$ have been used in Refs. [10,11,16–21] and that their large wave number energy spectrum has the form $E(k)\sim k^{-5+2s}$ for $1/2<s<2$ with the appropriate large scale bound. We choose $s\geqslant1$ to ensure that $u_\phi(r)$ does not increase with increasing r and $s<2$ to ensure that the energy spectrum is steeper than k^{-1}. The initial scalar field $\theta_0=\theta\,(\mathbf{x},t=0)$ is characterized by a regular interface between $\theta_0\neq0$ and $\theta_0=0$ with minimal distance r_0 and maximal L from the rotation axis. By regular structure we mean that the interface has no irregularities on scales smaller than L. Nothing else needs to be specified about the initial scalar field $\theta_0(\mathbf{x})$. Such a patchy initial condition where all the nonzero scalar is confined within a regular interface mimics well initial conditions in certain laboratory experiments where scalar is injected in the flow in the form of blobs (e.g., Ref. [9]).

As time proceeds, the patch winds around the vortex and builds up a spiral structure and decays due to diffusion. The characteristic time Ω_0^{-1} is the inverse angular velocity of the vortex at L. This defines a Péclet number $\mathrm{Pe}=\Omega_0L^2\kappa^{-1}$. We nondimensionalize Eq. (2.1) by using the following transformations:

$$L^{-1}r\to r,\quad \Omega_0t\to t,\quad \Omega_0^{-1}\Omega(r)\to\Omega(r),\quad L^2\nabla^2\to\nabla^2,$$

and Eq. (2.1) takes the form

$$\partial_t\theta+\Omega(r)\partial_\phi\theta=\frac{1}{\mathrm{Pe}}\nabla^2\theta. \tag{2.2}$$

In the nondimensionalized variables we have $\Omega(r)=r^{-s}$, and r_0 represents r_0/L since $L=1$. Considering finite diffusivity κ, the form of the general solution of the evolution equation(2.2) for any initial field θ_0 in the limit of large t, i.e., $t\gg1$, is the following [11]:

$$\theta(\mathbf{r},t)=\sum_n f_n(r,t)e^{in[\phi-\Omega(r)t]},$$

$$f_n(r,t)=f_n(r,0)\exp\left[-\frac{1}{3}n^2\Omega'^2\,\mathrm{Pe}^{-1}t^3\right], \tag{2.3}$$

where $r=|\mathbf{r}|$ and ϕ is the azimuthal angle and n is an integer. Ω' is the derivative of Ω with respect to r. The angular Fourier coefficients $f_n(r,t)$ are time dependent and the initial condition is fully specified by $f_n(r,0)$. We do not go in to the details of the solution of Eq. (2.1) which can be found in Ref. [11].

III. STRUCTURE FUNCTIONS OF PASSIVE DECAYING SCALAR IN ONE VORTEX

In this work we concentrate in finding the scaling properties of the structure functions of the scalar field in a planar turbulence consisting of coherent vortices. The two point equal time pth order structure function is defined as follows

$$\mathcal{S}_p(r,t)=\overline{\langle[\theta(\mathbf{x}+\mathbf{r},t)-\theta(\mathbf{x},t)]^p\rangle}=\overline{\langle[\delta\theta(\mathbf{r},t)]^p\rangle}. \tag{3.1}$$

S_p depends only on the magnitude of the distance between two points, when the ensemble average is taken over an isotropic and homogeneous distribution of the scalar field. The overbar denotes ensemble averaging and the brackets imply space averaging ($\langle\ \dots\ \rangle\propto\int d^2\mathbf{x}$).

Let us first calculate $\langle[\delta\theta(\mathbf{r},t)]^p\rangle$ for one 2D vortex. We use the binomial expansion as follows:

$$\begin{aligned}\langle[\theta(\mathbf{x}+\mathbf{r},t)-\theta(\mathbf{x},t)]^p\rangle &=\langle\theta^p(\mathbf{x}+\mathbf{r},t)\rangle+(-1)^p\langle\theta^p(\mathbf{x},t)\rangle\\ &\quad+\sum_{q=1}^{p-1}C_{qp}(-1)^q\langle\theta^{p-q}(\mathbf{x}+\mathbf{r},t)\,\theta^q(\mathbf{x},t)\rangle,\end{aligned} \tag{3.2}$$

where C_{qp} is the binomial coefficient of the expansion. In order to calculate $\langle[\delta\theta(\mathbf{r},t)]^p\rangle$ we first determine

$$\langle\theta^q(\mathbf{x},t)\,\theta^{p-q}(\mathbf{x}+\mathbf{r},t)\rangle=B_{pq}(\mathbf{r},t), \tag{3.3}$$

which is the qth term in the binomial expansion of $\langle[\delta\theta(\mathbf{r})]^p\rangle$ as shown in Eq. (3.2). Now we write the above as

$$\frac{1}{L_A^2}\int d^2\mathbf{x}\,\theta^q(\mathbf{x},t)\,\theta^{p-q}(\mathbf{x}+\mathbf{r},t)=B_{pq}(\mathbf{r},t), \tag{3.4}$$

where L_A is a large scale over which the spatial average may be calculated. The Fourier transform of Eq. (3.4) is given by

$$\hat{F}_{pq}(\mathbf{k},t)=\frac{1}{2\pi}\int e^{-i\mathbf{k}\cdot\mathbf{r}}B_{pq}(\mathbf{r},t)d^2\mathbf{r}. \tag{3.5}$$

