Stochastic resonance of small-world networks

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Stochastic resonance (SR) of a coupled array of bistable oscillators with small-world connectivity is numerically studied. At certain coupling strength, it is found that both temporal SR and spatial synchronization of the oscillators can be considerably improved by increasing the order of randomness of the network due to the long-range couplings. Moreover, our results show that a small fraction of long-range couplings is sufficient to obtain great improvement in SR and synchronization.

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I. INTRODUCTION

Stochastic resonance (SR) is a somewhat counterintuitive phenomenon which has attracted increasing attention over the last two decades (for a review, see Ref. [1]). In SR, noise shows a surprising ability to optimize the response of a nonlinear system to a subthreshold periodic signal. A canonical model for SR is an overdamped particle moving in a double-well potential driven by a small periodic signal and an additive noise. In this case, the output signal-to-noise ratio (SNR) shows nonmonotonic behavior as a function of the input noise strength. Since the original work of Benzi *et al.* [2], SR has been shown to occur in a large variety of systems, from biological and chemical to physical systems [3–11].

Among recent studies, an interesting and important topic is addressed on SR in coupled oscillator systems [12–14]. Lindner *et al*. found that the resonance behavior of an oscillator, measured by SNR, can be further enhanced by coupling it into an array of oscillators, and they named this phenomenon array enhanced stochastic resonance [12,13]. For a coupled system, an additional parameter, the coupling strength, is introduced which strongly affects the SR behavior of the system. When the coupling is weak, the individual oscillators behave almost independently. To the other extreme, if the coupling is very strong, the whole array moves as a single element. However, at an optimal value of the coupling strength, the best SNR and spatiotemporal synchronization are observed [12].

A coupled system can be considered as a network or graph, where the vertices represent the elements of the system and the edges represent the interactions or couplings between them. The topology of these networks may influence the cooperative behavior of the systems. For example, it has been found that the output of an array of double-well oscillators can be significantly improved when the oscillators are connected in a two-dimensional square array instead of a one-dimensional chain, and much lower coupling strengths are needed to obtain the best SNR. However, in the bulk of the investigations on SR in coupled systems, the connectivity between the identical oscillators was regular, either local or global.

In this paper, we study SR in the systems with both local (regular) and random connectivities, and investigate how the existence of randomness of the network connections affects the behavior of the systems. In order to do this, we use the idea of the so-called "small-world" networks, recently introduced by Watts and Strogatz [15]. Such networks can be obtained by randomly rewiring a fraction p of the connections of a regular lattice. Therefore they are indeed a kind of disordered networks which lie somewhere between the regular ($p=0$) and the completely random ($p=1$) networks, and the parameter p stands for a measure of the order of randomness of the connectivity. It has been shown that some real networks, e.g., the neural network of the worm *Caenorhabditis elegans*, are small-world networks [15].

Here we consider the double-well oscillators coupled in small-world networks. We find that the SNR of the output of a network can be further improved by increasing the order of randomness p of the network. In fact, increasing p implies that there are more "long-range" links in the network, which may lead to more efficient cooperations of the oscillators. Therefore the SNR, as well as the degree of synchronization, can be enhanced. There are a number of parameters (such as noise intensity, coupling strength, and the fraction of random connections) affecting the SR and synchronization behaviors, and various optimization features are observed in varying these parameters.

II. DESCRIPTION OF THE MODEL AND NUMERICAL RESULTS

The networks used here are constructed following Watts and Strogatz [15]. We first consider a one-dimensional regular lattice with periodic boundary condition, composed of N vertices with each vertex connected to its k nearest neighbors [Fig. 1(a)]. So, there are $\frac{1}{2}Nk$ edges in the entire graph. Then, with probability p, each edge is rewired at random [Fig. 1(b)]. Specifically, we set $N=100$ and $k=4$, and p takes different values between 0 and 1. Note that in the rewiring process, the numbers of both vertices and edges remain unchanged.

The model for this study is made of an array of coupled

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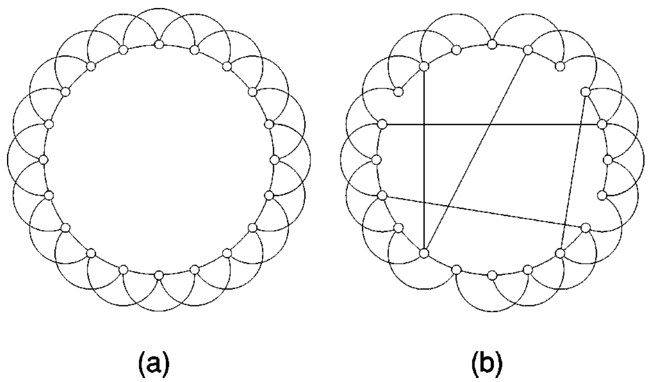

