Stochastic resonance via switching between the two stable limit cycles on a cylinder

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This paper deals with the determination of stochastic resonance (SR) in a driven bistable potential based on an embedding-based description in terms of an autonomous system of stochastic equations. It points out a criterion for SR on the basis of the relative position of limit cycles of the deterministic system.

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The notion stochastic resonance (SR) was originally introduced to explain the periodic occurrence of the earth ice age [1,2]. Since then, a vast majority of experimental and theoretical evidence of SR has been accumulated (for recent review, see [3]), among which the most frequently studied is about SR in a driven bistable potential. In subthreshold regime, SR has been well explored using different techniques [4–15], to quote but a few. As for SR in suprathreshold case, this had been deemed as impossible [6–7] until recently, Apostolico *et al.* give a noise failure mechanism of SR above the threshold [16].

In this paper, following the idea of embedding the nonautonomous equation in an autonomous system [8–10], we will further show the existence of SR in suprathreshold regime via numerically showing two stable limit cycles of the deterministic system above a bifurcate curve. Our point of view is that the mechanism of SR both for subthreshold and suprathreshold cases are essentially the same.

Moreover, we give a precise condition of the onset of SR by exploring the impact of limit cycles of the deterministic phase portraits both for subthreshold and suprathreshold cases. We find that good relative position of the stable limit cycle (SLC) to the unstable limit cycle (ULC) does not always mean the happening of SR though switching transitions display very good coherence. Actually, the effect of SR also crucially depends on the amplitude of the SLC. In our point of view, the onset of SR is just the competitive result of these two ingredients.

In our measurement, we calculate the average power spectrum and characterize SR by the response amplitude that is defined as $\beta = R_1/R_0$, where R_1 is the amplitude of the output at the frequency of the input signal in noisy background and R_0 is the height of the deterministic spectrum peak. We say that the phenomenon of SR occurs when the curve of β vs D displays as bell shaped with the height of the peak $\beta_{\max} > 1$ and admit that larger value of $\beta_{\max}$ means better effect of SR. All the quantities used here are dimensionless.

The model we considered here is the following quartic potential stochastic system:

$$\dot{x} = -V'(x) + A\cos\omega(t) + D\xi(t), \tag{1}$$

where $V(x) = x^4/4 - x^2/2, \xi(t)$ is the Gaussian white noise satisfying $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$. In Ref. [8], it has been pointed out that there exists a critical amplitude $A_c(\omega)$ such that for $0 < A \leqslant A_c(\omega)$, Eq. (1) in the deterministic case has two stable periodic orbits (attractors in $x-\theta$ phase space by setting $\theta = \omega t$), and for $A > A_c(\omega)$, the deterministic system is no more bistable. To give an explicit expression of the function $A_c(\omega)$ is an interesting but hard problem.

In this paper, we set $y = A\cos\omega t$, $z = A\sin\omega t$, then the deterministic dynamics of Eq. (1) may be equivalently characterized as

$$\begin{aligned} \dot{x} &= x - x^3 + y, \\ \dot{y} &= -\omega z, \\ \dot{z} &= \omega y. \end{aligned} \tag{2}$$

So, a solution to Eq. (2) is a curve winding on the cylinder E^2: $y^2 + z^2 = A^2$, and the two attractors in Ref. [8] are actually two SLC's on E^2. Numerical simulations show that in subthreshold regime $[A \leqslant (2\sqrt{3}/9)]$, for all the values of ω, Eq. (2) has exactly two SLC's (see Fig. 1). And in suprathreshold regime $[A > (2\sqrt{3}/9)]$, for small values of ω, system (2) has only one SLC, but as ω increases to a certain value, there are still two SLC's existing on E^2 (see Fig. 2). Thus, there exists a bifurcate curve L_1: $\omega = \omega_1(A)$ [equivalent to the critical amplitude $A_c(\omega)$ in Ref. [8]] such that for $0 < \omega < \omega_1(A)$, there is only one SLC on the cylinder, however, for $\omega \geqslant \omega_1(A)$, Eq. (2) still has two SLC's. In Fig. 3,

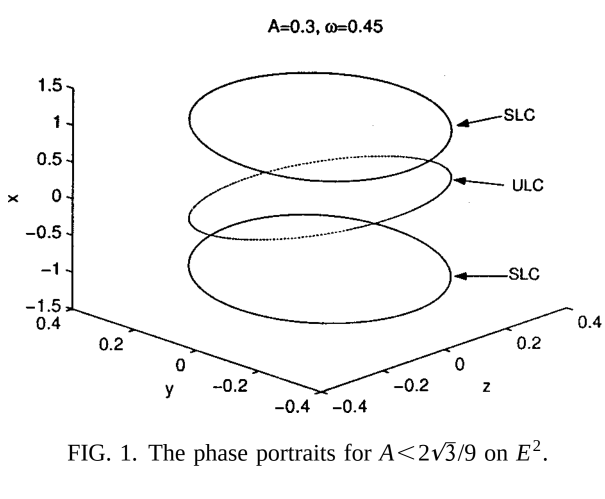

FIG. 1. The phase portraits for $A < 2\sqrt{3}/9$ on E^2.