Substituting Eq. (3.4) in Eq. (3.5) and after some standard manipulations we get

$$\hat{F}_{pq}(\mathbf{k},t)=\frac{1}{2\pi L_A^2}\int\theta^q(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}}d^2\mathbf{x}\times\int\theta^{p-q}(\mathbf{x}',t)e^{-i\mathbf{k}\cdot\mathbf{x}'}d^2\mathbf{x}'. \tag{3.6}$$

Now if we integrate Eq. (3.6) over a circular shell in k space we get

$$F_{pq}(k,t)=\int_0^{2\pi}dA(k)\hat{F}_{pq}(\mathbf{k},t), \tag{3.7}$$

where $dA(k)\equiv kd\phi_k$, $k=|\mathbf{k}|$, and ϕ_k is the angle of $\mathbf{k}$. $F_{pq}(k,t)$ could be called the generalized power spectrum of the scalar field in Fourier space. Substituting Eq. (2.3) and Eq. (3.6) in Eq. (3.7) we get the following

$$F_{pq}(k,t)=\frac{1}{(2\pi L_A^2)}\int dA(k)\int d^2\mathbf{x}\;\; e^{i\mathbf{k}\cdot\mathbf{x}}\times\left\{\sum_n f_n(x,t)e^{[in(\phi-\Omega(x)t)]}\right\}^q\int d^2\mathbf{x}'e^{-i\mathbf{k}\cdot\mathbf{x}'}\times\left\{\sum_m f_m(x',t)e^{[im(\phi'-\Omega(x')t)]}\right\}^{p-q}, \tag{3.8}$$

where $x=|\mathbf{x}|$ and $x'=|\mathbf{x}'|$. After some standard manipulations Eq. (3.8) leads to

$$F_{pq}(k,t)=\frac{1}{(2\pi L_A^2)}\int kd\phi_k\int dx\,xJ_n(kx)2\pi(i)^n e^{in\phi_k}\times\sum_{n,n_1,n_2\ldots n_{q-1}}f_{n_1}f_{n_2}\cdots f_{n-n_1\ldots-n_{q-1}}e^{-in\Omega(x)t}\times\int dx'x'e^{im\phi_k}J_m^*(kx')2\pi(-i)^m\times\sum_{m,m_1,m_2\ldots m_{p-q-1}}f_{m_1}f_{m_2}\cdots f_{m-m_1\ldots-m_{p-q-1}}\times e^{-im\Omega(x')t}, \tag{3.9}$$

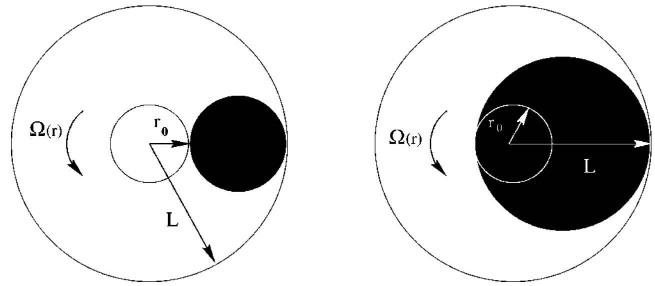

FIG. 1. The two cases (I and II) when the center of the vortex is, respectively, outside a scalar patch and inside it.

where we have changed summation variables in accordance with the following conditions:

$$n=n_1+n_2+n_3\ldots n_q$$

and

$$m=m_1+m_2+m_3+\ \ldots m_{p-q}. \tag{3.10}$$

All the f_n's and f_m's are functions of time t and of x and x', respectively. $J_n(kx)$ is the Bessel function that has been substituted in place of the integral representation

$$\int_0^{2\pi}e^{in\phi}\exp[-ikx\cos(\phi-\phi_k)]d\phi=2\pi(i)^n e^{in\phi_k}J_n(kx). \tag{3.11}$$

After integrating Eq. (3.9) with respect to ϕ_k and summing over m we get the following relation:

$$F_{pq}(k,t)=\frac{1}{(2\pi L_A^2)}\int dx\,kxJ_n(kx)2\pi(i)^n\times\sum_{n,n_1,n_2\ldots n_{q-1}}f_{n_1}f_{n_2}\cdots f_{n-n_1\ldots-n_{q-1}}e^{-in\Omega(x)t}\times\int dx'x'J_{-n}^*(kx')2\pi(-i)^{-n}\times\sum_{m_1,m_2\ldots m_{p-q-1}}f_{m_1}f_{m_2}\cdots f_{-n-m_1\ldots-m_{p-q-1}}\times e^{in\Omega(x')t}. \tag{3.12}$$

We now have two cases to study (see Fig. 1). In case I the vortex center is outside the scalar patch and its nearest distance to the scalar interface is r_0. In Case II the center of the vortex is inside the scalar patch and its nearest distance to the scalar interface is r_0. In view of the above we can write Eq. (3.12) as

$$F_{pq}(k,t)=\frac{1}{(2\pi L_A^2)}\left[\int_0^{r_0}+\int_{r_0}^{\infty}\right]dx\,kxJ_n(kx)2\pi(i)^n \times\sum_{n,n_1,n_2\dots n_{q-1}} f_{n_1}f_{n_2}\cdots f_{n-n_1\dots-n_{q-1}} \times e^{-in\Omega(x)t} \times\left[\int_0^{r_0}+\int_{r_0}^{\infty}\right]dx'x'J^*_{-n}(kx')2\pi(-i)^{-n} \times\sum_{m_1,m_2\dots m_{p-q-1}} f_{m_1}f_{m_2}\cdots f_{-n-m_1\dots-m_{p-q-1}} \times e^{in\Omega(x')t}. \tag{3.13}$$

