Scaling fields in the two-dimensional Abelian sandpile model

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We consider the unoriented two-dimensional Abelian sandpile model from a perspective based on two-dimensional (conformal) field theory. We compute lattice correlation functions for various cluster variables (at and off criticality), from which we infer the field-theoretic description in the scaling limit. We find perfect agreement with the predictions of a $c=-2$ conformal field theory and its massive perturbation, thereby providing direct evidence for conformal invariance and more generally for a description in terms of a local field theory. The question of the height 2 variable is also addressed, with, however, no definite conclusion yet.

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I. INTRODUCTION

Sandpile models have been invented by Bak, Tang, and Wiesenfeld [1] as prototypical examples for a class of models that show self-organized criticality. The main peculiarity of these models is that they possess a dynamics that drives them to a critical regime, robust against various perturbations. The ineluctable criticality as well as the robustness of these specific dynamics could provide a universal explanation of the ubiquity of power laws in natural phenomena. Various physical situations have been discussed following this idea; see the recent books [2,3].

Sandpile models are among the simplest models showing self-organized criticality. Although their physical relevance can be questioned, it is believed that they have all the features that should be present in more complicated and/or physical models. Therefore they constitute a useful playground where the most important features can be understood.

One of the most interesting models is the two-dimensional unoriented Abelian sandpile model (ASM) [1], which we first briefly recall (recent reviews are [4,5]). The model is defined on an $L\times M$ square lattice. At each site i, we assign a random variable h_i, taking its values in the set $\{1,2,3,4\}$. We think of h_i as a height variable, which counts the number of grains of sand at i. Thus a sand configuration is specified by a set of values $\{h_i\}_i$ of the height variables. A configuration is stable if all $h_i\leqslant 4$, and unstable if $h_i>4$ for one or more sites. The number of stable configurations is equal to 4^{LM}.

The discrete dynamics of the model takes a stable configuration $\mathcal{C}_t$ at time t to another stable configuration $\mathcal{C}_{t+1}$, and is defined in two steps. The first step is the addition of sand: one grain of sand is dropped on a randomly chosen site of $\mathcal{C}_t$, and this produces a new configuration $\mathcal{C}_t'$. The second step is the relaxation to $\mathcal{C}_{t+1}$. If $\mathcal{C}_t'$ is stable, we simply set $\mathcal{C}_{t+1}=\mathcal{C}_t'$. If not, the site where $h_i>4$ topples: it loses four grains of sand, and each of its neighbors receives one grain, something we write in the form $h_j\rightarrow h_j-\Delta_{ij}$ for all sites j, with Δ the discrete Laplacian. In the process, one neighbor can have its height $h>4$, in which case it too topples: it loses four grains of sand, each of its neighbors receiving one grain. And so on for each site that has a height $h>4$, until we reach a stable configuration. $\mathcal{C}_{t+1}$ is then set equal to this new stable configuration. The relaxation process is well defined: it always stops (sand can leave the system at the boundaries) and produces the same result $\mathcal{C}_{t+1}$ independently of the order in which the topplings are performed (the Abelian property).

One can let an initial distribution over the stable configurations evolve in time according to the dynamics, and examine its time limit. Under mild assumptions, one shows [6] that all initial distributions converge to a well-defined and unique distribution P^*, called the SOC (for self-organized critical) state. The theory of Markov chains and the Abelian property allow for a complete characterization of it: P^* is uniform on the set $\mathcal{R}$ of so-called recurrent configurations, and is zero elsewhere (the transient configurations). The number of recurrent configurations is $|\mathcal{R}|=\det\Delta \sim(3.21)^{LM}$, with Δ the discrete Laplacian on the $L\times M$ lattice with open boundary conditions. Although the counting of recurrent configurations is easy, the criterion that actually decides whether a given stable configuration is recurrent or transient is well known [6,7] but hard and nonlocal: in a generic case, one has to scan the whole configuration in order to decide whether it is recurrent or not. Explicit calculations are therefore difficult (and few).

From the point of view of critical systems and conformal field theory, one is interested in the thermodynamic limit $\lim_{L,M\rightarrow\infty}P^*$. The result should be a probability measure on the space of spatially unbounded configurations, or equivalently on the infinite collection of random variables h_i. Despite the fact that these variables are strongly coupled—the couplings are even nonlocal because of the recurrence condition—their correlation functions seem to be of the usual, local form. In the scaling limit, one could therefore hope to recover a local field theory.

There are indications that indeed a conformal field theory emerges, as in ordinary critical, equilibrium lattice models. In [7], a connection with spanning trees was established, which suggests a relationship with the $q=0$ limit of the q-state Potts model, and hence with a $c=-2$ conformal field theory, a value confirmed by calculation of the universal finite size correction to the free energy on a finite strip [7]. The two-site probability $\mathrm{Prob}[h_i=h_j=1]$ was shown in [8] to decay algebraically, with an exponent that can be easily accommodated in a $c=-2$ free Grassmanian scalar field theory [9]. The two-site probabilities for height variables on the boundary of a half-plane domain have also been computed in [10,11], and show the same algebraic falloff as the height 1 variables in the bulk.

Beyond these concordant elements, no systematic investigation in the sandpile model has been made, to our knowledge, which can solidly confirm the connection with a $c=-2$ conformal field theory. It is our purpose to provide a more explicit link between the two. We do this by computing multisite probabilities of various height variables, and by comparing them with the conformal predictions. More specifically, we compute the scaling limits of the two-, three-, and four-site correlations of height 1 variables, but also of other lattice variables, namely, finite subconfigurations that can be handled by the technique developed in [8].

In fact, we compute these correlations in an off-critical extension of the Abelian sandpile model. We evaluate them in the scaling regime, extract the scaling limit, and then establish a correspondence with a field theory. In this way we strengthen the field-theoretic connection away from criticality, by relating a massive perturbation of the ASM to the massive extension of the $c=-2$ fermionic field theory. One can therefore probe more deeply the structure of both pictures, leaving little doubt about the identifications that are to be made.

The conclusion these calculations allow us to draw is that the $c=-2$ theory, and its massive extension, seems to provide a field-theoretic description of the height profile of the sandpile model. At least for the cluster variables examined in this paper, this is a statement that we could verify explicitly. Other important spatial, nondynamical features of the SOC state must be studied. These include boundary features and avalanche distributions. The latter are undoubtedly much more difficult to handle, because they lie at a higher level of nonlocality than the height variables, since they depend on height values in unbounded regions. Whether they can be accounted for by the nonlocal sectors of the $c=-2$ conformal theory remains a largely open question.

II. LATTICE CALCULATIONS IN THE SANDPILE MODEL

As recalled above, explicit calculations in the bulk of the lattice are notoriously hard, because of the nonlocal nature of the SOC state (probability measure) P^*.

All four one-site probabilities $\mathrm{Prob}[h_i=a]$, for $a=1,2,3,4$, have been computed exactly in the thermodynamic limit, but the calculation for $a\geqslant 2$ [12] is already formidably more complicated than for $a=1$ [8]. The only two-site probability that has been computed is again for the unit height variables [8].

The technique used to compute the correlation of two unit height variables is a particular case of a beautiful idea put forward by Majumdar and Dhar [8]. It is based on the important notion of forbidden subconfigurations (FSCs), and its relation to recurrent configurations. A cluster F of sites, with its heights h_i, is a FSC if, for each site $j\in F$, the number of sites in F and connected to j is bigger than or equal to h_j. Simple examples of FSCs are two adjacent 1s (11), a linear arrangement (121), or a cross-shape arrangement with four 1s surrounding a central site with any height value. A configuration is then recurrent if and only if it contains no FSC [6,7].

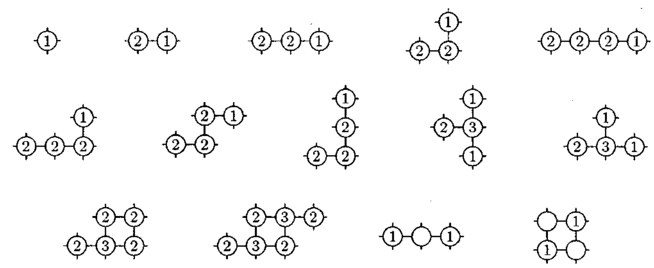


FIG. 1. On the first two lines are shown the ten smallest weakly allowed cluster variables, up to orientations, which contain no more than four sites. Taking the different orientations into account makes a total of 57 clusters of weight smaller than or equal to 4. In addition, calculations involving the four clusters on the last line will be considered in the text. All these clusters will be numbered S_0 to S_{13} from left to right and top to bottom. The reason for including the last two clusters is explained in Sec. VII.

The idea used in [8] allows us to compute the probability of occurrence, in the SOC state, of any cluster that becomes an FSC if any of its heights is decreased by one unit. A simple case is a height 2 next to a height 1, but more examples are given in Fig. 1. Following [13], let us call them weakly allowed cluster variables.

Let S be such a cluster. The authors of [8] show how one can define a new sandpile model, with its own toppling rules (and a new matrix Δ'), such that the number of its recurrent configurations is the number, in the original model, of recurrent configurations that contain S. From this, a simple determinantal formula follows, $\mathrm{Prob}(S)=\det\Delta'/\det\Delta$. Because the new sandpile model is obtained by modifying the original one in the region localized around S, the ratio of the two determinants reduces to a finite determinant, even for an infinite lattice.

This technique has been used to compute the probabilities of various subconfigurations, like those in Fig. 1. The simplest one is the cluster reduced to one site, with height equal to 1. In this case, the new model is obtained by changing the toppling rules at four sites (the height 1 and three neighbors). A 4×4 determinant then yields $P(1)\equiv\mathrm{Prob}[h_i=1]=(2/\pi^2)(1-2/\pi)\sim 0.074$. Allowing for disconnected clusters leads to multisite correlations such as the two-site correlation of unit heights:

$$\mathrm{Prob}[h_i=h_j=1]=P(1)^2\left[1-\frac{1}{2r^4}+\cdots\right],\qquad r=|i-j|\geqslant 1. \tag{2.1}$$

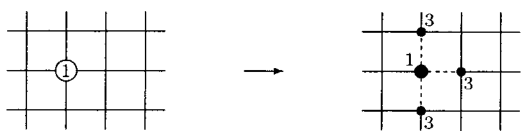
It was also remarked in [8] that more general clusters—for instance, a single site with height equal to 2—can be handled using the same ideas, but the corresponding probabilities become infinite series, the terms of which involve weakly allowed clusters, of increasing size. Unfortunately, these series seem to be slowly convergent.

In general, the way the original sandpile model is modified is by removing some of the bonds linking S to its nearest neighborhood, and at the same time by reducing the threshold at which the sites become unstable (4 in the original model), so that the threshold at every site remains equal to its connectivity. These modifications affect all the sites of S, plus a certain number of sites that are nearest neighbors of S. All together they form a set we call M_S, the cardinal of which depends on the shape of S. The new toppling matrix is then given by $\Delta'=\Delta+B$, where the symmetric matrix B has entries $B_{ij}=1$ if the bond linking i to j has been removed, $B_{ii}=-n$ if n bonds off the site i have been removed, and is zero otherwise. Then the probability of S (in the original model) is

$$P(S)=\frac{\det\Delta'}{\det\Delta}=\det(\mathbb{I}+GB)=\det(\mathbb{I}+GB)\big|_{M_S}. \tag{2.2}$$

Because B is zero outside the finite set M_S, the determinant is finite, in fact of size $|M_S|$, but requires knowledge of the Green function $G\equiv\Delta^{-1}$ of the Laplacian at all sites belonging to M_S.

In the above example where S is just one site with a height equal to 1, the modifications can be pictorially described as follows:



The dashed segments represent the removed bonds, and the numbers on the right lattice indicate the thresholds at which the sites become unstable and topple.