FIG. 1. (a) A one-dimensional lattice with periodic boundary conditions. Each vertex is connected to its k neighbors, where in this case $k=4$. (b) A small fraction of the links (in this case five of them) are rewired to new sites chosen at random.

double-well oscillators which are driven by a periodic force and white noise. The couplings between the oscillators are considered to have the small-world topologies mentioned above. The dynamics of each oscillator is described by

$$dx_i=\left[kx_i-k'x_i^3+A\sin\omega t+\sum_j \varepsilon_{ij}(x_j-x_i)\right]dt$$
$$+DdW_i(t),\quad i=1,2,\dots,N, \tag{1}$$

where k and k' are positive to ensure a double-well potential, A and ω are the amplitude and the frequency of the external periodic force which serves as the input signal, $W_i(t)$ ($i=1,2,\dots,N$) are independent standard Wiener processes, and D is the noise strength. ε_{ij} is the coupling between the two oscillators i and j, and its value is determined by the coupling pattern of the system. If these two oscillators are coupled to each other, we have $\varepsilon_{ij}=\varepsilon$, and otherwise, $\varepsilon_{ij}=0$. All the quantities in Eq. (1) are dimensionless. Throughout this work, we choose the parameters $k=2.1078$, $k'=1.4706$, $A=1.3039$, and $f=\omega/2\pi=0.1162$, which were used in Ref. [12], and take ε, D, and p as our varying control parameters to exam the response of the array system to the external signal and driving noise.

For each set of values of p and ε, we typically generate 50 different networks. A specific network defines a coupling pattern of the system. We take the full Runge-Kutta weak method [16] for numerically integrating the stochastic differential equation (1). The time series of the oscillators are recorded over 32 periods of the external force. Then, the power spectral density (PSD) is calculated and averaged over different elements of the array and the networks with different connectivities. It should be mentioned that because we are only interested in *interwell* motion of the oscillators, the effect of intrawell motion is filtered out by setting the output to be binary values of ± 1 [12].

The PSD has two components, i.e., the output signal and a background noise. The SNR, denoted by R, can be simply defined as [17]

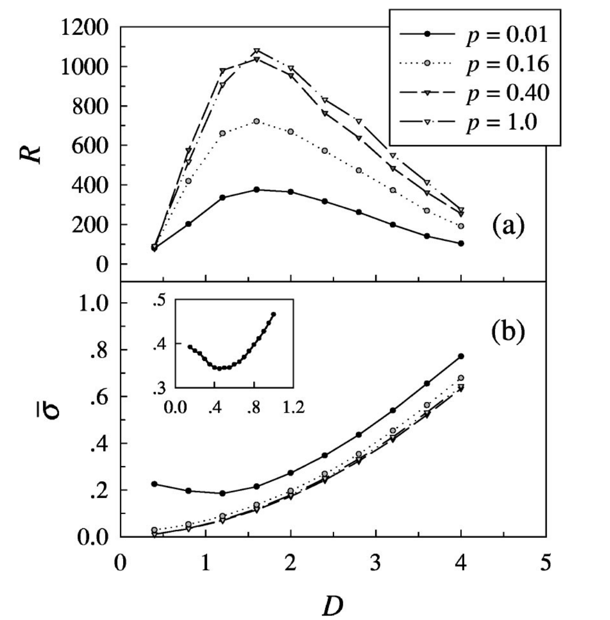

FIG. 2. (a) The signal-to-noise ratio R, defined in Eq. (2); (b) $\bar{\sigma}$ defined in Eq. (4) versus the input noise intensities D for several values of p. $\varepsilon=3.98$. The inset in (b) is $\bar{\sigma}$ versus D with $\varepsilon=0.16$ and $p=0.1$. All quantities are dimensionless.

$$R=\frac{(\text{signal power})}{(\text{noise power})}=\frac{(\text{total power})-(\text{noise power})}{(\text{noise power})}. \tag{2}$$

The SNR measures the temporal periodicity of the output of the array, i.e., the response of the oscillators to the periodic forcing. Figure 2(a) illustrates the SNR versus the input noise strength D for the cases of $\varepsilon=3.98$ and $p=0.01$, 0.16, 0.40, and 1.0. The curves show typical SR character in the sense that there exists an optimal noise strength, at which the SNR is maximized. From the figure, one can also see that the SNR peak rises very rapidly as the randomness of the network increases.

In Eq. (1), noise is applied locally, i.e., the noise is uncorrelated from site to site. Intuitively, this noise tends to make the array more spatially disordered. At the same time, the couplings between the elements help the whole array be synchronized. The competition between the noise and the couplings determines the spatial organization of the system.