In Eq. (3.13) we have four terms of the form as shown below

$$\int_0^{r_0}dx\int_0^{r_0}dx'(\dots)+\int_0^{r_0}dx\int_{r_0}^{\infty}dx'(\dots) +\int_{r_0}^{\infty}dx\int_0^{r_0}dx'(\dots)+\int_{r_0}^{\infty}dx\int_{r_0}^{\infty}dx'(\dots), \tag{3.14}$$

with the integrands denoted by $(\dots)$ being the same as Eq. (3.13) for all the terms of the above. For case I, terms containing $\int_0^{r_0}(\dots)$ are zero since $f_n=0$ in the regime $0<x<r_0$ for all n, even for $n=0$, since there is no scalar patch in the region $0<x<r_0$. Therefore, contributions only come from the term

$$\int_{r_0}^{\infty}dx\,x\int_{r_0}^{\infty}dx'x'(\dots).$$

For case II we can legitimately replace θ by $\theta-f_0$ and get the same result as in case I. Hence the only contributing term is the following:

$$F_{pq}(k,t) =\frac{1}{(2\pi L_A^2)}\int_{r_0}^{\infty}dx\,kxJ_n(kx)2\pi(i)^n \times\sum_{n,n_1,n_2\dots n_{q-1}} f_{n_1}f_{n_2}\cdots f_{n-n_1\dots-n_{q-1}}e^{-in\Omega(x)t} \times\int_{r_0}^{\infty}dx'x'J^*_{-n}(kx')2\pi(-i)^{-n} \times\sum_{m_1,m_2\dots m_{p-q-1}} f_{m_1}f_{m_2}\cdots f_{-n-m_1\dots-m_{p-q-1}} \times e^{in\Omega(x')t}. \tag{3.15}$$

It is because f_0 decays with a time scale that is much larger than the decay time of the nonzero harmonics [16] that f_0 is considered to be a constant and, therfore, subtracted away from the θ in case II. The same reasoning can be applied for case I. Hence all the n_i's and m_i's are nonzero in the above equation and in the rest of the paper.

To take into account the fact that diffusion gradually wipes out the spiral structure of the scalar field near the vortex center (a fact not taken in to account in Ref. [16] where the spiral structure is assumed to exist wholly intact until finally destroyed by viscosity), we follow Flohr and Vassilicos [11] and define a critical radius ρ, which gives a measure of this diffused region. This critical radius is defined in the limit $\mathrm{Pe}\to\infty$ for times $t\ll\mathrm{Pe}^{1/3}$, which are such that

$$f_n(r,t)=f_n(r,0)\quad\text{for}\quad r\ll\rho,$$

but

$$|f_n(r,t)|\ll|f_n(r,0)|\quad\text{for}\quad r\gg\rho,$$

see Eq. (2.3). Hence, we set $\frac{1}{3}n^2\Omega'^2(\rho)\mathrm{Pe}^{-1}t^3=1$, which implies

$$\rho(t)=\left[\frac{1}{3}n^2s^2\,\mathrm{Pe}^{-1}\,t^3\right]^{1/[2(s+1)]}.$$

This critical radius is time dependent and grows with time. It can be thought of as a diffusive length scale over which the harmonics in n have diffused and the spiral structure has been smeared out.

In view of the above, the integrals in Eq. (3.15) can be further divided as

$$\int_{r_0}^{\infty}dx=\int_{r_0}^{\rho}dx+\int_{\rho}^{\infty}dx.$$

The only significantly nonzero contribution comes from the range $\rho<x<1$ in the integrals and similarly for x'. Hence we get the following:

$$F_{pq}(k,t) =\frac{1}{(2\pi L_A^2)}\int_{\rho}^{\infty}dx\,kxJ_n(kx)2\pi(i)^n \times\sum_{n,n_1,n_2\dots n_{q-1}} f_{n_1}f_{n_2}\cdots f_{n-n_1\dots-n_{q-1}}e^{-in\Omega(x)t} \times\int_{\rho}^{\infty}dx'x'J^*_{-n}(kx')2\pi(-i)^{-n} \times\sum_{m_1,m_2\dots m_{p-q-1}} f_{m_1}f_{m_2}\cdots f_{-n-m_1\dots-m_{p-q-1}} \times e^{in\Omega(x')t}. \tag{3.16}$$

Now for large kx, i.e., $kx\gg1$, we can use the asymptotic expansion for the Bessel function

$$J_n(kx)\sim\left(\frac{1}{2\pi kx}\right)^{1/2}[(-i)^{n+(1/2)}e^{ikx}+(i)^{n+(1/2)}e^{-ikx}]. \tag{3.17}$$

This is appropriate for our analysis if the Fourier modes are to resolve at the very least the distance r_0 from the center of the vortex to the scalar patch interface, i.e., $1<kr_0$. After substituting Eq. (3.17) in Eq. (3.16) we use the method of stationary phase to evaluate the integrals where the phase is given by

$$\Phi=kx-n\Omega(x)t. \tag{3.18}$$

The approximation for a general integral of this type is known to be

$$I(x)=\int_a^b f(t)\exp[ix\Psi(t)]dt \sim \sqrt{\frac{\pi}{2x|\Psi''(t^*)|}}f(t^*)\exp[ix\Psi(t^*)\pm\pi/4], \tag{3.19}$$

where t^* is the t where the derivative of the phase is zero. The condition of stationary phase gives

$$0=\Phi'=-k-n\Omega'(x_n)t, \tag{3.20}$$

which picks out points x_n where the contribution to the integral is maximum. Finally what we get is