In fact, in the modified lattice shown on the right, the only site to which the 1 is connected has a height bigger than or equal to 2. So one could as well decrease its height and its threshold by 1, and remove the connection. In this way, the site with a height originally equal to 1 is completely cut off from the rest of the lattice, defining a different modification of the original ASM.[1] Either of them can be used to compute correlations involving heights 1.

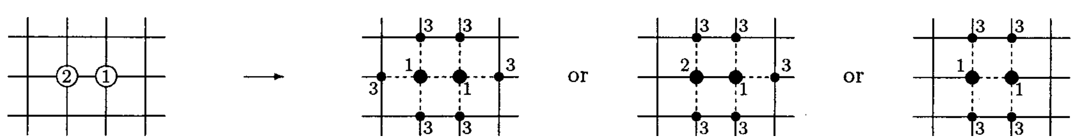
Correspondingly, the matrix B that specifies the modifications is a 4×4 or a 5×5 matrix given by (in an obvious ordering)

$$B=\begin{pmatrix}-3&1&1&1\\1&-1&0&0\\1&0&-1&0\\1&0&0&-1\end{pmatrix}$$

or

$$\begin{pmatrix}-3&1&1&1&1\\1&-1&0&0&0\\1&0&-1&0&0\\1&0&0&-1&0\\1&0&0&0&-1\end{pmatrix}. \tag{2.3}$$

For bigger clusters, there is a fair amount of ambiguity in the way the modifications are made in order to freeze the cluster heights to what we want. These modifications can affect regions of different sizes, and so can be more or less computationally convenient. The least economical solution is the analog of the second modification explained above for the unit height. It is also the easiest to describe: one simply cuts the cluster off the rest of the lattice, removing all bonds inside the cluster and all connections between the cluster and the outside lattice. There are many other choices of intermediate efficiency. For the second cluster in Fig. 1, for instance, namely, a 2 next to a 1, one may consider the following three modifications (among others):



with corresponding B matrices of dimension 8, 7, or 6. For bigger clusters, the difference can be computationally noticeable, and so choosing the modifications that affect the smallest possible region makes the calculation of determinants easier.[2] So one should cut as few links as possible, a prescription that makes sure that the modified ASM remains conservative where the original one is: the removal of a bond

[1]Strictly speaking, in this second modified ASM, the removal of the bond connecting the height 1 to its western neighbor should be supplemented by the creation of a bond connecting the height 1 to a sink site, so that sand brought in by seeding can be evacuated. The part of the modifications that affects the sink site plays no role whatsoever, so we may ignore it completely. See Appendix B for a detailed argument.

[2]The reader familiar with the technique knows that these determinants can be reduced by appropriate summations of rows (or columns). The gain in size is equal to the size of the cluster one considers, but it has its price, because it renders the entries of the reduced determinant more complex. This gain is the same no matter how the ASM is modified.

off one site is always accompanied by the lowering by 1 of the threshold at that site or, equivalently, the B matrix has row and column sums equal to 0.

When the cluster $S=\cup_k S_k$ is disconnected, the matrix B is the direct sum of submatrices B_k. The probability Prob(S) (the correlation of the subparts S_k) involves the Green function $G(i,j)=G(0,i-j)$ at all sites i,j of S, and thus depends on the relative locations and orientations of the various S_k's, and in particular on their separation distances. As the original sandpile model is invariant under lattice translation, the probabilities retain the translation invariance. For S containing two heights equal to 1, separated by a distance r, the evaluation of the 8×8 determinant yields the dominant term r^{-4} given in Eq. (2.1), independently of the angular distance of the two sites.

Precisely in the case in which S is disconnected and contains different pieces separated by large distances, a simple but important observation can be made. Because the probability of S is going to depend on the Green function $G(z_k,z_{k'})\sim\log|z_k-z_{k'}|$ evaluated at points where the subparts are located, one could expect at first sight a logarithmic dependence in the separation distances. However, due to the property that sand is conserved in the modified ASM, the probability in fact depends only on the derivatives (or finite differences) of the Green function. This removes the logarithmic dependences and turns all correlations into rational series in the various distances $(z_k-z_{k'})$.

III. THE MASSIVE SANDPILE MODEL

The previous section summarized the calculation of correlations of cluster variables in the standard ASM. Even though it is critical, and self-organized in the dynamical sense, one can drive it off criticality by switching on relevant perturbations. There are various ways of doing it, but one of the simplest is to add dissipation, whose rate is controlled by a parameter t. In effect this introduces a mass $m\sim\sqrt{t}$, or equivalently a nonzero reduced temperature t. The resulting model can be described as a massive (or thermal) perturbation of the massless, critical sandpile model. For the purpose of comparing the correlations in the ASM with those of a local field theory, the inclusion of some neighborhood of the critical point is important as it strengthens the connection.

The way a mass can be introduced in the model is most straightforward, and corresponds to a dissipation of sand each time a toppling takes place. We define the perturbed ASM by its toppling matrix (we suppress the explicit dependence on x)

$$\Delta_{i,j}=\begin{cases} x & \text{if } i=j,\\ -1 & \text{if } i \text{ and } j \text{ are nearest neighbors},\\ 0 & \text{otherwise.}\end{cases} \tag{3.1}$$

The external driving rate of the sandpile remains the same (one grain per unit of time), but the threshold beyond which the sites become unstable is increased from 4 to x. As a consequence, the height variables now take values between 1 and x. Each time a site topples, $t=x-4\geqslant 0$ grains of sand are dissipated.

In order to assess the robustness of their SOC features, perturbations of sandpile models have often been discussed, with various and sometimes surprising conclusions; see, for instance, [14] for a review. Among the many contributions on the subject, [15] was one of the first attempts to see how the nonconservation of sand in the toppling rules can alter the critical properties of the model. In particular, the massive perturbation defined above in Eq. (3.1) corresponds to the globally dissipative model studied in that paper, and for which the authors found that the avalanche distributions decay exponentially. More recently, the same perturbation was reconsidered in [16], in which the exponential decay of the two-site probability for unit height variables, our Eq. (4.4) below, was proved.

The advantage of the perturbation (3.1) is that it allows the same calculations as the nonperturbed model, in the way that has been recalled in Sec. II. One can in particular compute the correlation functions from the same formulas, with, however, two minor changes. The first one is of course that one uses the massive Green function, with a mass fixed by $\sqrt{x-4}$. The second one concerns the B matrices that define the modified ASM. Because the height variables now take values from 1 up to x, the diagonal entries of B corresponding to sites of the cluster S must be set equal to $1-x$, in order to lock the heights into their minimal values. (As a consequence, note that sand is conserved at those sites, in the modified model.)

When doing concrete computations, one needs the value of the Green function at points close to the origin (at sites belonging to the same connected subpart), and at points far from the origin (at sites located in different connected parts). For the former, one uses a development around $t=0$ (in powers of t with $\log t$ terms), whereas for the latter one performs a double expansion in inverse powers of the distances, and in (half-integral) powers of the perturbing parameter t. For arbitrary positions, this development is cumbersome as it depends also on the angular positions. In the calculations to be presented in the following sections, we have therefore restricted ourselves to configurations of clusters that require the knowledge of the Green function only at points close to a principal or a diagonal axis, for which all useful expressions are collected in Appendix A.

The field theory enters as a description of the long distance regime of the ASM correlations (perturbed or not). As usual, this requires at the same time an adjustment of the correlation length, or equivalently of the mass. So we are interested in computing the scaling regime of correlations. To reach it, we take simultaneously the long distance limit $R=r/a\to\infty$ and the critical limit[3] $x-4=a^2M^2\to 0$, so that the product $\sqrt{x-4}R\to Mr$ defines the effective mass M and

[3] We deliberately take the stand of formally continuing all the expressions from integer values of x to arbitrary values $x\geqslant 4$. Thus we do not define a family of well-defined sandpile models, parametrized by a real number $x\geqslant 4$. For x rational, this can easily be done; however, the limit for x going to four by rational values is not the usual, original model defined for $x=4$. We suspect that the model one gets in this specific limit is a model in which the height variables are completely decoupled. See [17] for a related discussion.

the macroscopic distances r. The scale $a\to 0$ controls the way the limit is taken, and can be thought of as a lattice spacing.

In the actual calculations of correlation functions, large determinants are needed, with entries given by Green function values, themselves expressed as power series. In the scaling regime $x\sim 4$, it is convenient to expand all matrix entries and the correlations as power series of $\sqrt{t}$. The first nonzero term in a correlation should then be directly related to its scaling limit.

We will finish this section by commenting on the way the calculations have been done, before presenting in the next section the results for the unit height variables.

Suppose that we want to compute the joined probability for having a certain cluster S at the origin, say, and another cluster S' at some site i. Each cluster comes with its own set M_S or $M_{S'}$ which contains the sites where the ASM has been modified, the modifications themselves being specified by the matrices B and B'. According to the discussion of the previous section, this probability is equal to a determinant:

$$\mathrm{Prob}[S(0),\ \ S'(i)]=\det\left(\mathbb{I}+\begin{pmatrix} G_{00} & G_{0i}\\ G_{i0} & G_{ii}\end{pmatrix}\begin{pmatrix} B & 0\\ 0 & B'\end{pmatrix}\right)$$
$$=\det\begin{pmatrix}\mathbb{I}+G_{00}B & G_{0i}B'\\ G_{i0}B & \mathbb{I}+G_{ii}B'\end{pmatrix}. \qquad (3.2)$$

The G blocks collectively denote Green function values evaluated at two sites belonging to the set $M_S\cup M_{S'}$, with in addition $G_{i0}=(G_{0i})^t$.

We do not want to know the exact value of this determinant, but rather the terms that are dominant in the scaling region, when i is far from the origin. Using the standard development of a rank n determinant in terms of the matrix entries,

$$\det A=\sum_{\sigma\in S_n}\epsilon(\sigma)A_{1,\sigma(1)}A_{2,\sigma(2)}\cdots A_{n,\sigma(n)}, \qquad (3.3)$$

one may distinguish in Eq. (3.2) several types of term. The permutations σ that do not mix the sites of the cluster S with the sites of S' produce terms that do not depend on the distance $|i|$ separating S from S', and thus contribute a term equal to $[\mathrm{Prob}(S)][\mathrm{Prob}(S')]$.

The other permutations necessarily involve an even number of entries from the off-diagonal blocks. As all such entries are combinations of Green functions, they decay exponentially with the distance. Therefore the two-point function will be dominated by those terms in the determinant that are quadratic in the off-diagonal Green functions. With the help of the formulas in Appendix A, these Green functions are all reducible to the single $G(i)=G_{0,i}$, and its derivatives.

The quadratic terms come from the permutations that send one site of the first cluster onto one site of the second cluster, and vice-versa (with possibly two other sites). The contributions of all those permutations can be summed up to yield a formula written in terms of the minors of the diagonal blocks:

$$\mathrm{Prob}[S(0),\ \ S'(i)]=\mathrm{Prob}(S)\mathrm{Prob}(S')$$
$$-\mathrm{Tr}\{[\mathrm{Mi}(\mathbb{I}+G_{00}B)]^t(G_{0i}B')$$
$$\times[\mathrm{Mi}(\mathbb{I}+G_{ii}B')]^t(G_{i0}B)\}+\cdots. \qquad (3.4)$$

Here $\mathrm{Mi}(A)=(-1)^{i+j}\det(A_{\hat{i},\hat{j}})$ denote, up to signs, the minors of A of maximal order ($A_{\hat{i},\hat{j}}$ is the matrix A with the ith row and the jth column removed). Formula (3.4) is exact modulo quartic, sextic, etc. terms in the off-diagonal Green functions. It has been used to compute all two-cluster correlations considered in this article.