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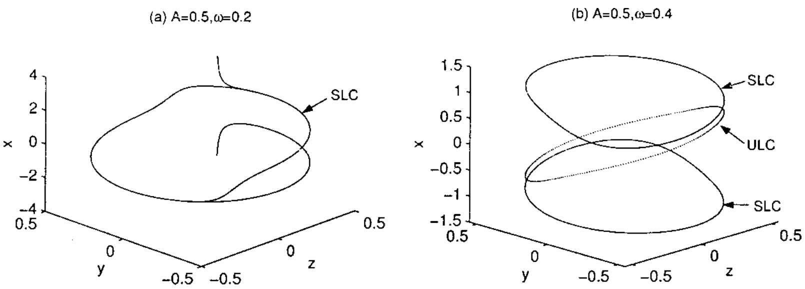

FIG. 2. The phase portraits for $A>2\sqrt{3}/9$ on E^2.

we numerically plot this bifurcate curve. For convenience, we still call L_1 together with the line of $\omega=0$ $[0<A \leqslant (2\sqrt{3}/9)]$ as the bifurcate curve. In such a picture, one can easily conclude that it is impossible for SR (even for residual SR) to occur when (A,ω) lies below L_1 since adding the noise can only destroy the original coherent rotation around the unique attractor (see Fig. 4). However, in the regime above the bifurcate curve, there is no essential difference between the mechanism of the SR in the suprathreshold case to that in the subthreshold cases. The bell-shaped curve of the response amplitude versus the noise intensity D in Fig. 5 for $A=0.4$, $\omega=0.1$ confirms this conclusion.

However, the phenomenon of SR does not always come to exist in the regime above the bifurcate curve (see Fig. 6). To precisely explore the condition of the onset of SR, it is better to give some analysis of the deterministic phase portraits. As shown in Fig. 7, one can see that the deterministic phase portrait may be characterized by two quantities, namely, the amplitude (variation of the height on the cylinder) of the SLC, denoted as A_{SLC} and the relative position of any one of the SLC's to the ULC which can be quantified by their smallest distance $d_{\min}$. Here, the roles of the input signal is just to modulate these two quantities. We observe the following:

(1) For a fixed value of A, increasing the signal frequency ω results in the increase of $d_{\min}$ together with the decrease of A_{SLC}. On the other hand, in the region above the curve L_1, for fixed value of ω, as $d_{\min}$ decreases with the increase of A, the value of A_{SLC} increases. The above relationship can be simply expressed as: $d_{\min} \propto \omega/A$ and $A_{\mathrm{SLC}} \propto A/\omega$.

(2) If the value of ω is fixed, the value of $d_{\min}$ in the case $A>2\sqrt{3}/9$ is much smaller than that in the case $A \leqslant 2\sqrt{3}/9$.

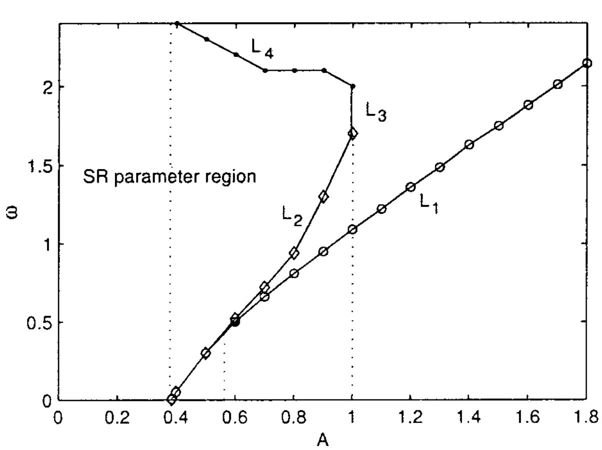

FIG. 3. The critical curve of L_i: $\omega=\omega_i(A)(i=1,2,3,4)$. The region for SR to exist can be clearly seen.