$$F_{pq}(k,t) \sim \sum_{n,n_1\ldots n_{q-1}} \frac{2\pi k}{n|\Omega''(x_n)|t} \times\left(\frac{x_n}{2\pi k}\right) f_{n_1}f_{n_2}\ldots f_{n-n_1-n_2-\ldots-n_{q-1}} \times \sum_{m_1,m_2\ldots m_{p-q-1}} f_{m_1}f_{m_2}\ldots \times f_{-n-m_1-m_2-\ldots-m_{p-q-1}}, \tag{3.21}$$

where $f_n=f_n(x_n,0)$ and similarly for f_m. Now from the condition of stationary phase (3.20) we can find

$$x_n=\left(\frac{snt}{k}\right)^{1/(s+1)}, \tag{3.22}$$

where we have used $\Omega(r)=r^{-s}$. The stationary phase contributes only when $\rho<x_n<1$ because the spiral structure only exists in that range of distances from the center of the vortex. The relation between the fractal codimension (Kolmogorov capacity) D of the scalar spiral and the power law of the decay of the azimuthal velocity of the vortex is [11,22]

$$D=\frac{s}{s+1}. \tag{3.23}$$

This D is such that $1/2<D<2/3$ because $1<s<2$ and gives a measure of the space-filling property of the spiral. Hence after doing the summations in Eq. (3.21) we can show that the power spectrum $F_{pq}(k,t)$ scales like

$$F_{pq}(k,t)\sim k^{-(3-2D)}t^{2(1-D)}[\text{const}+\text{ higher order terms}], \tag{3.24}$$

in the limit $\mathrm{Pe}\to\infty$ and in the range $t<k<\sqrt{\mathrm{Pe}/t}$ for times $1\ll t\ll \mathrm{Pe}^{1/3}$, which is the range of times for which the scalar patch has a well-defined spiral structure in the range of wave numbers $t<k<\sqrt{\mathrm{Pe}/t}$ (obtained from $\rho<x_n<1$). The higher order terms are functions of k/t, and decay faster than $(k/t)^{-1}$ in the range $t<k<\sqrt{\mathrm{Pe}/t}$, and can, therefore, be neglected.

We notice that as time runs forward the spiral range ($t<k<\sqrt{\mathrm{Pe}/t}$) shifts to higher values of k that is solely due to the vortex continuously wrapping the scalar field in to finer and finer spirals thus generating scales that have higher wave numbers. This range also shrinks as it shifts to higher values of k because of the action of diffusion. In the next section we generalize our results to the case of multiple spirals generated by a dilute collection of noninteracting vortices that may be representative of a turbulent velocity field with coherent vortical structure, perhaps obtained in the experiments of Jullien *et al.* [9].

IV. STRUCTURE FUNCTIONS OF PASSIVE DECAYING SCALAR IN A FLOW CONSISTING OF MANY IDENTICAL NONINTERACTING VORTICES

All the analysis in this section is carried out in dimensionalized variables so we invert the transformations of Sec. II. Let us consider many non interacting vortices randomly distributed over 2D space and sufficiently far apart so that we can safely describe the velocity field in terms of compact vortical structures characterized by

$$\Omega(x)=\Omega_0\left(\frac{x}{L}\right)^{-s} \quad \text{if } \frac{x}{L}\leqslant 1,$$

$$\Omega(x)=0 \quad \text{if } \frac{x}{L}>1,$$

where the x's are measured from the center of each vortex at $\mathbf{x}_m$ for all m and

$$\min|\mathbf{x}_m-\mathbf{x}_n|\geqslant L \quad \text{for all } m \text{ and } n. \tag{4.1}$$

For the calculation of the generalized power spectrum and structure functions we need only to consider the scalar patches within a distance L of each vortex because these scalar patches acquire a spiral structure and thereby dominate the scaling of the structure functions. The scalar field at distances larger than L from all vortices contributes an $O(r/L)$ term to the structure function for $r\ll L$ because the interfacial structure of the scalar field far from the vortices remain regular. As we show in this section, this term is negligible in the $r/L\ll 1$ limit compared to the r dependence of the structure functions caused by the scalar spiral structures around the vortices. It is, therefore, sufficient to consider that $\theta(\mathbf{x},t)$ consists only of the local scalar fields $\theta_m(\mathbf{x}-\mathbf{x}_m,t)$ in the vicinity of vortices labeled m and write

$$\theta(\mathbf{x},t)=\sum_m \theta_m(\mathbf{x}-\mathbf{x}_m,t). \tag{4.2}$$

Every scalar spiral in the right-hand side of Eq. (4.2) is localized within a distance L of $\mathbf{x}_m$ and the condition (4.1) ensures that they do not overlap each other.

Now we can generalize Eq. (3.6) to include the effect of many noninteracting vortices with nonoverlapping scalar spirals as shown below

$$\hat{F}_{pq}(\mathbf{k},t)=\frac{1}{2\pi L_A^2}\sum_m \int \theta_m^q(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}}d^2\mathbf{x} \times\int \theta_m^{p-q}(\mathbf{x}',t)e^{-i\mathbf{k}\cdot\mathbf{x}'}d^2\mathbf{x}'. \tag{4.3}$$

Since the integrals are independent of the m's we can take them out of the sum. Hence we get the same result as in Eq. (3.24) multiplied by the number of vortices per unit area, that is,

$$F_{pq}(k,t)\sim(kL)^{-(3-2D)}(\Omega_0 t)^{2(1-D)}\sum_m \frac{1}{2\pi L_A^2}. \tag{4.4}$$

This asymptotic relation is valid when $\Omega_0 t<kL<\sqrt{\mathrm{Pe}/\Omega_0 t}$ which is found from the condition $\rho<x_n<l$ in dimensionalized form and $1\ll\Omega_0 t\ll \mathrm{Pe}^{1/3}$ in the limit $\mathrm{Pe}\to\infty$.