In order to determine the dominant term in the perturbing parameter t, one still makes an expansion in powers of $\sqrt{t}$ (actually the expansions of elements of the diagonal blocks G_{00} and G_{ii} involve the two kinds of terms $t^{k/2}$ and $t^{k/2}\log t$). To this end, one develops all Green functions around $t=0$ using the formulas of Appendix A, and keeps the first nonzero term in the trace. Since the mass M or inverse correlation length is related to $\sqrt{t}$, a first nonzero contribution of the form $t^{(x_1+x_2)/2}F(G(i\sqrt{t}),G'(i\sqrt{t}),\ldots)$ determines the scaling limit of the correlation, and hence the corresponding field-theoretic two-point function, in terms of two fields of scale dimensions x_1 and x_2, in the usual way. In this respect the presence of a logarithmic singularity $\log t$ in the final result would be the signal that the scaling limit is ill defined. It turns out, in all the calculations we have performed, that the first nonzero term scales like t^2 (yielding $x_1+x_2=4$). Because the off-diagonal terms start off like $\sqrt{t}$—they are differences of Green functions at neighboring sites—it is enough to expand all Green functions up to order $t^{3/2}$, as has been done in Appendix A[4] (the three-cluster correlations require expansions to order t^2).

In fact this procedure has anticipated the results on one point. For the purpose of taking the scaling limit, it is the dominant term in t that we want to determine, while the above procedure determines the dominant term in t among the contributions that are quadratic in the Green functions. So one should also check that no higher than quadratic term in the Green functions brings a t^2 contribution. This can easily be done in the following way. Since the off-diagonal terms start off like $\sqrt{t}$, checking the quartic terms is enough, and one can stop the expansion of the off-diagonal blocks to $\sqrt{t}$ order. To that order, the two blocks $G_{0i}B'$ and $G_{i0}B$

[4]This would not be the case if the least economical modification was chosen (the one that cuts the cluster off the rest of the lattice). The B matrix would not have all row sums equal to zero, and consequently the off-diagonal Green functions would have nonzero terms of order 0 in t. This would force us to expand everything to order 2 (instead of 3/2) in t. So these modifications appear to be doubly inefficient.

have all their rows identical. Indeed, inside a given column, all entries are finite differences of Green functions evaluated at neighboring sites, and so differ by second order finite differences of Green functions, i.e., by terms of order t. Thus the determinant with $G_{0i}B'$ and $G_{i0}B$ as off-diagonal blocks can be reduced to a determinant where the two off-diagonal blocks have but their first row nonzero, and equal to linear combinations of Green functions. Such a determinant has no term that is quartic in the off-diagonal block entries.

To end this section, we give the expansion analogous to Eq. (3.4) that pertains to the calculation of three-cluster correlations. Its proof relies on the same arguments as above regarding permutations. For three clusters rooted at sites i,j,k, it reads

$$\begin{aligned}\mathrm{Prob}[S(i),\ S'(j),\ S''(k)]\\ &= -2\,\mathrm{Prob}(S)\mathrm{Prob}(S')\mathrm{Prob}(S'')+\mathrm{Prob}(S)\mathrm{Prob}[S'(j),S''(k)]+\mathrm{Prob}(S')\mathrm{Prob}[S(i),S''(k)]\\ &\quad+\mathrm{Prob}(S'')\mathrm{Prob}[S(i),S'(j)]+\mathrm{Tr}\{[\mathrm{Mi}(\mathbb{I}+G_{ii}B)]^t(G_{ij}B')[\mathrm{Mi}(\mathbb{I}+G_{jj}B')]^t(G_{jk}B'')\\ &\quad\times[\mathrm{Mi}(\mathbb{I}+G_{kk}B'')]^t(G_{ki}B)\}+\mathrm{Tr}\{[\mathrm{Mi}(\mathbb{I}+G_{ii}B)]^t(G_{ik}B'')[\mathrm{Mi}(\mathbb{I}+G_{kk}B'')]^t(G_{kj}B')[\mathrm{Mi}(\mathbb{I}+G_{jj}B')]^t(G_{ji}B)\}+\cdots.\end{aligned} \tag{3.5}$$

This formula gives all terms of the determinant that are cubic in the off-diagonal Green functions. They are to be expanded around $t=0$ as discussed above.

IV. UNIT HEIGHT VARIABLES

The simplest cluster variable is S_0, namely, the unit height variable. We give in this section its multisite correlation functions, in various configurations, as computed along the lines exposed above.

The one-point function, namely, the probability that a fixed site has height equal to 1, poses no problem (and is in any case of little interest for the comparison with a field theory). Making everything very explicit for once, it is given by

$$\mathrm{Prob}(S_0)\equiv P(1)=\det\left(\mathbb{I}+\begin{pmatrix}G(0,0)&G(1,0)&G(1,0)&G(1,0)\\G(1,0)&G(0,0)&G(1,1)&G(2,0)\\G(1,0)&G(1,1)&G(0,0)&G(1,1)\\G(1,0)&G(2,0)&G(1,1)&G(0,0)\end{pmatrix}\times\begin{pmatrix}1-x&1&1&1\\1&-1&0&0\\1&0&-1&0\\1&0&0&-1\end{pmatrix}\right). \tag{4.1}$$

Here $G(m,n)$ is $(\Delta^{-1})_{i,0}$ for the site $i=(m,n)$, and we have used the symmetries of the Green function. The site ordering is O, N, E, and S.

This can easily be computed in terms of complete elliptic functions (see Appendix A), although the result is not particularly transparent:

$$\begin{aligned}P(1)=\frac{1}{256}[(x-4)G(0,0)-1][x^2G(0,0)-16G(1,1)\\-(x+4)][(x^2-8)G(0,0)-8G(1,1)-(x-4)]^2.\end{aligned} \tag{4.2}$$

It goes to $2[2G(1,1)-2G(0,0)+1][G(1,1)-G(0,0)]^2=2(\pi-2)/\pi^3$ in the limit $x\to4$.

More interesting is its graph, which shows that Prob(S_0) increases when x goes away from 4 before falling off algebraically when x keeps growing. The graph of Prob(S_0) as a function of x is reproduced in Fig. 2 as the long-dashed curve.

A. Two-point correlation

The joint probability for having a 1 at the origin, say, and another 1 at a site i is equal to the 8×8 determinant

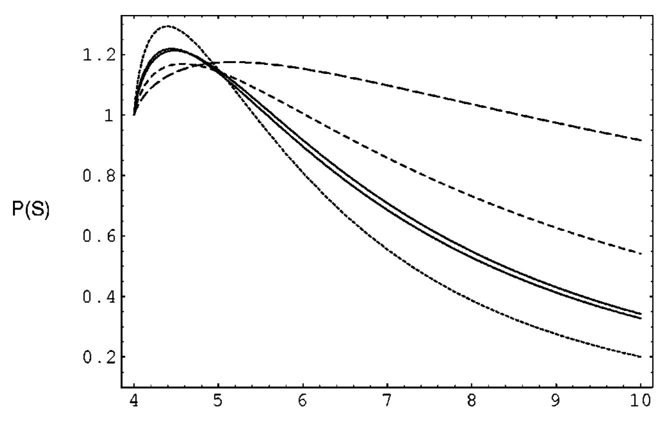


FIG. 2. Unnormalized probabilities of the clusters S_0 up to S_4 as functions of the perturbing parameter x. In this figure, all probabilities have been normalized to 1 at $x=4$ to prevent some of them from melting into the horizontal axis. The curves from top to bottom refer to S_0 (long dashes) down to S_4 (shortest dashes). The two solid lines correspond to S_2 and S_3.

$$\mathrm{Prob}[h_0=h_i=1]=\det\begin{pmatrix} \mathbb{I}+G_{00}B & G_{0i}B \\ G_{i0}B & \mathbb{I}+G_{ii}B \end{pmatrix}, \tag{4.3}$$

where B is the matrix used in Eq. (4.1). Because the two clusters are identical, $G_{ii}=G_{00}$.

As mentioned earlier, the expansion of the Green function at arbitrary points tends to be complicated, so we have restricted ourselves to configurations where the Green functions close to a principal axis or a diagonal axis only are required. For the two-site correlation, this leaves only the two possibilities $i=(m,0)$ and $i=(m,m)$. Using the formula (3.4), we found the same answer in both cases:

$$\mathrm{Prob}[h_0=h_i=1]-[P(1)]^2=-t^2[P(1)]^2\left\{\frac{1}{2}K_0''^2(\sqrt{t}|i|)-\frac{1}{2}K_0(\sqrt{t}|i|)K_0''(\sqrt{t}|i|)+\frac{1}{2\pi}K_0'^2(\sqrt{t}|i|)+\frac{1+\pi^2}{4\pi^2}K_0^2(\sqrt{t}|i|)\right\}+\cdots, \tag{4.4}$$

with $|i|=m$ or $\sqrt{2}m$ depending on whether i is real or on the diagonal. The function K_0 is the modified Bessel function. Note that the $P(1)$ appearing in the left-hand side (LHS) (in the subtraction term) is the off-critical probability, while that in the (RHS) can be taken to be the critical one.

This formula has a number of instructive and comforting features. The spatial dependence is only through the function K_0, that is, the scaling form of the massive lattice Green function. The other functions, denoted D_i or P_i in Appendix A, and representing the lattice corrections to the scaled, continuum Green function, actually do not enter. Moreover, the fact that the answer is the same for the two positions of i suggests that the probability is invariant under rotations, in agreement with the rotational invariance of the cluster S_0 itself. This is related to the first point, since the functions $D_{i>0}$ and $P_{i>0}$ represent anisotropic terms in the lattice Green function.

Another reassuring feature is that the correlation (4.4) scales like t^2, to the dominant order, and that all logarithmic terms $\log t$ have dropped out at that order. This requires massive cancellations because logarithmic terms occur in all entries of the blocks G_{00} and G_{ii}, which store Green function values around the origin (see Appendix A). We have also checked that Eq. (4.4) is exact up to higher order in t: all terms of order lower than t^2 vanish identically [apart from the zeroth order term $P(1)^2$], and there are no terms quartic or higher in the Green function that contribute a t^2 term. Thus Eq. (4.4) is exact to order t^2.

That the correlation scales like the fourth power of the mass was expected since the critical correlation decays like $|i|^{-4}$ [8]. It is easily recovered from Eq. (4.4) by taking the limit $t\to 0$, in which the term in $K_0''^2(|i|\sqrt{t})\sim 1/t^2|i|^4$ is the only one to survive, reproducing the result (2.1).

What the above suggests is that the scaled unit height variable goes over, in the scaling limit, to a massive field ϕ_0 with scale dimension 2,

$$\lim_{a\to 0}\frac{1}{a^2}[\delta(h_{z/a}-1)-P(1)]=\phi_0(z), \quad i=\frac{z}{a}\to\infty,$$
$$t=a^2M^2\to 0 \quad \text{with} \quad i\sqrt{t}=Mz, \tag{4.5}$$

and whose two-point function reads

$$\langle\phi_0(0)\phi_0(z)\rangle=-M^4[P(1)]^2\left\{\frac{1}{2}K_0''^2(M|z|)-\frac{1}{2}K_0(M|z|)K_0''(M|z|)+\frac{1}{2\pi}K_0'^2(M|z|)+\frac{1+\pi^2}{4\pi^2}K_0^2(M|z|)\right\}. \tag{4.6}$$

B. Three-point correlation

We made the same calculations for the three-site probability, using the formula (3.5). The use of the Green functions on the horizontal or the diagonal axis leaves essentially two possibilities: either the three insertion points i, j, and k are aligned, or else they form an isoceles right triangle. In both cases, the probabilities scale like t^3, with all logarithms of t canceled out. The explicit results, however, differ in these two cases.