Intuitively, if the value of $d_{\min}$ is too large, it is hard for the particle to switch between the two SLC's unless the noise intensity is sufficiently large. However, the switches may hardly show any coherence under such large noise perturbation. This is why in Fig. 6 one can only see decreased response amplitudes with the increase of D. So for SR to be observed, the SLC should lie at a relative good position to the ULC. To confirm this, we plot the curves of β vs D for $A=0.4$ and different values of ω in Fig. 7(b-4), respectively. One can see that as $d_{\min}$ decreases to a certain value, the curve of β versus D exhibits single peak (see the curve for $\omega=0.3$) and the height of the peak $\beta_{\max}$ increases with the further decrease of $d_{\min}$. Especially, when the system parameters (A,ω) lies on the bifurcate curve L_1, $d_{\min}$ is close to zero, and the value of $\beta_{\max}$ reaches its maximum. This is essentially true for the near threshold values of A.

It seems that the smaller the value of $d_{\min}$ is, the better will be the effect of SR. However, it is not the whole story. As illustrated in Fig. 7(c-1), the value of $d_{\min}$ is also close to zero, but the corresponding curve of β vs D [see the curve for $\omega=0.5$ in Fig. 7(c-4)] shows that the response amplitude decreases first to a minimum with the increase of D, though later it passes through a bell-shaped maximum with D further increasing, the value of $\beta_{\max}$ exceeds no more than one. In Ref. [16], this phenomenon was termed as residual SR, but in

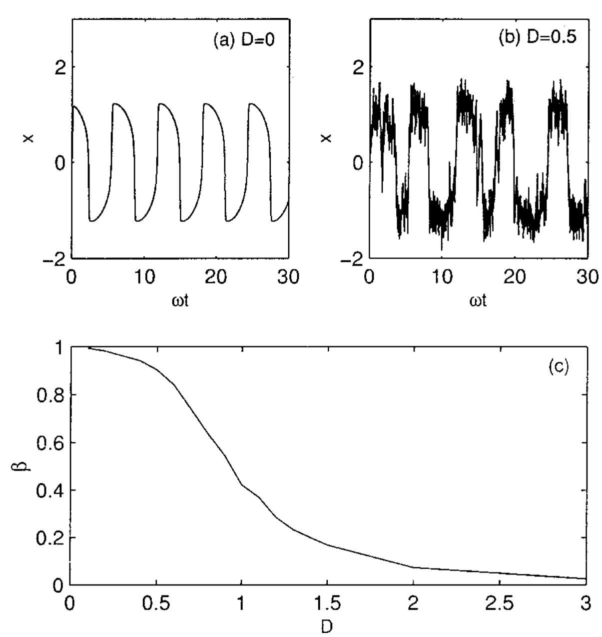

FIG. 4. The trajectories for $D=0$(a), 0.5(b), and the curve of β vs D (c) for the case of $A=0.6$, $\omega=0.1$.

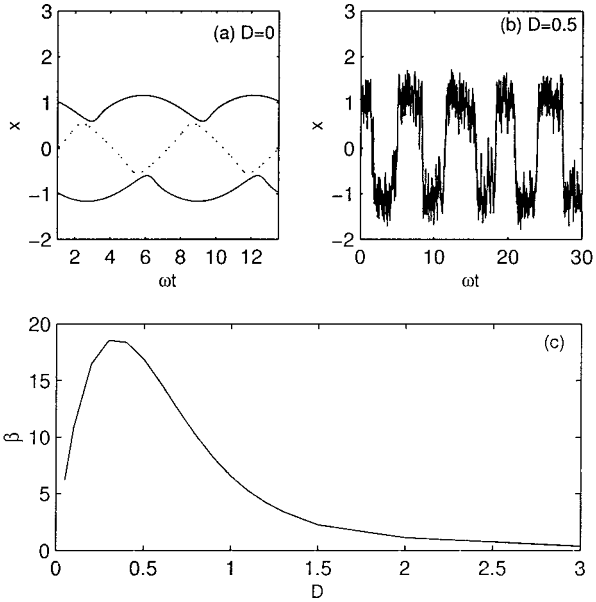

FIG. 5. The trajectories for $D=0$(a), 0.5(b), and the curve of β vs D (c) for the case of $A=0.4$, $\omega=0.1$.

our terminology, we say that there is no SR for this group of parameters since $\beta_{\max}<1$, which means that the signal is not amplified by the noise. Now, let the signal frequency increase, and we observe that at a certain value of ω, $\beta_{\max}$ is larger than one again [see the curve for $\omega=0.8$ in Fig. 7(c-4)] which means the occurrence of the SR. After $\beta_{\max}$ reaches a maximum, it decreases with the further increase of ω. For sufficiently large signal frequency, the effect of SR becomes very weak [see the curve for $\omega=1.5$ in Fig. 7(c-4)].