Assuming the distribution of the scalar spirals over the two dimensional space to be homogeneous and isotropic, the power spectrum $F_{pq}(k,t)=2\pi k\overline{\hat{F}_{pq}(\mathbf{k},t)}$ where the bar implies ensemble averaging over the distribution of many spirals (because vortices are noninteracting and spirals are, therefore, statistically independent from each other, it does make sense for the average over space already included in the definition of $\hat{F}_{pq}(\mathbf{k},t)$ to be taken over an idealised space where there is only one spiral, and for the ensemble average to be taken over the distribution of many spirals). From Eq. (3.2) we can show that for a homogenous distribution of scalar spirals the odd order structure functions vanish. Only the even order structure functions do not vanish, that is for $p=$even. Hence from Eq. (3.1)

$$\begin{aligned}\mathcal{S}_p(r,t)&=\overline{\langle[\theta(\mathbf{x}+\mathbf{r},t)-\theta(\mathbf{x},t)]^p\rangle}\\&=\overline{\langle[\delta\theta(\mathbf{r},t)]^p\rangle}\\&\sim\int\left(2-\sum_{q=1}^{p-1}C_{qp}e^{i\mathbf{k}\cdot\mathbf{r}}\right)\overline{\hat{F}_{pq}(\mathbf{k},t)}k\,dk\,d\phi\\&\sim\int(1-e^{i\mathbf{k}\cdot\mathbf{r}})F_{pq}(k,t)dk\,d\phi\\&\sim\int[1-J_0(kr)]F_{pq}(k,t)dk,\end{aligned} \tag{4.5}$$

where $J_0(kr)$ is same as Eq. (3.11) with $n=0$. After substituting Eq. (4.4) in Eq. (4.5) and integrating we find

$$\mathcal{S}_p(r,t)\sim\left(\frac{r}{L}\right)^{2(1-D)}(\Omega_0 t)^{2(1-D)} \text{ in the ranges } \frac{1}{\Omega_0 t}>\frac{r}{L}>\sqrt{\frac{\Omega_0 t}{\mathrm{Pe}}} \text{ and } 1\ll\Omega_0 t\ll\mathrm{Pe}^{1/3}. \tag{4.6}$$

V. STRUCTURE FUNCTIONS OF STATISTICALLY STATIONARY PASSIVE SCALAR

To achieve a statistically stationary passive scalar field we may imagine that, as scalar patches take spiral shapes and decay, more scalar patches are introduced in to the flow as may well happen in an experimental setup in the laboratory. This procedure soon leads to a situation where many scalar spirals coexist in the flow all in different stages of their evolution. Assuming the rate of regular injection of the scalar blobs to balance exactly the rate of scalar dissipation, we can expect to have a statistically stationary scalar field. In this case, the averaging over many spirals in different stages of their evolution (which is involved in the calculation of the generalized power spectra and structure functions) may be assumed, in the spirit of Lundgren [10], to be equivalent to averaging over the lifetime of a single spiral. Hence, to obtain the generalized power spectra of the statistically stationary scalar we average the previous section's results over time in the range $1<\Omega_0 t<\mathrm{Pe}^{1/3}$, which represents the life time of the spirals. The spiral structure lies in the range $\rho<x_n<L$ which implies $\Omega_0 t<kL<\sqrt{\mathrm{Pe}/\Omega_0 t}$. This spiral range of wave numbers together with the time range of the spiral gives the range of $\Omega_0 t$ over which we can average for a given value of kL. This leads to a time averaged $F_{pq}(k,t)$, which takes the form

$$F_{pq}(k)\sim(kL)^{-1} \text{ where } 1<kL<\mathrm{Pe}^{1/3}, \tag{5.1}$$

$$F_{pq}(k)\sim(kL)^{-(7-6D)}\mathrm{Pe}^{2(1-D)} \text{ where } \mathrm{Pe}^{1/3}<kL<\mathrm{Pe}^{1/2}. \tag{5.2}$$

Equations (5.1) and (5.2) are the result of averaging (4.4) over the time ranges $1<\Omega_0 t<kL$ and $1<\Omega_0 t<\mathrm{Pe}/(kL)^2$, respectively. These time ranges are determined by the respective wave number ranges in Eqs. (5.1) and (5.2). Finally Eqs. (5.1) and (5.2) give the following structure functions:

$$\mathcal{S}_p(r)\sim\text{const}+\ln\left(\frac{r}{L\,\mathrm{Pe}^{-1/3}}\right) \text{ where } L\,\mathrm{Pe}^{-1/3}<r<L, \tag{5.3}$$

$$\mathcal{S}_p(r)\sim\left(\frac{r}{L\,\mathrm{Pe}^{-1/3}}\right)^{6(1-D)} \text{ where } L\,\mathrm{Pe}^{-1/2}<r<L\,\mathrm{Pe}^{-1/3}. \tag{5.4}$$

In Eqs. (5.3) and (5.4) $\mathcal{S}_p(r)$ is a time average of $\mathcal{S}_p(r,t)$ in Eq. (4.6). Note that $\zeta_p=0$ with a logarithmic correction in the range $L\,\mathrm{Pe}^{-1/3}<r<L$ and that $\zeta_p=6(1-D)\in]2,3[$ in the dissipative range $L\,\mathrm{Pe}^{-1/2}<r<L\,\mathrm{Pe}^{-1/3}$.