The degree of spatial synchronization can be quantified by the mean square deviation

$$\sigma(t)=\langle x_i(t)^2\rangle-\langle x_i(t)\rangle^2=\frac{1}{N}\sum_{i=1}^{N}x_i(t)^2-\left[\frac{1}{N}\sum_{i=1}^{N}x_i(t)\right]^2. \tag{3}$$

$\sigma(t)$ is a periodic function of time whose period is half a forcing period. $\sigma(t)$ is a good quantity for measuring spatial synchronization of oscillators. For a fixed t, large $\sigma(t)$ represents large deviations between various oscillators, and small $\sigma(t)$ demonstrates strong collective motion and, consequently, better synchronization. Extremely, $\sigma(t)=0$ shows complete synchronization. For measuring SNR we discretize the variable values to ± 1. Now, $\sigma(t)$ is defined by the con-

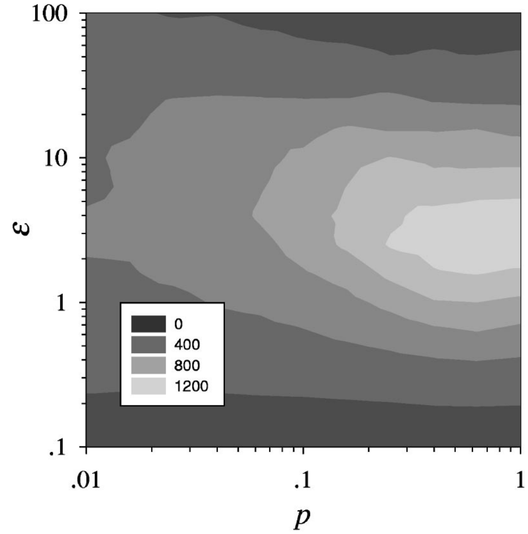

FIG. 3. The best SNR, R_s, as a function of p and ε. Both axes are in log scale. All quantities are dimensionless.

tinuous variables, such provides additional information of measuring. We are not interested in the oscillation of $\sigma(t)$, and then the time average of $\sigma(t)$,

$$\bar{\sigma}=\frac{1}{T}\int_0^T \sigma(t)dt, \tag{4}$$

will be used to quantitatively measure the synchronization of the array. Now the quantity SNR defined in Eq. (2) represents the order parameter for the order behavior of the output, while $\bar{\sigma}$ defined in Eq. (4) denotes the spatial synchronization of the output. These two quantities will be the central focus of our investigation. In Fig. 2(b) we plot $\bar{\sigma}$ versus D for different p and fixed $\varepsilon=3.98$. We also show a curve for the case $\varepsilon=0.16$ and $p=0.1$ in the inset of Fig. 2(b), where minimum $\bar{\sigma}$ (the best synchronization) can be clearly seen for certain optimal noise strength. However, if ε and/or p are not very small, the couplings between the oscillators will be strong enough to pull the oscillators to the same potential well even without the help of noise, and, in this case, the best synchronization occurs at very low noise level. This feature is also observed in Fig. 2(b).

In Fig. 3, we show the best SNR, R_s, as a function of the coupling strength ε and the randomness of the network p. By the best SNR we mean the largest SNR over D for each pair of (ε,p). Note, both ε and p are in log scale. It can be found that at certain values of ε and large p, R_s has very large values, where SR is greatly enhanced.

From Fig. 3, it is obvious that R_s is small for both too small and too large ε. And large R_s can be identified at certain optimal coupling strength as shown in Fig. 4. The explanation of the peaked curves in Fig. 4 has already been given in Ref. [12]. In this paper, we are most interested in the influence of parameter p on the SR behavior. Increasing p means that more edges are rewired and thus more long-range connections are introduced. In this case, the elements become much closer to each other or, in other words, each

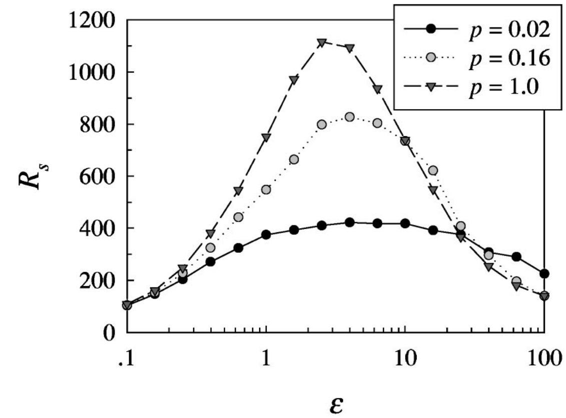

FIG. 4. The best SNR, R_s, versus ε for several values of p. All quantities are dimensionless.

oscillator has more (not necessarily nearest) "neighbors." This is indicated by the rapid decrease of the characteristic-path length of the network l with the increasing of p [15]. Here l is defined as the number of edges in the shortest path between two vertices, averaged over all pairs of vertices. $l(p)$ measures the typical separation between two vertices and hence serves as the characteristic length scale of the network. So, as p increases, the motion of an individual oscillator can influence a *larger* number of other oscillators more quickly. Therefore when the coupling strength is not very large, the SR phenomenon can be considerably enhanced [see Fig. 5(a)]. On the other hand, if the couplings are very strong, all oscillators behave almost as a single one, and increasing p may even help this trend. So, in this case, one may expect that the disorder of the network may reduce SR [see Fig. 5(b)].