When they form a linear arrangement, be it on the horizontal or diagonal axis, the result for the connected probability (i.e., products of lower correlations are subtracted) reads

$$\begin{aligned}\mathrm{Prob}[h_i=h_j=h_k=1]_{\text{aligned, connected}}=\frac{M^6}{4}[P(1)]^3\Big\{&[K_0(12)-K_0''(12)][K_0(13)-K_0''(13)][K_0(23)-K_0''(23)]\\&+K_0''(12)K_0''(13)K_0''(23)+\frac{1}{\pi}[K_0''(12)K_0'(13)K_0'(23)-K_0'(12)K_0''(13)K_0'(23)+K_0'(12)K_0'(13)K_0''(23)]\\&-\frac{1}{\pi^2}[K_0(12)K_0'(13)K_0'(23)-K_0'(12)K_0(13)K_0'(23)+K_0'(12)K_0'(13)K_0(23)]-\frac{1}{\pi^3}K_0(12)K_0(13)K_0(23)\Big\}.\end{aligned} \tag{4.7}$$

We have written the answer in the scaled form, that is, after the scaling limit in which the sites i,j,k go over to the macroscopic positions z_1,z_2, and z_3. The notation $K_0(ij)$ stands for $K_0(M|z_i-z_j|)$.

For the triangular configuration, we chose the insertion points $i=(0,0)$ and $k=(2m,0)$ to be real, and put $j=(m,m)$ on the diagonal. The result is slightly different in this case, and reads, in the same notation,

$$\begin{aligned}
&\mathrm{Prob}[h_i=h_j=h_k=1]_{\text{triangular, connected}}\\
&=-\frac{M^6}{4}[P(1)]^3\Big\{2\,K_0''(12)K_0(13)K_0''(23)-K_0''(12)K_0(13)K_0(23)-K_0(12)K_0(13)K_0''(23)\\
&\quad+\frac{1}{\pi}[\sqrt{2}[K_0''(12)-K_0(12)]K_0'(13)K_0'(23)+\sqrt{2}K_0'(12)K_0'(13)[K_0''(23)-K_0(23)]\\
&\quad+K_0'(12)[2\,K_0''(13)-K_0(13)]K_0'(23)]+\frac{\sqrt{2}}{\pi^2}[K_0'(12)K_0'(13)K_0(23)-K_0(12)K_0'(13)K_0'(23)]\\
&\quad+\frac{2}{\pi^3}K_0(12)K_0(13)K_0(23)\Big\}. \qquad (4.8)
\end{aligned}$$

Exactly the same result was found, as expected, for the rotated configuration where i is at the origin, $j=(m,0)$ on the real axis, and $k=(m,m)$ on the diagonal.

The same comments as for the two-site correlation apply here but for one point. If indeed the three-site probability scaling $\sim t^3$ around the critical point is consistent with the dimension 2 of a unit height variable, one observes that the probabilities themselves vanish in the critical limit ($M\to 0$). Thus the scaling limit of three unit height variables in the usual, unperturbed, ASM vanishes:

$$\lim_{\text{scaling}} \mathrm{Prob}[h_i=h_j=h_k=1]_{x=4,\text{connected}}=0. \qquad (4.9)$$

We have checked this result by using the critical Green functions, and found that the probability for three sites aligned along the real axis,

$$\begin{aligned}
&\mathrm{Prob}[h_i=h_j=h_k=1]_{\text{real, connected}}\\
&=-\frac{P(1)^2}{\pi^3}\left[\frac{1}{z_{12}^3z_{23}^3z_{13}^2}-\frac{1}{z_{12}^3z_{23}^2z_{13}^3}-\frac{1}{z_{12}^2z_{23}^3z_{13}^3}\right]\\
&\quad+\frac{P(1)^3}{8}\left[\frac{1}{z_{12}^4z_{23}^4}+\frac{1}{z_{12}^4z_{13}^4}+\frac{1}{z_{23}^4z_{13}^4}\right]\\
&\quad+\frac{3P(1)^3}{4}\left[\frac{1}{z_{12}^4z_{23}^2z_{13}^2}+\frac{1}{z_{12}^2z_{23}^4z_{13}^2}+\frac{1}{z_{12}^2z_{23}^2z_{13}^4}\right]\\
&\quad+(\text{higher order}), \qquad (4.10)
\end{aligned}$$

indeed, decays like a global power -8 of the separation distances. Moreover, the same calculation for the three sites aligned on the diagonal axis produces different coefficients. Thus the dominant term of the critical lattice three-point function is not isotropic, contradicting the expected rotational invariance, and so should not survive the scaling limit.

C. Four-point correlation

Finally, we have also determined the four-site probability for unit height variables, at the critical point only, as otherwise the number of terms grows quickly. So in this case we have used throughout the calculations the expansions at $x=4$ of the Green functions, also given in Appendix A.

We have examined two different arrangements of the insertion points, when they are all aligned on the real axis, and when they lie at the vertices of a square. When they are all aligned on the real axis, the connected four-site probability takes a very simple form, at the dominant order,

$$\begin{aligned}
&\mathrm{Prob}[h_i=h_j=h_k=h_l=1]^{\text{real}}_{\text{connected}}\\
&=-\frac{P(1)^4}{4}\left\{\frac{1}{(z_{12}z_{34}z_{13}z_{24})^2}+\frac{1}{(z_{13}z_{24}z_{14}z_{23})^2}\right.\\
&\quad\left.+\frac{1}{(z_{14}z_{23}z_{12}z_{34})^2}\right\}+\cdots, \qquad (4.11)
\end{aligned}$$

where the ellipsis represents terms of global power smaller than or equal to -10 (they disappear in the scaling limit), and $z_{13}=i-k,\dots$ (real). The other case, for which $i=(0,0)$, $j=(m,0)$, $k=(0,m)$, and $l=(m,m)$ are the vertices of a square of side length m, is much more rigid as it depends on a single distance m. The result we found for this situation is

$$\mathrm{Prob}[h_i=h_j=h_k=h_l=1]^{\text{square}}_{\text{connected}}=-\frac{3}{8}\frac{[P(1)]^4}{m^8}+\cdots. \qquad (4.12)$$

Before presenting the results for the other cluster variables of Fig. 1, we examine the above correlations for the unit height random variable from the point of view of the conformal field theory that is the most natural candidate, namely, the $c=-2$ theory, and its massive extension.

V. CONFORMAL FIELD THEORY

The $c=-2$ conformal field theory was studied first in the context of polymers [18], and a bit later served as the simplest example of a logarithmic conformal field theory [19]. Since then it has been extensively examined by many authors [20–26]. Reference [24] in particular presents a clear and rather complete account of the structure of the $c=-2$ theory as a rational conformal field theory. Even if it is considered as the simplest situation where logarithms can occur, it contains many subtle aspects and probably possesses many different and inequivalent realizations. The one that is relevant here is perhaps the most natural one.

The underlying field theory is formulated in terms of a pair of free Grassmanian scalars $\theta^\alpha=(\theta,\bar{\theta})$ with action

$$S=\frac{1}{2\pi}\int \varepsilon_{\alpha\beta}\partial\theta^\alpha\bar{\partial}\theta^\beta=\frac{1}{\pi}\int \partial\theta\,\overline{\partial\theta}, \tag{5.1}$$

where ε is the canonical symplectic form, $\epsilon_{12}=+1$.

The zero modes of $\theta,\bar{\theta}$, call them ξ and $\bar{\xi}$, have been much discussed. Because the action does not depend on them, the expectation value of anything that does not contain θ and $\bar{\theta}$ explicitly, but only their derivatives, vanishes identically. In particular, the partition function itself vanishes, so the correlation functions are normalized by $Z'=\int\mathcal{D}\theta'\mathcal{D}\overline{\theta'}\,e^{-S}$, where the primed fields exclude the zero modes ξ and $\bar{\xi}$. This normalization implies, for instance ($\epsilon^{\alpha\beta}=-\epsilon_{\alpha\beta}$),

$$\langle 1\rangle=0,\quad \langle\bar{\xi}\xi\rangle=1, \tag{5.2}$$

$$\langle\theta^\alpha(z)\theta^\beta(w)\rangle=\epsilon^{\alpha\beta},\quad \langle\theta^\alpha(z)\theta^\beta(w)\bar{\xi}\xi\rangle=\epsilon^{\alpha\beta}\log|z-w|, \tag{5.3}$$

$$\langle\partial\theta^\alpha(z)\partial\theta^\beta(w)\rangle=0,\quad \langle\partial\theta^\alpha(z)\partial\theta^\beta(w)\bar{\xi}\xi\rangle=\frac{\epsilon^{\alpha\beta}}{2(z-w)^2}, \tag{5.4}$$

$$\begin{aligned}\langle\theta^\alpha(z_1)\theta^\beta(z_2)\theta^\gamma(z_3)\theta^\delta(z_4)\rangle &=\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\log|z_{12}z_{34}|-\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\log|z_{13}z_{24}|\\ &\quad+\epsilon^{\alpha\delta}\epsilon^{\beta\gamma}\log|z_{14}z_{23}|.\end{aligned} \tag{5.5}$$

As far as derivatives of fields are concerned—as will be the case in the ASM, at least at the conformal point—one can insert the two zero modes in the correlators, as in Eq. (5.4), to take care of the integral over constant fields. The functional integral over nonconstant fields then yields the usual form for the correlators, obtained from Wick's theorem and the kernel of the Laplacian. Equivalently, one can define the functional integral for derivative fields by keeping the zero modes out, or consider the so-called η–ξ system [18].

The stress-energy tensor components $T=2{:}\partial\theta\,\partial\bar{\theta}{:}$ and $\bar{T}=2{:}\bar{\partial}\theta\,\bar{\partial}\bar{\theta}{:}$ have operator product expansions (OPEs) characteristic of a conformal theory with central charge $c=-2$. The fields θ and $\bar{\theta}$ are primary fields with conformal dimensions (0,0), while the bosonic composite field ${:}\theta\bar{\theta}{:}$ has the following OPE with T:

$$T(z){:}\theta\bar{\theta}{:}(w)=\frac{-1}{2(z-w)^2}+\frac{\partial{:}\theta\bar{\theta}{:}(w)}{z-w}+\cdots. \tag{5.6}$$

It shows that the conformal transformation of ${:}\theta\bar{\theta}{:}$ does not close on itself (and its descendants) but also involves the identity and its descendants, which form a conformal module on their own. Thus the identity and ${:}\theta\bar{\theta}{:}$ generate a Virasoro module, which is reducible but not fully reducible. This is a characteristic feature of logarithmic conformal theories [19]. The field ${:}\theta\bar{\theta}{:}$ is called the logarithmic partner of the identity. It is neither a primary field nor a descendant (see below for a field that is primary and descendant without being null).

The fact that there are two fields with zero scaling dimension is the main source of unusual features (and confusing subtleties), one of them being the existence of two degenerate vacua $|0\rangle$ and $|\xi\bar{\xi}\rangle$ (there are two more of fermionic nature, $|\xi\rangle$ and $|\bar{\xi}\rangle$). The above prescription about the insertion of the zero modes can be viewed in the operator formalism as the taking of operator matrix elements between two distinct ingoing and outgoing vacua.

In conclusion, the theory specified by the action (5.1) is a logarithmic conformal theory with central charge $c=-2$. It contains a nonlogarithmic local sector, which retains the central charge value $c=-2$, and in which derivative fields only are considered. Anticipating the analysis to be given below, our results suggest that the ASM scaling fields related to height variables lie precisely in this $c=-2$ nonlogarithmic conformal theory.

It should also be noted that either theory, logarithmic or nonlogarithmic, contains additional nonlocal (twisted) sectors. Although they could play an important role in the sandpile models, for the description of other lattice variables than heights, we will not discuss them here, and refer to [24] for further details.