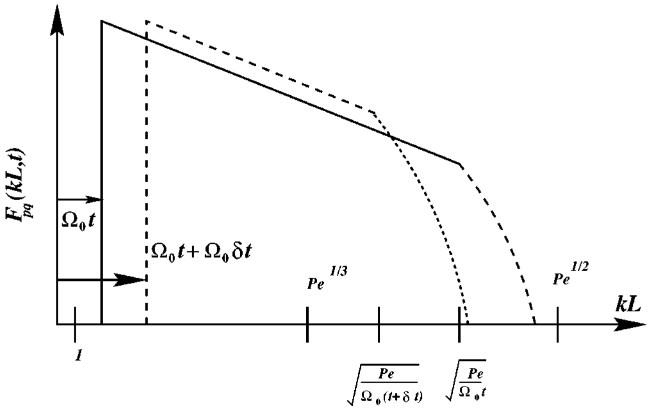

FIG. 2. The time evolution of the scalar power spectrum $F_{pq}(kL,t)$ in a log-log plot.

The structure functions $\mathcal{S}_p(r)$ are not time dependent and may be interpreted as characterizing a scalar field in a statistically steady state achieved with an external large-scale source of scalar (scalar forcing). This scalar forcing may consist of regularly placing in the flow scalar blobs with large-scale smooth interfacial structure. In the spirit of Birkhoff's Ergodic theorem [23] we should expect Lundgren's time average assumption to be relevant for the calculation of structure functions when the scalar field is statistically stationary.

VI. PHENOMENOLOGY

In this section we extract the phenomenology underlying the calculations and results of the previous sections and show that the essential ingredient of this phenomenology are the locality of scalar interscale transfer (6.1) and the Lundgren time-averaging operation. Indeed, as we show in this section, Eqs. (5.1) and (5.2) can be retrieved by a simple phenomenological argument based on these two ingredients.

Let us return to the time-dependent wind-up of scalar spirals. As time proceeds, i.e., as $\Omega_0 t\to\Omega_0(t+\delta t)$, then $kL\to kL+L\,\delta k$ because of the differential rotation(which amounts to local shear) in every steady vortex and the entire scalar spectrum is shifted towards higher wave numbers with time (see Fig. 2). That is to say that the shearing advection to which the scalar patches are subjected in every steady vortex is such that the generalized power spectra obey

$$F_{pq}(kL+\delta kL,\Omega_0 t+\Omega_0\delta t)d(kL+\delta kL)=F_{pq}(kL,\Omega_0 t)d(kL). \qquad (6.1)$$

The amount of scalar variance in the wave number band $d(kL)$ around wave number kL is simply transported to wave number band $d(kL+\delta kL)$ around wave number $kL+\delta kL$ after an incremental time duration $\Omega_0\delta t$ (see Fig. 2). As shown in Ref. [20], the distance l between consecutive coils of the scalar spiral in every vortex at a distance r from the center scales as

$$l\sim\frac{L}{\Omega_0 t}\left(\frac{r}{L}\right)^{1+1/s}. \qquad (6.2)$$

Letting time vary by a small amount δt, the distance between two coils changes by

$$\frac{\delta l}{l}\simeq-\frac{\delta t}{t},$$

the minus sign indicating that l is decreasing. Identifying k with $2\pi/l$ for the purpose of Eq. (6.1) so that $\delta k/k=-\delta l/l$, it follows that (6.1) becomes

$$F_{pq}\left[kL\left(1+\frac{\delta t}{t}\right),\Omega_0 t\left(1+\frac{\delta t}{t}\right)\right]dk\,L\left(1+\frac{\delta t}{t}\right)=F_{pq}(kL,\Omega_0 t)d(kL). \qquad (6.3)$$

The solution of this equation is

$$F_{pq}(kL,\Omega_0 t)=L(kL)^{-1}\mathcal{F}_{pq}\left(\frac{\Omega_0 t}{kL}\right), \qquad (6.4)$$

where $\mathcal{F}_{pq}$ are arbitrary dimensionless functions. As indicated in Fig. 2 this form of the generalized spectra is valid in the limit $\mathrm{Pe}\to\infty$ and in the wave number range $\Omega_0 t\ll kL\ll\sqrt{\mathrm{Pe}/\Omega_0 t}$ and time range $1\ll\Omega_0 t\ll\mathrm{Pe}^{1/3}$. Note that $1\ll\Omega_0 t\ll\mathrm{Pe}^{1/3}\ll\sqrt{\mathrm{Pe}/\Omega_0 t}\ll\mathrm{Pe}^{1/2}$. The inverse of $\Omega_0 t$ represents the decaying outer length scale of the spiral range and the inverse of $\sqrt{\mathrm{Pe}/\Omega_0 t}$ represents the growing microscale of diffusive attrition. A wave number in the range $1\ll kL\ll\mathrm{Pe}^{1/3}$ during the time-period $1\ll\Omega_0 t\ll\mathrm{Pe}^{1/3}$ does not have the time to be affected by diffusive attrition and only receives scalar variance activity from lower wave numbers until $\Omega_0 t=kL$. We, therefore, refer to $1\ll kL\ll\mathrm{Pe}^{1/3}$ as the advective wave number range. The time averaged generalized power spectra in this range are given by

$$F_{pq}(kL)=\frac{1}{kL-1}\int_1^{kL}d(\Omega_0 t)L(kL)^{-1}\mathcal{F}_{pq}\left(\frac{\Omega_0 t}{kL}\right), \qquad (6.5)$$

and $\mathcal{F}_{pq}$ must be increasing functions of $\Omega_0 t/kL$ because the differential rotation's shearing process causes the power spectra to shift from small to large wave numbers (see Fig. 2). Hence we retrieve Eq. (5.1), i.e.,

$$F_{pq}(kL)\sim(kL)^{-1}$$

in the advective range $1\ll kL\ll\mathrm{Pe}^{1/3}$, which is well defined in the limit $\mathrm{Pe}\to\infty$.