The SR behavior of Figs. 3–5 shows the temporal order of the system output. For demonstrating the spatial synchro-

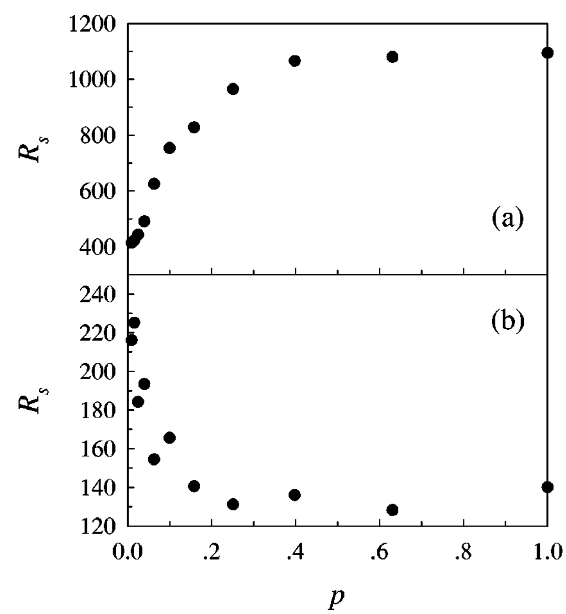

FIG. 5. The best SNR, R_s, as functions of p. The coupling stength is (a) $\varepsilon=3.98$ and (b) $\varepsilon=100$. All quantities are dimensionless.

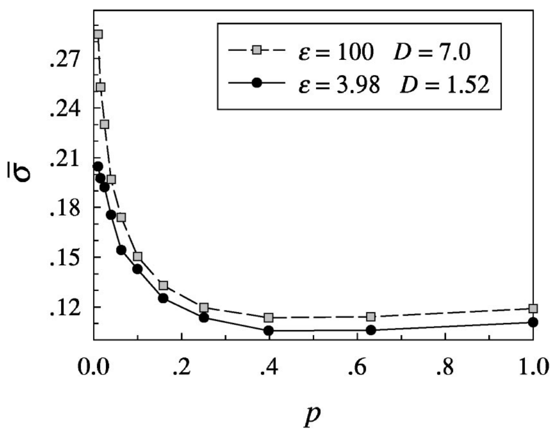

FIG. 6. The dependence of $\bar{\sigma}$ on p. All quantities are dimensionless.

nization features we show $\bar{\sigma}$ versus p in Figs. 6(a) and 6(b) with the coupling strength $\varepsilon = 3.98$ and 100, respectively. In Fig. 6(a), we take $D = 1.52$, while in Fig. 6(b), $D = 7.0$. For other noise levels, $\bar{\sigma}$ has similar behavior [see Fig. 2(b)]. From these figures, we observe that $\bar{\sigma}$ always decreases as p increases. This feature is different from the dependence of R_s on p [Figs. 5(a) and 5(b)].

Moreover, l drops very rapidly as p increases from 0 [15]. This implies that only a few long-range connections are sufficient to considerably shorten the distance between the elements in an array. For our choosing of parameters ($N = 100$ and $k = 4$), only about one-fifth of the total edges needs to be rewired in order to obtain a characteristic-path length comparable to that of a completely random network. As a result, the collective behaviors of the systems also show such small-world property as shown in Figs. 5 and 6.

III. SUMMARY AND DISCUSSION

In conclusion, with proper coupling strength, the random connectivity of the networks may induce improvement in both SR and spatial synchronization due to the long-range couplings. However, in order to obtain a good result, only a few long-range couplings are needed, i.e., the connectivity has a small-world topology. It should be mentioned here that a higher dimensional regular lattice can be considered as a lower dimensional lattice with additional *specific* long-range connections. These long-range connections can also reduce the distance between two vertices. So, one can expect to find improvement in SR of higher dimensional lattices over their lower dimensional counterparts. As evidenced, an increase of SNR of two-dimensional lattices compared with that of one-dimensional chains has been observed recently [14]. In addition, a negative role to the SR behavior played by increasing random connections at large coupling strength is observed and intuitively understood.

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