We will also need the off-critical, massive extension of the above conformal theory. It corresponds to a perturbation by the logarithmic partner of the identity

$$S(M)=\frac{1}{\pi}\int {:}\partial\theta\,\overline{\partial\theta}{:}+\frac{M^2}{4}{:}\theta\bar{\theta}{:}. \tag{5.7}$$

The zero mode problem no longer arises in the massive theory, so that one can normalize the correlation functions by the full partition function $Z(M)=\int \mathcal{D}\theta\mathcal{D}\bar{\theta}\, e^{-S(M)}$. One then obtains

$$\langle \theta(z)\,\bar{\theta}(w)\rangle = K_0(M|z-w|),$$
$$\langle \theta(z)\,\theta(w)\rangle = \langle \bar{\theta}(z)\,\bar{\theta}(w)\rangle = 0, \tag{5.8}$$

and, for instance,

$$\langle \partial\theta(z)\partial\bar{\theta}(0)\rangle = -\frac{M^2}{4}\frac{\bar{z}}{z}[2K_0''(M|z|)-K_0(M|z|)]. \tag{5.9}$$

On account of $K_0(x)\sim -\log x$ for small arguments, the massless limit of the previous equation exists and reproduces the expression given in Eq. (5.4) with the zero modes inserted. This is expected since the effect of the zero mode insertion is formally to change the normalization factor from Z' to Z. On the other hand, the same does not apply to the correlations of the fields θ^α themselves, as the normalizing functional $Z(M)$ goes to zero as $M\to 0$.

As mentioned above, the cluster variables we consider in this article are all related to derivative fields. The previous remark then implies that the off-critical ASM multi-site probabilities have a smooth massless limit, equal to the critical probabilities. The scaling form of the off-critical probabilities will be related to the above massive free theory, while the critical ones will be computable in terms of the nonlogarithmic conformal field theory using the insertion prescription.

VI. SCALING FIELDS FOR CLUSTER VARIABLES

Let us now reconsider the multisite probabilities for height 1 computed in Sec. IV. The two-site probability suggested that the unit height variable is described by a field with scaling dimension 2, which should in addition be scalar since a unit height variable is rotationally invariant. If one assumes that this field is local in $\theta,\bar{\theta}$, the only possibilities are $:\partial\theta^\alpha\,\bar{\partial}\theta^\beta:$, $:\theta\bar{\theta}\,\partial\theta^\alpha\,\bar{\partial}\theta^\beta:$, and $M^2{:}\theta\bar{\theta}{:}$. The second set of fields $:\theta\bar{\theta}\,\partial\theta^\alpha\,\bar{\partial}\theta^\beta:$ must be excluded, because, as explained above, they would produce logarithms in correlation functions, contradicting the observation we made in Sec. II that, in the massless sandpile model ($x=4$), the multisite probabilities are never logarithmic (at least those one can compute from the Majumdar-Dhar technique, i.e., from finite determinants).

It is not difficult to see that

$$\phi_0 = -P(1)\left[:\partial\theta\bar{\partial}\bar{\theta}+\bar{\partial}\theta\partial\bar{\theta}: + \frac{M^2}{2\pi}:\theta\bar{\theta}:\right] \tag{6.1}$$

is indeed the right combination: its two-point function is exactly the form given in Eq. (4.6), which was obtained by taking the scaling limit of the two-site probability computed on the lattice.

In order to confirm this identification, the field-theoretic three-point function of ϕ_0 can be computed and compared with the lattice result. In the same notation as in Sec. IV, one finds for an arbitrary arrangement of the insertion points

$$\begin{aligned}\langle \phi_0(z_1)\,\phi_0(z_2)\,\phi_0(z_3)\rangle = -\frac{M^6}{16}\times\Bigg\{&\frac{1}{2}\left(\frac{z_{13}\bar{z}_{23}}{\bar{z}_{13}z_{23}}+\text{c.c.}\right)K_0(12)[2K_0''(13)-K_0(13)][2K_0''(23)-K_0(23)]+\text{perm}\\ &+\frac{\pi-2}{\pi^2}\left[\left(\frac{z_{13}\bar{z}_{23}}{|z_{13}||z_{23}|}+\text{c.c.}\right)K_0(12)K_0'(13)K_0'(23)+\text{perm}\right]+\frac{1}{\pi}\Bigg[\left(\frac{z_{12}^2\bar{z}_{13}\bar{z}_{23}}{|z_{12}|^2|z_{13}||z_{23}|}\right)\\ &\times[2K_0''(12)-K_0(12)]K_0'(13)K_0'(23)+\text{perm}\Bigg]+\frac{\pi^3-4}{\pi^3}K_0(12)K_0(13)K_0(23)\Bigg\},\end{aligned} \tag{6.2}$$

where the permutations that must be added are the two exchanges $z_1\leftrightarrow z_3$ and $z_2\leftrightarrow z_3$. One easily checks that it reproduces the three-site probabilities reported in Sec. IV for the two arrangements examined there. The massless limit of Eq. (6.2) vanishes, as clearly follows from Eq. (6.1) for $M=0$, since the three-point function will necessarily involve a Wick contraction of a $\partial\theta^\alpha$ with some $\bar{\partial}\theta^\beta$.

Finally, the four-point function can be compared. For convenience, we give the field-theoretic result in the massless regime:

$$\langle \phi_0(z_1)\phi_0(z_2)\phi_0(z_3)\phi_0(z_4)\rangle_{M=0} = \frac{P(1)^4}{4|z_{12}|^4|z_{34}|^4} + \frac{P(1)^4}{4|z_{13}|^4|z_{24}|^4} + \frac{P(1)^4}{4|z_{14}|^4|z_{23}|^4} - \frac{P(1)^4}{8}\left\{ \frac{1}{(z_{12}z_{34}\bar{z}_{13}\bar{z}_{24})^2} + \frac{1}{(z_{13}z_{24}\bar{z}_{14}\bar{z}_{23})^2} + \frac{1}{(z_{14}z_{23}\bar{z}_{12}\bar{z}_{34})^2} + \text{c.c.}\right\}, \tag{6.3}$$

where only the last term within the curly brackets represents the connected part of the four-point function. When the four insertions lie on the real axis, it clearly reproduces the lattice result (4.11), and when they are the vertices of a square of side m, $z_1=0, z_2=m, z_3=im, z_4=(1+i)m$, it reduces to

$$\langle \phi_0(z_1)\phi_0(z_2)\phi_0(z_3)\phi_0(z_4)\rangle_{M=0,\text{square,connected}} = -\frac{3}{8}\frac{[P(1)]^4}{m^8}, \tag{6.4}$$

and again matches the connected four-site probability (4.12).

We believe these comparisons provide enough evidence to assert that the unit height random variable of the sandpile model goes over, in the scaling limit, to the field ϕ_0 defined in Eq. (6.1). In the conformal limit, $\phi_0 \sim :\partial\theta\overline{\partial\theta}+\bar{\partial}\theta\partial\bar{\theta}: = \partial\bar{\partial}:\theta\bar{\theta}:$ is a primary field with conformal dimensions (1,1), but is also a descendant of $:\theta\bar{\theta}:$.

The rest of this section presents analogous results for the other cluster variables pictured in Fig. 1.

We have repeated, for the other 13 clusters in Fig. 1, the same calculations we performed for the unit height variable. More precisely, for each of the cluster variables S_1 up to S_{13}, we have computed its joint probability with a unit height, namely, $\text{Prob}[S_0(0),S_k(i)]$, with i on the principal and on the diagonal axis. From these two probabilities one can write down an ansatz for the field ϕ_k with which the cluster S_k gets identified in the scaling limit. These identifications were subsequently checked to reproduce all two-site probabilities $\text{Prob}[S_k(0),S_\ell(i)]$, for all pairs $k,\ell = 0,1,2,\ldots,13$, on both the principal and the diagonal axes. In addition, at least one rotated (or mirrored) version of each cluster has been examined, although not systematically (only the correlation with S_0 on both axes). The results we found for the rotated clusters are in agreement with the rotations of the fields assigned to the unrotated clusters, so that the field of the rotated cluster is the rotated field. Finally, mixed three-cluster probabilities involving unit heights and S_1 clusters have also been computed. They all confirmed the field identifications.

All calculations have been performed exactly, i.e., not numerically. The two-cluster probabilities take a form similar to Eq. (4.4), where the coefficients are in general complicated rational expressions of π. Keeping these coefficients in an exact form allows the check of the field identifications to be made in an exact way. For simplicity, however, the results presented below are given numerically.

The features of the two-cluster probabilities are the same as for the unit height variables. We found that all of them scale like t^2, with all logarithmic singularities canceled out. This implies that all cluster variables go in the scaling limit to fields with scaling dimension 2:

$$\lim_{a\to 0}\frac{1}{a^2}[\delta(S(i))-P(S)] = \phi_S(z), \quad i=\frac{z}{a}\to\infty,$$

$$t \equiv x-4 = a^2M^2 \to 0 \quad \text{with} \quad i\sqrt{t}=Mz. \tag{6.5}$$

This is somewhat surprising as one might have expected the dimension of the scaling fields to increase with increasing size of the clusters.

All cluster variables we have considered have a scaling limit that corresponds to a field of the following form:

$$\phi(z) = -\Big\{ A:\partial\theta\overline{\partial\theta}+\bar{\partial}\theta\partial\bar{\theta}: + B_1:\partial\theta\partial\bar{\theta}+\bar{\partial}\theta\overline{\partial\theta}: + iB_2:\partial\theta\partial\bar{\theta} - \bar{\partial}\theta\overline{\partial\theta}: + C\,P(S)\frac{M^2}{2\pi}:\theta\bar{\theta}:\Big\}. \tag{6.6}$$

The (real) coefficients A, B_1, B_2, and C are given in Table I for each cluster. The factor $P(S)$ in front of the term $:\theta\bar{\theta}:$ is the probability of S evaluated at $x=4$. Note that the field is not invariant under a rotation of $\pi/2$ as soon as B_1 or B_2 is nonzero, but is invariant under a rotation of π no matter what the coefficients are. So, in particular, the scaling limit of the cluster variables does not in general yield conformal fields, but sums of pieces with different tensor structures.

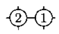
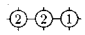
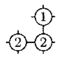
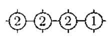
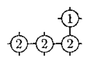
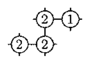
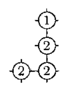
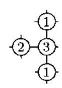
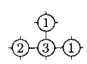
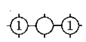
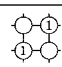
As far as numerical values are concerned, the last column of the table is particularly striking: all entries are integers, simply equal to the size of the cluster. This makes the coefficient of the $:\theta\bar{\theta}:$ terms particularly simple and apparently regular. The reason for this is unclear.

The other numbers mentioned in the table are not in themselves particularly interesting. As mentioned above, all these numbers are complicated expressions. For instance, the first three numbers on the line corresponding to S_9 (the last cluster of size 4) are in fact equal to

$$\text{Prob}(S_9) = \frac{2\,621\,440}{27\,\pi^7} - \frac{21\,389\,312}{81\,\pi^6} + \frac{24\,279\,040}{81\,\pi^5} - \frac{14\,968\,672}{81\,\pi^4} + \frac{1\,809\,776}{27\,\pi^3} - \frac{258\,037}{18\,\pi^2} + \frac{10\,061}{6\,\pi} - \frac{663}{8}, \tag{6.7}$$

$$A = \left(\frac{3\pi-8}{\pi^2}\right)\left(\frac{655\,360}{27\,\pi^5} - \frac{3\,389\,440}{81\,\pi^4} + \frac{2\,259\,952}{81\,\pi^3} - \frac{81\,566}{9\,\pi^2} + \frac{5765}{4\,\pi} - \frac{8647}{96}\right), \tag{6.8}$$

TABLE I. For each cluster in Fig. 1, the table gives the values of the parameters A, B_1, B_2, and C specifying the field that describes the scaling behavior of the given cluster [see Eq. (6.6)]. Note in particular that the coefficient C is equal to the size of the cluster.

cluster S	$P(S)$	A	B_1	B_2	C
	0.0736362	0.0736362	0	0	1
	0.0103411	0.0201433	−0.00619014	0	2
	0.00141994	0.00449027	−0.00208908	0	3
	0.00134477	0.00389417	0	−0.000534524	3
	0.00019246	0.000893234	−0.000502885	0	4
	0.000179829	0.000752599	−0.000172832	−0.000122784	4
	0.000173323	0.000695941	−0.00012949	−0.000153467	4
	0.000179829	0.000752599	0.000172832	−0.000122784	4
	0.000173106	0.000692147	0.000135489	0	4
	0.000173106	0.000692147	−0.000135489	0	4
S_{10}	0.0000572863	0.000255127	−0.0000470539	−0.0000259802	5
S_{11}	0.00000731457	0.000042272	−0.0000130978	−0.0000071978	6
$S_{12}=$ 	0.00496687	0.00969315	−0.00129393	0	2
$S_{13}=$ 	0.00404859	0.010213	0	−0.00312009	2

$$B_1=\left(\frac{3\pi-8}{\pi^2}\right)\left(\frac{305\,152}{81\pi^4}-\frac{359\,056}{81\pi^3}+\frac{17\,554}{9\pi^2}-\frac{13\,693}{36\,\pi}+\frac{2663}{96}\right). \tag{6.9}$$

A gross feature of Table I is that the (nonzero) numbers are roughly constant for all clusters of the same size, namely, the probabilities and the coefficients do not change much with the shape of the clusters, but depend essentially on their size only. Roughly speaking, these numbers (except C) get divided by 10 when the size increases by 1.