In the advective-diffusive range $\mathrm{Pe}^{1/3}\ll kL\ll\mathrm{Pe}^{1/2}$ a wave number kL experiences the advection process from $\Omega_0 t=1$ until $kL=\sqrt{\mathrm{Pe}/\Omega_0 t}$ when molecular diffusion sets in. Hence the time averaged generalized power spectra are given by

$$F_{pq}(kL)=\frac{1}{\frac{\mathrm{Pe}}{kL^2}-1}\int_1^{\mathrm{Pe}/kL^2}d(\Omega_0 t)L(kL)^{-1}\mathcal{F}_{pq}\left(\frac{\Omega_0 t}{kL}\right) \qquad (6.6)$$

in the advective diffusive range and using $\mathcal{F}_{pq}(\Omega_0 t/kL) \sim(\Omega_0 t/kL)^{2(1-D)}$ [see Eq. (4.4)] we retrieve Eq. (5.2), i.e.,

$$F_{pq}(kL)\sim \mathrm{Pe}^{2(1-D)}(kL)^{-7+6D}$$

in the advective-diffusive range $\mathrm{Pe}^{1/3} \ll kL \ll \mathrm{Pe}^{1/2}$, which is well defined in the limit $\mathrm{Pe}\to\infty$. Note that the diffusive micro-length-scale $L\,\mathrm{Pe}^{-1/2}$ is the Taylor microscale of the scalar field (first introduced by Corrsin in 1951). Note also that $7-6D\in]3,4[$ and that the experimental results of [9] seem to show a steeper power-law wave number spectrum at wave numbers larger than where the k^{-1} spectrum is observed.

VII. CONCLUSIONS AND DISCUSSION

In a two-dimensional isotropic and homogeneous collection of noninteracting compact and time-independent singular vortices with a large wave number energy spectrum $E(k)\sim k^{-\alpha}$ with $1<\alpha\leq 3$, the structure functions of an advected and freely decaying scalar field have the following scaling behavior in the limit where $\mathrm{Pe}\to\infty$:

$$\mathcal{S}_p(r,t)\sim\left(\frac{r}{L}\Omega_0 t\right)^{2(1-D)}, \tag{7.1}$$

where $\sqrt{\Omega_0 t/\mathrm{Pe}} \ll r/L \ll 1/\Omega_0 t$ and $1\ll\Omega_0 t\ll \mathrm{Pe}^{1/3}$, and the fractal codimension D of the scalar interfaces is such that $1/2\leq D<2/3$.

By applying the Lundgren time-average assumption we obtain predictions for the structure functions of a statistically stationary scalar field in the same 2D velocity field and the same limit $\mathrm{Pe}\to\infty$

$$\mathcal{S}_p(r)\sim \mathrm{const}+\ln\left(\frac{r}{L\,\mathrm{Pe}^{-1/3}}\right), \tag{7.2}$$

in the range $L\,\mathrm{Pe}^{-1/3}\ll r\ll L$ and

$$\mathcal{S}_p(r)\sim\left(\frac{r}{L\,\mathrm{Pe}^{-1/3}}\right)^{6(1-D)}, \tag{7.3}$$

in the range $L\,\mathrm{Pe}^{-1/2}\ll r\ll L\,\mathrm{Pe}^{-1/3}$. The logarithmic term in Eq. (7.2) corresponds to k^{-1} generalized power spectra. It may be worth mentioning that the 2D velocity fields of Holzer and Siggia [24] where they observe a well-defined k^{-1} scalar power spectrum are replete with spiral scalar structures.

Predictions of k^{-1} scalar power spectra in the limit $\mathrm{Pe}\to\infty$ have been made for scalar fields in smooth (i.e., at least everywhere continuous and differentiable) homogeneous and isotropic random velocity field with arbitrary dimensionality and time dependence by Chertkov *et al.* [25] who generalized and unified the results of Batchelor [26] and Kraichnan [27]. Experimental investigations of the high Péclet number k^{-1} scalar power spectrum have been inconclusive in 3D turbulent flows even though Prasad and Sreenivasan have claimed such a spectrum in a 3D wake [28]. However, k^{-1} scalar power spectra have been observed at high Péclet numbers in numerical simulations of scalar fields in 2D and 3D chaotic flows [29–31] and in 2D velocity fields obeying the stochastically forced Euler equation restricted to a narrow band of small (integral scale) wave numbers [24]. More recently, k^{-1} scalar power spectra have been observed at $\mathrm{Pe}=10^7$ in 2D or quasi 2D statistically stationary turbulent flows in the same range where the velocity field's energy spectrum is k^{-3} by Jullien *et al.* [9] who have also observed logarithmic scalings in that range for all order structure functions [similarly to Eq. (7.2), but without the ability to establish the $L\,\mathrm{Pe}^{-1/3}$ scaling factor and range]. The theory of Chertkov *et al.* [25] does not apply to this experiment because homogeneous and isotropic random velocity fields that are everywhere continuous and differentiable have energy spectra steeper than k^{-4} (see the Appendix). The present paper's theory, however, applies when the energy spectrum scales as $k^{-\alpha}$ with $1<\alpha\leq 3$ but is limited to time-independent velocity fields. Nevertheless, the phenomenology developed in Sec. VI also holds for time-dependent velocity fields and we now apply and generalize it to spatially smooth chaotic flows (and also, by the way, to frozen straining velocity field structures).