The above phenomenon seems somewhat puzzling. However, it is not a surprise if the amplitude of the SLC is taken into account. Comparing the value of A_{SLC} in Fig. 7(b-1)

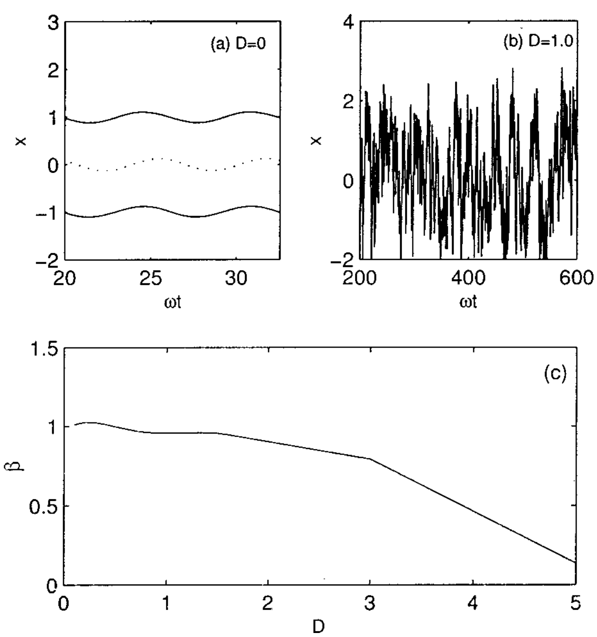

FIG. 6. The trajectories for $D=0$(a), 1.0(b), and the curve of β vs D for the case of $A=0.4$, $\omega=3$.

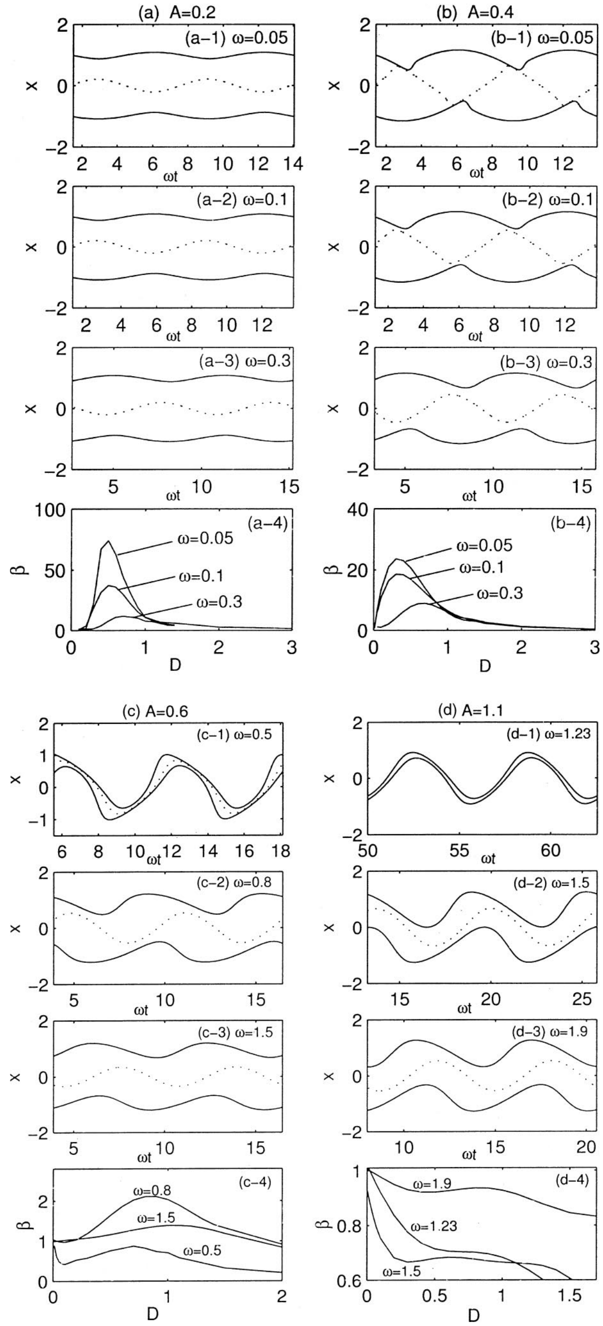

FIG. 7. The deterministic trajectories (phase portraits cut along one generator of the cylinder and developed to a plane; solid line: SLC, dashed line: ULC) and the curves of β vs D for the cases of $A=0.2$ (a), $A=0.4$ (b), $A=0.6$ (c), $A=1.1$ (d), and various values of ω.

