Renormalization group analysis of autoregressive processes and fractional noise

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A renormalization group analysis is applied to autoregressive processes with an infinite series of coefficients. A simple fixed point is given by a random walk, and a second class is found that is proportional to the high order coefficients of fractional autoregressive integrated moving average (ARIMA) processes. The approach might be useful to detect nonstationarity in autoregressive processes.

The renormalization group (RG) approach has been successfully applied to determine large scale and long time properties of a large number of different physical systems with given microscopic dynamics [1]. A very illuminating approach dates back to Kadanoff [2], who invented the decimation of degrees of freedom associated with an iterative mapping to an effective model (see [3,4] for examples). Since the dynamics of natural systems is often either too complex or not known, these are frequently described using autoregressive (AR) models to provide analyses, simulations, and forecasts [5]. To incorporate long time memory, Hosking extended AR processes to fractional AR (FAR) models [equivalently, autoregressive integrated moving average processes ARIMA $(0,d,0)$], or the so-called fractional noise] [6–8]. For FAR models, the infinite series of regression coefficients is determined by a single fractal dimension d, and the power spectrum is self-similar for small frequencies, $S(f) \sim f^{-2d}$. In this publication, a standard RG analysis is applied to an AR process with an infinite number of regression coefficients. A simple fixed point is given by the random walk, and a second class is proportional to Hosking's FAR process for high order coefficients. The RG might be useful to detect nonstationarity of AR processes.

The autoregressive process AR(p) for the observable x_t at discrete time t and for a time interval Δ is

$$x_t = \sum_{n=1}^{p} a_n x_{t-n} + \epsilon_t \qquad (1)$$

where ϵ_t is uncorrelated white noise and the a_n, $n = 1, \ldots, p$, are the regression coefficients; in the RG, p is infinite. To determine the long term behavior a renormalization group analysis is constructed which includes these main steps. (i) Nonoverlapping adjacent pairs of time steps are combined to blocks. (ii) The dynamics of these blocks is transformed to an AR process. (iii) Subsequent iterations of this procedure yield fixed points which are identified with the long term behavior.

In the first RG step, the process x_t at two neighboring time steps is combined, $x'_\tau = (x_t + x_{t-1})/\sqrt{2}$, where $t = 2\tau$ is an even time step and $\sqrt{2}$ is chosen to preserve the variance. The AR process for x'_τ within the block τ is written by add-

ing the ARs for x_t and x_{t-1}. Terms involving differences within the blocks, e.g., $x_{t-2} - x_{t-3}$, are neglected since they represent high frequency variability. The intensities at even and odd time steps on the right-hand side (rhs) of Eq. (1) are eliminated by approximating x_t and x_{t-1} with $x'_\tau/\sqrt{2}$. The dynamics for the combined process becomes

$$x'_\tau = \frac{a_1}{2} x'_\tau + \frac{1}{2} \sum_{m=1}^{\infty} (a_{2m-1} + 2a_{2m} + a_{2m+1}) x'_{\tau-m}$$
$$+ \frac{1}{\sqrt{2}} (\epsilon_{2\tau} + \epsilon_{2\tau-1}). \qquad (2)$$

The first term on the rhs (from the dynamics of x_t) leads to a nonlinearity in the RG scheme. In the second RG step, the blocks are identified with the original time $\tau \to t$ and $m \to n$. Accordingly, the time step width is enhanced to $\Delta' = 2\Delta$ at each RG step. Thus, the dynamical equation for the renormalized process x'_t can be written as an AR process

$$x'_t = \sum_{n=1}^{\infty} a'_n x'_{t-n} + \epsilon'_t \qquad (3)$$

with the renormalized regression coefficients

$$a'_n = \frac{1}{2 - a_1} (a_{2n-1} + 2a_{2n} + a_{2n+1}) \qquad (4)$$

and the renormalized noise

$$\epsilon'_t = \frac{\sqrt{2}}{2 - a_1} (\epsilon_{2\tau} + \epsilon_{2\tau-1}). \qquad (5)$$

The variance $\overline{\epsilon^2}$ of the noise increases by the factor $4/(2 - a_1)^2$ at each RG step (with the actual a_1). The singularity at $a_1 = 2$ originates in the block size of two time steps and the particular design of the RG. The denominator $2 - a_1$ in Eqs. (4) and (5) is essential for the behavior of the RG map. For a finite number p of regression coefficients, each renormalization step reduces p by a factor of two (approximately).

As a first application let us consider the relation between stationarity of the AR process and the RG mapping. A stochastic process is called stationary if the mean and the variance are time independent, and the correlation depends only on the lag. (More stringent definitions demand that the distribution is time invariant.) Stationarity of the AR process x_t

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requires that all roots z_j of the characteristic polynomial $\phi(z)=1-\Sigma_{n=1}^p a_n z^n$ are outside the unit circle $|z_j|>1$ [9], which might be complicated in general. Stationarity for $p=1$ is given if $|a_1|<1$. For $p=2$ the two coefficients have to be within the triangle $a_2-a_1<1$, $a_1+a_2<1$, and $|a_2|<1$. The renormalized process of an AR(2) process is AR(1) with a single coefficient given by $a_1'=(a_1+2a_2)/(2-a_1)$ according to Eq. (4). The stationarity condition for the initial AR(2) process leads to $|a_1'|<1$, i.e., the renormalized process (4) is also stationary. For a stationary AR(3) process one can show that after two RG steps $|a_1''|<1$ (all other coefficients vanish). Therefore, we assume, although there is no general proof at the moment, that the RG preserves stationarity also for higher orders p. A violation of the stationarity condition after renormalization might give a hint of nonstationarity in the initial AR process. This approach is extremely simple, since a sufficient number of iterations of the renormalization reduces the number of coefficients to a single final coefficient.