The zeros in the table or the equality (up to signs) of coefficients can be understood from the transformations of the clusters and the corresponding fields under the symmetry group of the lattice. One easily sees that the field ϕ in Eq. (6.6) changes under rotations and reflections according to the following rules:

$$(A,B_1,B_2,C)\to\begin{cases}(A,-B_1,-B_2,C) & \text{under a } \pi/2 \text{ rotation,}\\ (A,B_1,-B_2,C) & \text{under an } x \text{ or } y \text{ reflection.}\end{cases} \tag{6.10}$$

By convention, all clusters are assumed to be anchored to their lower left site. The rotations are performed about an axis passing through that site.

First of all, the only one to have $B_1=B_2=0$ is the unit height. Indeed, it is the only cluster that preserves its shape under rotations and reflections, and so one can expect the corresponding field to be a scalar under (continuous) rotations and reflections.

There are clusters whose fields have $B_2=0$, and they are precisely those clusters that are invariant under a reflection through the horizontal axis. The same can be said of the rotated clusters for a reflection through the vertical axis. That S_9 has a coefficient $B_2=0$ can be understood along the same lines, although it is not manifestly invariant under reflections. An x reflection of S_9 followed by a rotation by π and a translation of two lattice sites brings it to itself, except that a height 2 and a height 1 have been swapped. However, the assignment of heights within a cluster is irrelevant in the actual computation: the modified ASM is defined in terms of certain bonds being removed. Since each site whose height is being constrained loses three out of its four bonds, the actual height assignment is irrelevant. In effect, the set M_S that includes all the sites affected by the modifications and the modification matrix B itself can be chosen (have been chosen) invariant under a y reflection.

In the same way, one sees that S_5 and S_7 have equal coefficients, up to signs. As represented in Table I and Fig. 1, they are related by a rotation of $\pi/2$ and an x reflection, with the consequence that their B_1 coefficients are opposite but the B_2 are equal. The same can be said of S_8 and S_9, with the same remark as above regarding the locations of the height values within the clusters.

From these remarks, one easily finds the fields corresponding to different orientations of a cluster. The cluster S_6, for instance, comes in eight different orientations (all anchored to the same site). All of them have the same coefficient $A\sim 0.000\,695\,941$ and $C=4$, whereas pairs of clusters have coefficients (B_1,B_2), or $(-B_1,B_2)$, or $(B_1,-B_2)$, or $(-B_1,-B_2)$. As a consequence, the sum over the corresponding eight fields reduces to a projection onto the scalar part, and involves the A and C terms only.

In a sense, the fact that the fields reflect so well the geometric symmetries of the clusters is surprising. As discussed at length in Sec. II, the actual calculations are based on adequate modifications of the original ASM on a set we called M_S, which contains not only sites belonging to the cluster itself, but also sites in its close neighborhood. Thus each cluster drags with itself an invisible shadow, made of the sites in the set $M_S \backslash S$. The shadow is a computational artifact, but is nevertheless crucial. Moreover, it usually breaks or alters the geometric symmetries of the cluster it goes with. The insertion of a height 1, for instance, somewhere in the lattice, really requires us to consider a four-cluster pictured in Sec. II. Here the shadow consists of three neighbors of the central site, and clearly breaks the rotational invariance.

We will conclude this section by observing that the height h variables, for h bigger than 4, can be handled in the massive ASM exactly like the unit height variables, even more simply. The reason is that a height equal to $5,6,\ldots,x$ can never be in a forbidden subconfiguration, so that the set of recurrent configurations containing a height equal to $h>4$ at some site i is equal to the set of recurrent configurations on the lattice with i removed. Therefore the modifications needed to freeze the height of a site to $h>4$ must simply reduce the threshold at that site to 1, and cut it off from the rest of the lattice. This can be implemented by the following matrix:

$$B=\begin{pmatrix} 1-x & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.11}$$

The corresponding probability is simply given by $\mathrm{Prob}[h_i=h>4]=G(0,0)$, and is logarithmically divergent at $x=4$. For that reason, one considers instead the probability that h_i exceeds 4:

$$\mathrm{Prob}[h_i>4]=(x-4)G(0,0)=\frac{2(x-4)}{\pi x}K\left(\frac{4}{x}\right), \tag{6.12}$$

which goes to 0 when $x\to 4$. (The matrix B corresponding to this has -4 as the first diagonal entry, rather than $1-x$.) K is a complete elliptic function (see Appendix A).

As for the above clusters, one can compute the correlations of this random variable $\delta(h_i>4)$ with itself or with the other clusters, and see what field-theoretic description it has in the scaling limit. Again, the result is simple. The lattice calculation of its own correlation yields

$$\mathrm{Prob}[h_0>4,h_i>4]-\mathrm{Prob}[h_0>4]^2=-\frac{t^2}{4\pi^2}K_0^2(\sqrt{t}|i|)+\cdots, \tag{6.13}$$

which suggests the scaling limit

$$\delta(h_i>4)-\langle\delta(h_i>4)\rangle \xrightarrow{\text{scaling}} \phi=\frac{M^2}{2\pi}:\theta\bar{\theta}:. \tag{6.14}$$

Correlations with the other cluster variables confirm this limit. It nicely fits the expectation that the field should vanish at the critical point.

VII. THE HEIGHT 2 VARIABLE

We have so far focused on the class of weakly allowed cluster variables, whose correlations can be handled by the technique developed in [8], and in turn computed from a finite determinant. The authors point out in that article that non weakly allowed cluster variables can in fact be viewed as infinite series of weakly allowed clusters. It dramatically complicates their treatment, since a correlation involving a single non weakly allowed cluster requires the computation of an infinite number of correlations of weakly allowed clusters, of finite but unbounded size.

In this section, we address the question of the field assignment for the height 2 variable, in the light of the results of the previous sections. We will consider the height 2 variable, both from the perturbative point of view that we have just summarized, and from the conformal point of view.

That a height 2 variable can be treated as an infinite sum of weakly allowed cluster variables can be seen as follows [8]. Consider the set of recurrent configurations $\mathcal{C}$ with a height 2, at the origin say. That set can be divided up into two disjoint subsets according to whether the configurations remain recurrent when the 2 is replaced by a 1, or become transient upon that replacement.

The number of those that remain recurrent is the same as the number of recurrent configurations that have a height 1 at the origin, because, vice versa, a recurrent configuration with a 1 remains recurrent if the 1 is replaced by a 2. So the contribution to $P(2)\equiv\text{Prob}[h_0=2]$ from this first subset is exactly equal to $P(1)$.

For those configurations that become transient, it must be that the 2 belongs to a weakly allowed cluster. This weakly allowed cluster can be of various sizes and shapes, and a straight enumeration according to their size leads directly to the clusters of Fig. 1 (except the first one and the last two) and their various orientations. In this way, the second subset is itself divided into an infinite number of disjoint subsets, according to which weakly allowed cluster S the height 2 at the origin is part of. The subset labeled by S (fixed size, shape, and orientation) contributes to $P(2)$ a term equal to $P(S)$.

Putting all together, one obtains, observing that the number $P(S)$ does not depend on the orientation of S, the formula

$$\begin{aligned}P(2)=P(1)+\sum_{\text{w.a.c. } S}P(S)&=P(1)+4P(S_1)+4P(S_2)\\&\quad+8P(S_3)+4P(S_4)+8P(S_5)+\cdots\end{aligned}\tag{7.1}$$

where the summation is over the weakly allowed clusters which are ''anchored'' to a height 2. As pointed out in [8], the convergence is very slow. From Table I, the terms up to S_9 furnish the lower bound $P(2)\geqslant 0.13855$, well below the exact value $P(2)\sim 0.1739$ [12].

The argument recalled above leading to the perturbative formula for $P(2)$ works similarly for any correlation. The result can be expressed as an identity between random variables,

$$\delta(h_i-2)=\delta(h_i-1)+\sum_{\text{w.a.c. } S}\delta(S(i)).\tag{7.2}$$

Modulo the issue of convergence, this identity is valid when inserted in expectation values.

The results of the previous section suggest that all random variables on the RHS have the same scaling form, given by the field in Eq. (6.6). Assuming this at all orders and taking the scaling limit of the previous identity lead to a scaling field for height 2 of the same form as the scaling field for height 1, namely,

$$\delta(h_i-2)\xrightarrow{\text{scaling}}\alpha{:}\partial\theta\overline{\partial\theta}+\bar\partial\theta\partial\bar\theta{:}+\beta M^2{:}\theta\bar\theta{:}.\tag{7.3}$$

This follows from the observation we made earlier that the other terms $\partial\theta\partial\bar\theta\pm\bar\partial\theta\overline{\partial\theta}$ change sign under a rotation by $\pi/2$. The sum over the orientations of a cluster makes these terms cancel against each other, leaving a scalar field, as it should be.

The natural conclusion one might draw from this is that, at the critical point, the heights 1 and 2 scale in the same way and in fact go over, in the scaling limit, to the same—up to normalization—primary field of conformal dimensions (1,1). This is direct though tenuous evidence in favor of such a statement, which was in fact made in [11], based on an extrapolation to the bulk of a similar statement on the corresponding boundary variables, itself relying on the boundary two-point functions. As plausible and likely as it may be, the extrapolation remains uncontrolled, as there are well-known examples of lattice observables that go to different fields, depending on whether they lie on a boundary or in the bulk. Thus neither argument is convincing, but both point to the same field assignment for the height 2 variable (and probably similarly for heights 3 and 4).

This seems reasonable and likely. It is therefore surprising to observe that it does not appear to be consistent with a naive interpretation of the operator product expansions. To simplify, we consider the critical point, and the corresponding conformal field theory.

The two lattice variables, a height 1 and a height 2, can be taken far apart and subsequently brought closer to each other until they occupy neighboring sites, thus forming the cluster variable we called S_1. In the field-theoretic picture, this amounts to taking the two corresponding fields closer and closer to each other, until they become coincident, at which point they form a new composite field. The information about what composite fields a pair of fields can form when they come close to each other and are asymptotically coincident is contained in their operator product expansion.