with that in Fig. 7(c-1), one can see that the former one is much smaller than the later. This is just the reason to clear the puzzle, as one knows that larger amplitude of the SLC corresponds larger value of R_0, which means stronger coherence of the deterministic rotation around the SLC. Therefore, to be more precise, we need two quantities, i.e., A_{SLC} and $d_{\min}$ to completely characterize the SR phenomenon. In the later case, though the particle may easily be perturbed from one SLC to another when a small amount of noise is included, the added noise also destroys the original coherent motion around the SLC, the larger R_0 is, the more obvious will be the destructive role of noise. So, at first the noise-induced coherent switch is suppressed, and correspondingly, the spectrum peak as well as the response amplitude decreases with the increase of D. After β reaches a local minimum, further increasing D causes the switch between the two SLC's displaying better degree of coherence. Such coherent motion is, however, not as strong as that of the deterministic rotation around the SLC, so the response amplitude β is still smaller than one. Therefore, only residual SR, but no real SR, happens in this case. As ω increases, though, the increase of $d_{\min}$ is not favorable for the particle to switch between the two SLC's the decrease of A_{SLC} reduces the value of R_0, this is good for boosting the response amplitude. Then, at a certain ω, the compromising result is that the switch between the two SLC's shows a better degree of coherence than that of the original rotation around the SLC. Consequently, the value of β is larger than one and increases to a maximum, which characterize the occurrence of SR. After $\beta_{\max}$ reaches its maximum, further increasing ω destroys the compromising effect and the SR effect is decreased.

The conclusion is the effect of SR not only depends on the relative position of the SLC to the ULC but also crucially on the amplitude of the SLC. To further confirm this, let us consider the cases of a much larger A $[A>(2\sqrt{3}/9)]$ and a much smaller A $[A\leqslant(2\sqrt{3}/9)]$. Here, we take $A=1.1$ 0.2, respectively. In Figs. 7(d) and 7(a), the curves of β vs D corresponding to different values of ω are plotted. In the case $A=1.1$, because of large values of A_{SLC}, one can see that for all the values of ω, no SR is observed; while in the case of $A=0.2$, the amplitudes of the SLC's in three pictures are all very small, so though the relative positions of the SLC's to the ULC's are not favorable for the happening of switch, there are still good effects of SR.

The influence of the amplitude of SLC on the effect of SR may also be reflected by varying the value of A with ω being fixed. Comparing Figs. 7(a) with 7(b), one can see that for fixed value of ω, the smaller the value of A is, the smaller is the amplitude of SLC, and then the better is the effect of SR.

Informed from the Fig. 7(c), varying the value of A, one may expect that there exists a critical curve L_2: $\omega=\omega_2(A)$ such that if (A,ω) lies below this curve, no SR exists. In Fig. 3, we numerically plot this curve, where one can see that for near-threshold values of A (about $2\sqrt{3}/9<A<0.58$), L_2 coincides with L_1 very well, but outside this range, it gradually deviates from L_1 until A increases to about 1.0. Further increasing A, the phenomenon of SR could not possibly happen for all ω because of a too high deterministic spectrum peak. Besides, we are informed from Fig. 6 that for every fixed value of A, if ω is sufficiently large, it is also hard for SR to occur since the relative position of the SLC to the ULC is too bad. So, we assert that there are other two critical curves L_3: $A=A_c$ and L_4: $\omega=\omega_4(A)$ such that when $A>A_c$ or $\omega>\omega_4(A)$, SR can hardly be observed. In Fig. 3, we plot these two critical curves. It shows that when (A,ω) lies on L_2 or in the region formed by the curves $L_i(i=1,2,3,4)$, the ω *axis* and the A *axis*, SR really exists. Outside this region, no SR could exist, but residual SR may occur when (A,ω) lies in the region formed by L_1, L_2, and L_3.

In summary, via demonstrating the collision of limit circles above (or on) a bifurcate curve, we have given the mechanism of SR in suprathreshold regime (as well as in subthreshold regime). Furthermore, we have shown that the SR above the bifurcate curve crucially depends on two opposite tendencies. Improving the relative position of the SLC to the ULC up to collision may cause the switching motion, and hence, the SR to happen easier. Meanwhile, increased amplitude of the SLC boosts the deterministic spectrum peaks, which is not good for the increasing of response amplitude. So, the onset of SR is just the competitive outcome of these opposite effects. At last, we have precisely given a SR parameter region of the bistable potential.

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