A central issue in a RG analysis is the identification of fixed points $a_n^\star$ of Eq. (4) since these represent the long term behavior of the system. The sum of the fixed point regression coefficients $a_n^\star$ is unity, $\Sigma_{n=1}^\infty a_n^\star=1$, provided that the sum is finite and $a_1\neq0$. A first type of fixed point is obtained for a single nonvanishing coefficient $a_1\neq0$ that obeys the iteration $a_1'=a_1/(2-a_1)$ as given by Eq. (4). Note that this map becomes simply $\alpha'=2\alpha-1$ for the inverse, $\alpha=1/a_1$. The unstable fixed point $a_1^\star=1$ represents a random walk. The second, trivial fixed point $a_1^\star=0$ is attractive due to the reduction of memory by the RG.

A more complex set of fixed points with infinitely nonvanishing parameters is reached by starting from fractional noise (FAR processes). FAR processes are defined to model long time memory with a single parameter, the dimension d, which leads to an infinite series of AR coefficients. The dynamics is defined as [7]

$$(1-B)^d x_t = \epsilon_t \tag{6}$$

with the backshift operator B, $Bx_t=x_{t-1}$. This can be identified as an AR process with coefficients

$$a_n = -\frac{\Gamma(n-d)}{\Gamma(-d)\Gamma(n+1)}. \tag{7}$$

Noninteger dimensions d lead to fractional processes. The coefficients obey the recursion $a_{n+1}=(n-d)a_n/(n+1)$, starting from $a_1=d$. For large n, $a_n\sim-n^{-1-d}/\Gamma(-d)$. For small dimension $d\to0$ and large n, the FAR process coefficients are $a_n\sim d/n$. The FAR process (7) has long time memory with a power spectrum $S(f)\sim f^{-2d}$ for $f\to0$. The FAR process is stationary for $|d|<1/2$ and $1/f$ noise is obtained for $d=1/2$.

A main result of the present study pertains to the behavior of FAR processes subjected to the RG procedure (4). For small dimension, the FAR coefficients remain fixed in the RG, $a_n^\star\sim d/n$ ($n\to\infty$). For finite dimension, a closed expression for the remaining $a_n^\star$ could not be derived. Numerical
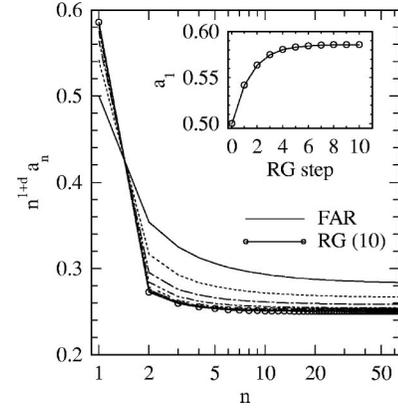
FIG. 1. Coefficients a_n multiplied by n^{1+d} for the initial FAR process (solid) and up to 10 RG steps (dashed, step 10 with circles). The inset shows the convergence of a_1 for these 10 RG steps.

analysis shows that the FAR coefficients with dimension d converge to a fixed series that depends on d. The higher order coefficients are proportional to the initial FAR coefficients and reach $a_n^\star\sim An^{-1-d}$. Thus Eq. (4) leads to

$$a_n^\star = \frac{1}{2-a_1^\star}\,4a_{2n}^\star, \qquad n\to\infty, \tag{8}$$

which requires $a_1^\star=2(1-2^{-d})$. Furthermore, the numerical result hints that $A=d(1-d)$. As an example, this large n approximation for a_3 and $d=1/2$ deviates from the numerical result by less than 4%.

The RG iterations of the initial FAR coefficients a_n for $d=1/2$ are shown in Fig. 1 up to $s=10$ RG steps (multiplied by n^{1+d}). To illustrate the rate of convergence, the inset shows the iterations for a_1. The power spectrum is $S(f)=\overline{\epsilon^2}/|1-\Sigma_n a_n\exp(2\pi in\Delta f)|^2$, where f is defined up to the Nyquist frequency $f_c=1/2\Delta$, with the renormalized time step Δ and the renormalized noise variance $\overline{\epsilon^2}$, both at the actual RG step. Figure 2 shows $S(f)$ for the original FAR
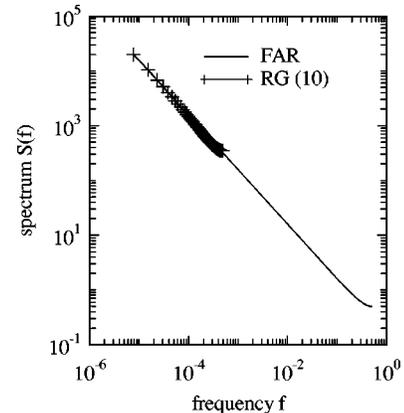
FIG. 2. Power spectra $S(f)$ for the initial FAR coefficients (solid) and the RG result after 10 steps (+); the frequency f is in units of the inverse initial time step Δ.

process and for the renormalized process after $s = 10$ RG steps. The small frequency part remains fixed.

In conclusion, the RG procedure proposed here yields a method to determine the long time behavior of AR processes. The fixed points of the RG depend on the initial AR coefficients. A finite set is given by the random walk and an infinite series of fixed coefficients is proportional to fractional AR processes for high coefficients. The power spectrum shows that the RG conserves the self-similar small frequency properties of the FAR process. Other classes of fixed points are possible but could not be determined since the general solution of the fixed point equation has yet to be found. A further question is the relation between the stationarity properties of an AR process and the renormalized process. The result for low order processes up to AR(3) indicates that a violation of nonstationarity after renormalization is a hint of nonstationarity of the original process. Since the RG reduces the number of coefficients, this is a possible application of the approach.

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