Thus it seems natural to expect that the field assigned to the cluster variable -②–①- be in the OPE of the field corresponding to height 1 with the field corresponding to height 2. If one assumes, as argued above, that the heights 1 and 2 scale to the same field, the required OPE is simply

$$\begin{aligned}&{:}\partial\theta\overline{\partial\theta}+\bar\partial\theta\partial\bar\theta{:}(z){:}\partial\theta\overline{\partial\theta}+\bar\partial\theta\partial\bar\theta{:}(w)\\&\quad=-\frac{1}{2|z-w|^4}+\frac{{:}\bar\partial\theta\overline{\partial\theta}{:}(w)}{(z-w)^2}+\frac{{:}\partial\theta\partial\bar\theta{:}(w)}{(\bar z-\bar w)^2}\\&\qquad+(\text{less singular}),\end{aligned}\tag{7.4}$$

where, from dimensional analysis, the less singular terms involve fields of scale dimension strictly larger than 2. One sees from Eq. (7.4) that the only fields with scale dimension 2 that can be formed in the fusion of a height 1 with a height 2 are the nonscalar parts of the field making the cluster -②–①-. The scalar part of it, ${:}\partial\theta\bar\partial\bar\theta+\bar\partial\theta\partial\bar\theta{:}$, is missing. [Note that it must be so, since otherwise the unit height variables, represented by that scalar field, would have a nonzero (connected) three-point function.]

One may observe that the only dimension 2 scalar fields whose fusion produces all field components of -2-1- are logarithmic fields like $:\theta\bar{\theta}(\partial\theta\bar{\partial}\bar{\theta}+\bar{\partial}\theta\partial\bar{\theta}):$. The change to this logarithmic scalar field has, however, heavy consequences as correlations involving heights 2 would automatically contain logarithmic functions of the separation distances, in addition to the usual rational functions.

Note that for exactly the same reasons one could question the field assignment of the height 1 variable itself, despite the fact that the field ϕ_0 has successfully passed so many tests. Although one cannot bring two heights 1 side by side, one can bring them fairly close to each other, as in the last two clusters of Fig. 1 (or Table I), in fact close enough so as not to lose the OPE argument. But then the fields associated with the two clusters S_{12} and S_{13} must be contained in the fusion of two heights 1, i.e., in the fusion (7.4), which we know is not the case.

Perhaps sandpile models are so special that one should reject the fusion altogether, on the basis that height variables have hair, because a particular height imposes restrictions on what can stand close to it. For example, a height 1 forces all its neighbors to be higher than or equal to 2, and a height 2 does not allow two of its neighbors to have a height 1. This might explain the inconsistency noticed above, but at the same time it denies the very possibility of a field assignment. We believe that this issue should be clarified.

VIII. CONCLUSION

The power of conformal field theory could bring a much better understanding of the sandpile model, if some of its observables could be identified with conformal fields. This is a nontrivial task even for the height variables, which are probably the easiest variables to account for in a field-theoretic setting. In addition, and in order to strengthen the connection with a field theory, the neighborhood of the critical point should be investigated. In this article, we have taken the first steps toward a systematic study of this relationship, at and off criticality.

The off-critical extension of the sandpile that we considered is defined by allowing dissipation, i.e., loss of sand each time a site topples. The dissipation rate is controlled by a parameter $t \geqslant 0$ and corresponds to a relevant perturbation of the usual Abelian undirected sandpile model.

We have examined multisite probabilities for the simplest local cluster variables in the off-critical sandpile model. By explicit calculations, we have shown that their scaling form can be fully reproduced by a free field theory of massive Grassmanian scalars. In the massless, critical limit, this theory is a logarithmic conformal field theory with central charge $c=-2$. The local fields assigned to the various cluster variables, however, all belong to a nonlogarithmic bosonic sector. The massive regime, with a mass $M \sim \sqrt{t}$ directly related to the perturbing parameter, corresponds to a thermal perturbation of the conformal theory, i.e., a mass term specified by a logarithmic field.

We have determined the field assignment for the 14 cluster variables pictured in Fig. 1, and checked their consistency against the correlation functions. On the other hand, at the critical point, we have noted a disagreement between these assignments and the naive fusion rules of the conformal theory.

We do not claim that all features of the sandpile models will be comprehensible within a field theory, but some of them definitely are. In this respect, other issues than the height variables can be raised: boundary phenomena against boundary conformal field theory, the question of the modular noninvariance on a torus (with leaking sites), etc. Also, the relevance and the role of logarithmic fields and twist fields in the $c=-2$ logarithmic conformal field theory must be further examined.

ACKNOWLEDGMENTS

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APPENDIX A: GREEN FUNCTIONS

In this Appendix, we collect a number of expressions we have used for the computations of correlations in the sandpile model.

The central object here is the Green function G of the massive discrete Laplacian on $\mathbb{Z}^2$, which is the solution of the Poisson equation $\Delta G=1$, with Δ being the finite difference operator given in Eq. (3.1). The solution is easily obtained by Fourier transform:

$$\begin{aligned} G(m,n) &= G((m',n'),(m+m',n+n')) \\ &= \int\int_0^{2\pi} \frac{d^2k}{4\pi^2} \frac{e^{ik_1m+ik_2n}}{x-2\cos k_1-2\cos k_2}, \\ & \quad (m,n),(m',n') \in \mathbb{Z}^2. \end{aligned} \tag{A1}$$

As explained in the text, values of G are needed at points that are either close to the origin, or else very far from the origin, and in this last case we have restricted ourselves to points close to a principal or a diagonal axis. We treat these three cases in turn.

1. The Green function at points close to the origin

By using the invariance of G under the reflection symmetries of the lattice and its defining equation $\Delta G=1$, the Green function can be given everywhere in terms of its values on a diagonal. By a suitable change of variables and one integration [27], the diagonal values can be recast into

$$G(m,m)=\frac{(-1)^m}{\pi x}\int_0^{\pi} dt \frac{\cos 2mt}{\sqrt{1-(16/x^2)\sin^2 t}}. \tag{A2}$$

This can be resolved in terms of the complete elliptic functions [28]

$$K(p)=\int_0^{\pi/2}dt\,\frac{1}{\sqrt{1-p^2\sin^2 t}}$$
$$=\left[1+\frac{q^2}{4}+\frac{9q^4}{64}+\cdots\right]\log\left(\frac{4}{q}\right)-\left[\frac{q^2}{4}+\frac{21q^4}{128}+\cdots\right], \tag{A3}$$

$$E(p)=\int_0^{\pi/2}dt\,\sqrt{1-p^2\sin^2 t}$$
$$=\left[1-\frac{q^2}{4}-\frac{13q^4}{64}-\cdots\right]+\left[\frac{q^2}{2}+\frac{3q^4}{16}+\cdots\right]\log\left(\frac{4}{q}\right), \tag{A4}$$

where $q=\sqrt{1-p^2}$, and where the expansions are given for $p\lesssim 1$ close to 1.

In terms of our perturbing parameter $t=x-4$, one finds, for instance,

$$G(0,0)=\frac{2}{\pi x}K\left(\frac{4}{x}\right), \tag{A5}$$

$$G(1,1)=\frac{1}{4\pi x}\left\{(x^2-8)K\left(\frac{4}{x}\right)-x^2\,E\left(\frac{4}{x}\right)\right\}, \tag{A6}$$

$$G(2,2)=\frac{1}{24\pi x}\left\{(x^4-16x^2+48)K\left(\frac{4}{x}\right)-x^2(x^2-8)E\left(\frac{4}{x}\right)\right\}, \tag{A7}$$

$$G(3,3)=\frac{1}{120\pi x}\left\{(x^6-24x^4+158x^2-240)K\left(\frac{4}{x}\right)\right.$$
$$\left.-x^2(x^4-16x^2+46)E\left(\frac{4}{x}\right)\right\}, \tag{A8}$$

which can then be expanded around $x=4$ by using Eqs. (A3) and (A4). They all have the same logarithmic singularity at $x=4$ as $G(0,0)$, so that the differences $G(m,n)-G(0,0)$ remain finite when $x\to 4$. In particular, the critical limit of the subtracted diagonal Green function is simply [27]

$$\lim_{x\to 4}[G(m,m)-G(0,0)]=-\frac{1}{\pi}\sum_{k=1}^{m}\frac{1}{2k-1}. \tag{A9}$$

2. The Green function on the far diagonal

For m large, the use of elliptic functions is impractical to extract the asymptotic behavior in m. Making the change of variables $z=e^{it}$, the formula (A2) becomes an integral over a contour that can be deformed to enclose the cut lying between the two roots $\pm u$ of the denominator, with $u=x/4-\sqrt{x^2/16-1}$. This yields

$$G(m,m)=-\frac{1}{2\pi}\int_{-u}^{u}dz\,\frac{z^{2m}}{\sqrt{(z^2-u^2)(z^2-1/u^2)}}. \tag{A10}$$

The asymptotic expansion of this kind of integral was studied in [29], from which one finds, using their notation,

$$G(m,m)=\sqrt{\frac{u}{x(1-u^2)}}\frac{1}{\sqrt{2\pi m}}u^{2m}\left\{1+\frac{\widetilde{A}_{1>}}{8m}+\frac{3}{64}\frac{\widetilde{A}_{2>}-\frac{5}{6}}{m^2}\right.$$
$$\left.+\frac{15}{512}\frac{\widetilde{A}_{3>}-\frac{7}{6}\widetilde{A}_{1>}}{m^3}+\cdots\right\}, \tag{A11}$$

where the coefficients $\widetilde{A}_{n>}$ are defined from the generating function

$$\widetilde{A}_{>}(z)=\frac{1}{\sqrt{[1+(1-u^2)/(1+u^2)z][1+(1+u^2)/(1-u^2)z]}}$$
$$=\sum_{n=0}^{\infty}\widetilde{A}_{n>}z^n, \tag{A12}$$

and are thus themselves infinite (Laurent) series in u^2, and hence in $\sqrt{t}=\sqrt{x-4}$. It is not difficult to show that these coefficients start off like

$$\widetilde{A}_{n>}=(-1)^n\frac{(2n-1)!!}{(2n)!!}\left(\frac{2}{t}\right)^{n/2}+\mathcal{O}(t^{-n/2+1}), \tag{A13}$$

with the consequence that the m^{-n} term in Eq. (A11) takes the form

$$\frac{(2n-1)!!}{8^n}\frac{\widetilde{A}_{n>}+\cdots}{m^n}=\frac{(-1)^n}{n!}\left(\frac{(2n-1)!!}{2^n}\right)^2$$
$$\times\frac{1}{(2\sqrt{2t}m)^n}[1+(\text{series in }t)], \tag{A14}$$

that is, a first term that has the scaling form times corrections in t, independent of the distance m.

By combining the previous expansion with that of the prefactor of Eq. (A11),

$$\sqrt{\frac{u}{x(1-u^2)}}\frac{1}{\sqrt{2\pi m}}u^{2m}$$
$$=\left(\frac{1}{8\pi\sqrt{2t}m}\right)^{1/2}e^{-m\sqrt{2t}+m\sqrt{2t^3}/48+\cdots}[1+(\text{series in }t)], \tag{A15}$$

one eventually finds that the Green function can be written as

$$G(m,m)=\{D_0(m\sqrt{2t})+t\,D_2(m\sqrt{2t})+t^2\,D_4(m\sqrt{2t})$$
$$+\cdots\}e^{m\sqrt{2t^3}/48+\cdots}, \tag{A16}$$

where all functions D_i depend on the single scaling variable $m\sqrt{2t}$ (the square root of 2 has to be included, since the distance from the origin is $\sqrt{2}m$). Moreover, from Eq. (A14), the first function D_0 is explicitly given as

$$D_0(z)=\frac{1}{\sqrt{8\pi z}}e^{-z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{(2n-1)!!}{2^n}\right)^2\frac{1}{(2z)^n}=\frac{1}{2\pi}K_0(z), \tag{A17}$$

a modified Bessel function. This is to be expected and confirms that the scaling limit of the Green function is indeed equal to $(1/2\pi)K_0(Mr)$, the propagator of a massive scalar.