The starting point of our phenomenology is the locality of transfer (6.1). Pedrizzetti and Vassilicos [21] have shown that interscale transfer in 2D compact vortices is indeed local at a given scale when velocity gradients do not vary much in physical space over that scale. This is the case in the 2D axisymmetric vortices considered in this paper but also in spatially smooth velocity fields. In a spatially smooth chaotic flow the distance l between successive folds of the scalar interface decays exponentially as determined by the largest positive Lyapunov exponent λ, i.e., $l(t)\sim e^{-\lambda t}$, which implies $\delta l/l\simeq-\lambda\,\delta t$. Applying the locality of transfer property we get

$$F_{pq}[k(1+\lambda\,\delta t),t+\delta t]dk(1+\lambda\,\delta t)=F_{pq}(k,t)dk, \tag{7.4}$$

the solution of which is

$$F_{pq}(k,t)=k^{-1}\mathcal{F}_{pq}\left(\frac{e^{\lambda t}}{k}\right). \tag{7.5}$$

This form of the generalized spectra is valid for $\mathrm{Pe}\to\infty$ and as long as $1<e^{\lambda t}<k$, so that applying Lundgren's time-average operation from $t=0$ to $t=\lambda^{-1}\ln k$ gives

$$F_{pq}(k)=k^{-1}\frac{\lambda}{\ln k}\int_0^{\ln k/\lambda}\mathcal{F}_{pq}\left(\frac{e^{\lambda t}}{k}\right)dt\sim[k\ln k]^{-1}, \tag{7.6}$$

because $\mathcal{F}_{pq}$ is an increasing function of $e^{\lambda t}/k$. Power spectra of scalar fields in spatially smooth chaotic flows are believed to scale as k^{-1} in the limit $\mathrm{Pe}\to\infty$ [29–31] but our theory predicts $(k\ln k)^{-1}$. This is a small correction to the spectrum but an exponentially large correction to the scalar variance in the limit $\mathrm{Pe}\to\infty$.

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APPENDIX

For a statistically homogeneous velocity field with velocity components $u_i(\mathbf{x})$ we can define a correlation function

$$R_{ij}(\mathbf{r})=\overline{\langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle},$$

and its Fourier transform

$$\Phi_{ij}(\mathbf{k})=\frac{1}{(2\pi)^2}\int d\mathbf{r}R_{ij}(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}.$$

Incompressibility ($\nabla\cdot\mathbf{u}=0$) and statistical isotropy of a two-component or 2D velocity field $u_i(\mathbf{x})$, $i=1,2$, imply

$$\Phi_{ij}(\mathbf{k})=\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{E(k)}{\pi k},$$

where the average kinetic energy per unit mass of the velocity field is $E=\int_0^\infty dk\,E(k)$, $k\equiv|\mathbf{k}|$; $E(k)$ is the energy spectrum of the velocity field.

One dimensional energy spectra are defined as follows:

$$\phi_{ij}(k_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{ij}(r_1,0)e^{-ik_1r_1}dr_1,$$

which, because of isotropy, are completely characterized by a single component, say $\phi_{11}(k_1)$.

From the above and a few standard manipulations one can get

$$\int_1^\infty dx\frac{\sqrt{x^2-1}}{\pi x^2}\frac{E(xk_1)}{\phi_{11}(k_1)}=\frac{1}{2},$$

and in a range of wave numbers where both $E(k)$ and $\phi_{11}(k_1)$ are monotonically decreasing functions of k and k_1, respectively, it follows that $E(k)\sim k^{-p}$ if and only if $\phi_{11}(k_1)\sim k_1^{-p}$.

The pivotal assumption in Ref. [25] is that the velocity field $u_i(\mathbf{x})$ is Taylor-expandable up to at least first derivative terms everywhere in physical space. In particular, this means that $u_1(x_1,0)$ is differentiable with respect to x_1 everywhere on the x_1 axis. If the first derivative of $u_1(x_1,0)$ with respect to x_1 is also continuous everywhere along the x_1 axis then the Fourier transform $\hat{u}_1(k_1)$ of $u_1(x_1,0)$ must decay faster than k_1^{-2} [32]. If, however, the first derivative of $u_1(x_1,0)$ is not everywhere continuous, then it is discontinuous either on a set of well-separated points or on a more pathological set of points which accumulate (and are, therefore, not well separated) in a fractal-like or in a spiral-like manner [22,32]. In the case where discontinuities in the derivative field of $u_1(x_1,0)$ are well separated, $\hat{u}_1(k_1)$ decays as k_1^{-2} because the Fourier transform of well separated discontinuities between which the field is continuous decays as k_1^{-1} and the Fourier transform of the derivative of $u_1(x_1,0)$ is equal to $ik_1\hat{u}_1(k_1)$. In the other case where discontinuities are not well separated and accumulate, the decay of $\hat{u}_1(k_1)$ can be anywhere between k_1^{-1} and k_1^{-2} [32,33], but in this case $u_1(x_1,0)$ cannot be considered to be differentiable at those points where discontinuities of its derivative accumulate.

In conclusion, the differentiability of the velocity field everywhere in physical space implies that $\hat{u}_1(k_1)$ must decay at least as k_1^{-2} and, therefore, $\phi_{11}(k_1)\sim|\hat{u}_1(k_1)|^2\sim O(k_1^{-4})$, which in turn implies $E(k)\sim O(k^{-4})$.

The condition $E(k)\sim O(k^{-3})$ stated in the conclusion of Ref. [25] guarantees that the strain field is large scale but not that the velocity field is differentiable. The spectral condition required to use the pivotal assumption of differentiability in Ref. [25] should in fact be $E(k)\sim O(k^{-4})$.

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