For calculations in the ASM model, one still needs the Green functions at points close to the diagonal. The Poisson equation is not sufficient, because it would require the knowledge of the Green function all the way down to the horizontal axis, but a simple ansatz similar to Eq. (A16) leads to the following expressions, valid for $0\leqslant k\ll m$:

$$G(m,m+k)=\Bigg\{D_0(z)+k\,D_0'(z)\,\sqrt{\frac{t}{2}}+\left[D_2(z)+\frac{k^2}{4}D_0(z)\right]t+\left[\frac{k}{96}D_0(z)+\frac{k^3}{8}D_0'(z)-\frac{k^3}{12}D_0'''(z)+\frac{k}{2}D_2'(z)\right]\sqrt{2t^3}+\cdots\Bigg\}e^{zt/48+\cdots}, \tag{A18}$$

where $z=m\sqrt{2t}$ is the scaled distance. At the order where all the calculations have been performed, the terms shown in the previous expression are all that is needed.

The critical limit of the above expansions is more conveniently computed from Eq. (A9) by using the asymptotic expansion of the ψ function [28], or from the integral (A2). The result is

$$\lim_{x\to4}[G(m,m)-G(0,0)]=-\frac{1}{2\pi}\log m-\frac{1}{\pi}\left(\frac{\gamma}{2}+\log 2\right)-\frac{1}{48\,\pi\,m^2}+\frac{7}{1920\,\pi\,m^4}-\frac{31}{16\,128\,\pi\,m^6}+\cdots, \tag{A19}$$

with $\gamma=0.577\,21\ldots$ the Euler constant.

3. The Green function on the far principal axis

The calculations can be repeated on a principal axis. The integration of Eq. (A1) over k_2 followed by the change of variable $z=e^{ik_1}$ gives $G(m,0)$ as a contour integral over the unit circle. It can again be deformed to encircle the branch cut joining the two roots $v<u$ of the denominator that lie inside the unit circle, yielding

$$G(m,0)=\frac{1}{4\pi}\int_0^{2\pi}dk_1\,\frac{e^{ik_1m}}{\sqrt{(x/2-\cos k_1)^2-1}}=\frac{1}{i\pi}\int_v^u dz\,\frac{z^m}{\sqrt{(z-u)(z-1/u)(z-v)(z-1/v)}}, \tag{A20}$$

with $u<1$ and $v<1$ the two roots of $z^2-(x-2)z+1$ and $z^2-(x+2)z+1$, respectively, that is,

$$u=\frac{1}{2}[x-2-\sqrt{x(x-4)}],\qquad v=\frac{1}{2}[x+2-\sqrt{x(x+4)}]. \tag{A21}$$

The asymptotic behavior of this integral for large m can be found again in [29], with the result

$$G(m,0)=\sqrt{\frac{u}{4\pi(1-u^2)}}\frac{u^m}{\sqrt{m}}\Bigg\{1+\frac{A_{1>}}{4m}+\frac{3}{16}\frac{A_{2>}-\frac{5}{6}}{m^2}+\frac{15}{64}\frac{A_{3>}-\frac{7}{6}A_{1>}}{m^3}+\cdots\Bigg\}. \tag{A22}$$

The series within the curly brackets is similar to that of the previous subsection, with, however, $m/2$ substituted for m, and with the coefficients $A_{n>}$ as defined in [29], namely, by

$$A_>(z)=\frac{1}{\sqrt{[1+(1+uv)/(1-uv)z][1-(1+v/u)/(1-v/u)z][1+(1+u^2)/(1-u^2)z]}}=\sum_{n=0}^{\infty}A_{n>}z^n \tag{A23}$$

(the coefficients $\tilde{A}_{n>}$ used above correspond to the present $A_{n>}$ upon the identification $v=-u$).

The usual expansion around $x=4$ now yields

$$G(m,0)=\{P_0(m\sqrt{t})+t\,P_2(m\sqrt{t})+t^2\,P_4(m\sqrt{t})+\cdots\}e^{m\sqrt{t^3}/24+\cdots}, \tag{A24}$$

with $P_0(z)=(1/2\pi)K_0(z)$ as before.

For points close to the horizontal axis, one finds from the Poisson equation the expansions for $k \ll m$:

$$G(m,k)=\left\{ P_0(z)+\left[P_2(z)+\frac{k^2}{2}[P_0(z)-P_0''(z)]\right] t + \cdots \right\} e^{zt/24+\cdots}, \tag{A25}$$

with $z=m\sqrt{t}$ the scaled distance. The ASM calculations also need the values of $G(m\pm\ell,k)$ for small ℓ, and those can easily be obtained by expanding the previous result, yielding a Taylor series in $\sqrt{t}$.

The critical asymptotic expansion of $G(m,0)$ can also be computed from Eq. (A20). One has

$$[G(m,0)-G(m+1,0)]|_{x=4}=\frac{1}{4\pi}\int_0^{2\pi} dk_1 e^{ik_1 m}F(k_1), \tag{A26}$$

where $F(x)=(1-e^{ix})/\sqrt{(2-\cos x)^2-1}$. A repeated use of integration by parts then leads to

$$\begin{aligned}[G(m,0)-G(m+1,0)]|_{x=4}&=-\frac{1}{4\pi}\sum_{k\geqslant 1}\frac{(-1)^k}{(im)^k}[d_x^k F]_0^{2\pi}\\ &=\frac{1}{2\pi}\left\{\frac{1}{m}-\frac{1}{2m^2}+\frac{1}{2m^3}-\frac{1}{2m^4} + \cdots\right\},\end{aligned} \tag{A27}$$

from which one deduces the subtracted Green function itself as

$$\begin{aligned}\lim_{x\to 4}[G(m,0)-G(0,0)]&=-\frac{1}{2\pi}\log m-\frac{1}{\pi}\left(\frac{\gamma}{2}+\frac{3}{4}\log 2\right)\\ &\quad+\frac{1}{24\,\pi\,m^2}+\frac{43}{480\,\pi\,m^4}\\ &\quad+\frac{949}{2016\,\pi\,m^6}+\cdots.\end{aligned} \tag{A28}$$

APPENDIX B: ABOUT THE SINK SITE

The evacuation of sand is a crucial ingredient in the self-organized criticality of the sandpile models. In order for its dynamics to be well defined—any unstable configuration relaxes to a stable one—each site should be pathwise connected to a sink, where the sand goes that falls off the pile. The sink is usually omitted in all discussions, perhaps because in the ordinary ASM only the boundary sites are connected to the sink, and the large volume limit takes them to infinity. In the massive ASM, however, each site is connected to the sink. One might thus worry about its possible role in actual computations.

We show here that the sink has in fact no effect at all and can be omitted completely, be it in the usual or the massive ASM. The argument is simple and worth being made explicitly.

The discrete dynamics of the ASM recalled in the Introduction uses a toppling matrix that ignores the sink site s. To include it, one simply defines an extended toppling matrix $\Delta_{\rm e}$ by adding to Δ a row and a column:

$$\Delta_{\rm e}=\begin{pmatrix} 1 & 0 \\ V & \Delta \end{pmatrix}. \tag{B1}$$

The diagonal entry $(\Delta_{\rm e})_{\rm s,s}$ is set equal to 1, in order to freeze the height of the sink site. The rest of the first row is equal to 0, since the sink has no connection to the sites of the pile. On the other hand, the first column is not zero: $V_{i,\rm s}=-n_i$ if n_i grains of sand fall off the pile when site i topples. The number n_i is equal to $n_i=-\Sigma_j \Delta_{i,j}$, so that all row sums of $\Delta_{\rm e}$, except the first one, are zero. In the usual ASM, $V_{i,\rm s}$ is nonzero for the boundary sites only, whereas in the massive model all components are nonzero, with $V_{i,\rm s}=4-x$ for all bulk sites. The formula for the number of recurrent configurations remains valid, with obviously the same result, $\det \Delta_{\rm e}=\det \Delta$.

The same method for computing probabilities and correlations of weakly allowed clusters works as before. One uses an extended B matrix specifying the way the ASM needs be modified:

$$B_{\rm e}=\begin{pmatrix} 0 & 0 \\ W & B \end{pmatrix}. \tag{B2}$$

The first row is clearly always zero, but the first column can be nonzero, depending on the modifications. For those called the "least economical" ones in Sec. II, in which one cuts the cluster off the rest of the lattice, each site i of the cluster is left with a sole connection to the sink, so one sets $W_i=5-x$.

The probability of a cluster variable S is given by the usual formula, which as before reduces to a finite determinant

$$\begin{aligned}\text{Prob}(S)&=\frac{\det(\Delta_{\rm e}+B_{\rm e})}{\det \Delta_{\rm e}}=\det(\mathbb{I}+\Delta_{\rm e}^{-1}B_{\rm e})\\ &=\det(\mathbb{I}+\Delta_{\rm e}^{-1}B_{\rm e})|_{M_S\cup\{\rm s\}}.\end{aligned} \tag{B3}$$

However, the restriction to $M_S\cup\{\rm s\}$ of $\Delta_{\rm e}^{-1}B_{\rm e}$ is particularly simple,

$$\Delta_{\rm e}^{-1}B_{\rm e}=\begin{pmatrix} 1 & 0 \\ -\Delta^{-1}V & \Delta^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ W & B \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ \Delta^{-1}W & \Delta^{-1}B \end{pmatrix}, \tag{B4}$$

and manifestly leads to the usual result, with no sink,

$$\text{Prob}(S)=\det(\mathbb{I}+\Delta^{-1}B)|_{M_S}. \tag{B5}$$

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[2] P. Bak, *How Nature Works—the Science of Self-Organized Criticality* (Oxford University Press, Oxford, 1997).
[3] H. J. Jensen, *Self-Organized Criticality* (Cambridge University Press, Cambridge, 1998).
[4] D. Dhar, Physica A **263**, 4 (1999).
[5] D. Dhar, e-print cond-mat/9909009.
[6] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[7] S.N. Majumdar and D. Dhar, Physica A **185**, 129 (1992).
[8] S.N. Majumdar and D. Dhar, J. Phys. A **24**, L357 (1991).
[9] E.V. Ivashkevich and V.B. Priezzhev, Physica A **254**, 97 (1998).
[10] J.G. Brankov, E.V. Ivashkevich, and V.B. Priezzhev, J. Phys. I **3**, 1729 (1993).
[11] E.V. Ivashkevich, J. Phys. A **27**, 3643 (1994).
[12] V.B. Priezzhev, J. Stat. Phys. **74**, 955 (1994).
[13] S.N. Majumdar and D. Dhar, J. Phys. A **23**, 4333 (1990).
[14] C.J. Pérez, Á. Corral, A. Díaz-Guilera, K. Christensen, and A. Arenas, Int. J. Mod. Phys. B **10**, 1111 (1996).
[15] S.S. Manna, L.B. Kiss, and J. Kertész, J. Stat. Phys. **61**, 923 (1990).
[16] T. Tsuchiya and M. Katori, Phys. Rev. E **61**, 1183 (2000).
[17] A. Vázquez, Phys. Rev. E **62**, 7797 (2000).
[18] H. Saleur, Nucl. Phys. B **382**, 486 (1992).
[19] V. Gurarie, Nucl. Phys. B **410**, 535 (1993).
[20] H. Kausch, e-print hep-th/9510149.
[21] M.R. Gaberdiel and H. Kausch, Phys. Lett. B **386**, 131 (1996).
[22] F. Rohsiepe, e-print hep-th/9611160.
[23] V. Gurarie, M. Flohr, and C. Nayak, Nucl. Phys. B **498**, 513 (1997).
[24] M.R. Gaberdiel and H. Kausch, Nucl. Phys. B **538**, 631 (1999).
[25] H. Kausch, Nucl. Phys. B **583**, 513 (2000).
[26] E.V. Ivashkevich, J. Phys. A **32**, 1691 (1999).
[27] F. Spitzer, *Principles of Random Walk*, 2nd ed., Graduate Texts in Mathematics Vol. 34 (Springer, New York, 1976).
[28] J. Spanier and K.B. Oldham, *An Atlas of Functions* (Hemisphere, Washington, 1987).
[29] B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard University Press, Cambridge, MA, 1973), p. 254.