# Rigidity of random networks of stiff fibers in the low-density limit

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Rigidity percolation is analyzed in two-dimensional random networks of stiff fibers. As fibers are randomly added to the system there exists a density threshold $q=q_{\min}$ above which a rigid stress-bearing percolation cluster appears. This threshold is found to be above the connectivity percolation threshold $q=q_c$ such that $q_{\min}=(1.1698\pm0.0004)q_c$. The transition is found to be continuous, and in the universality class of the two-dimensional central-force rigidity percolation on lattices. At percolation threshold the rigid backbone of the percolating cluster was found to break into rigid clusters, whose number diverges in the limit of infinite system size, when a critical bond is removed. The scaling with system size of the average size of these clusters was found to give a new scaling exponent $\delta=1.61\pm0.04$.

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## I. INTRODUCTION

Scalar percolation [1] is a simple model describing the transfer of a scalar conserved quantity, e.g., an electric charge, across a randomly diluted system. In two dimensions the geometric exponents of connectivity scalar percolation are known exactly [1,2]. Elastic percolation is not in general equivalent to scalar percolation [3], and the similarities and differences between the scalar percolation and the two-dimensional central-force rigidity transition have been studied extensively [4–16]. The results obtained for the latter can be divided roughly into three different scenarios. The rigidity transition can be discontinuous, i.e., of first order [10,12–14,16]. This is true for the square lattice with randomly diluted diagonals, for Cayley trees, and for the random-bond model. The rigidity transition can be continuous and belong to a different universality class than that of the two-dimensional (2D) scalar percolation [9,11,14–16]. This is true for generic bond diluted lattices and for random fiber networks. Here the decisive quantities are the central forces and the multiple connectivity of the structures. The transition can also be continuous and belong to the 2D scalar-percolation universality class. In this case angular forces are present, only singly connected paths are required for rigidity, and hence the geometric properties of the elastic backbone are exactly the same as those of the scalar percolation problem [17,18]. What sets these three different cases apart? In the case of 2D scalar percolation the percolation cluster is broken into two separate parts once a critical connection is removed. Removing a critical connection in Cayley trees leads, on the other hand, to the so-called house of cards effect, and the percolation cluster is broken into infinitely many clusters [12–14,16]. In the case of diluted lattices and random networks, it is diffucult to analyze the effects on the percolation cluster of removing a critical connection, but one can easily construct situations in which the rigid cluster is broken into three, four, or an even higher number of rigid clusters once a critical connection is removed. There is also another related question: can one type of behavior change to another (e.g., a discontinuous transition to a continuous one, 2D central-force rigidity percolation to 2D scalar percolation) when the model is continuously interpolated between two known models. This has been answered in the case of a model which can vary between a braced square lattice and a triangular lattice [16]. It was shown that the continuous 2D central-force rigidity percolation applies when the model is even slightly deviated from the braced square lattice. The properties of the percolation cluster and different limits of the models introduced seem thus a possible way to probe the properties that differentiate the three basic cases.

Two-dimensional random network is a geometrical structure. It can be used, e.g., to model planar structures that are composed of randomly positioned thin linelike objects (e.g., fibers in a sheet of paper). The statistical properties of this kind of 2D random networks are well known [19]. In our model the 2D fiber network is generated by randomly placing one-dimensional (1D) objects of equal length on a plane so that both the $x$ and $y$ coordinates and the orientation angles of the fibers are taken from a uniform distribution. We use periodic boundary conditions in the $y$ direction, and a box of linear size $L$ plus one fiber length in the $x$ direction to minimize the boundary effects and to keep the fiber density unchanged on the boundaries. A typical 2D random network is shown in Fig. 1.

The model is discussed in more detail in [15,20]. The rigidity properties of this type of structure are discussed in [15] in the case when the fibers are not totally stiff (i.e., they

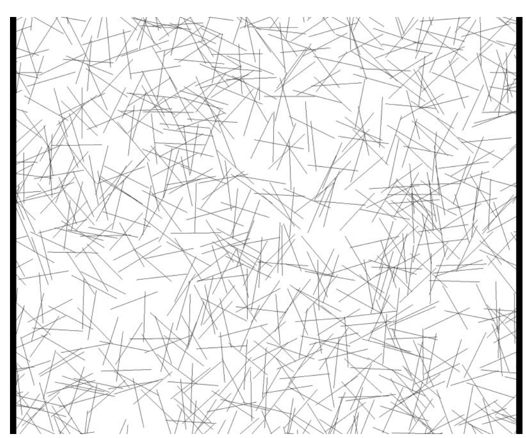

FIG. 1. A typical 2D random network with density $q=2q_c$ (see the text for the definitions of $q$ and $q_c$).

 

can bend at crossing points) nor are there connections between the directions of segments across crossing points (i.e., there are no angular forces between adjacent segments). At low densities, adding of angular constraints (e.g., upon drying in the case of paper composed of wood fibers) or stiffening of fibers in this kind of structure leads to a continuous rigidity transition in the central-force rigidity universality class. At very low densities almost all fibers need to be stiff to achieve rigidity, i.e., to resist bending. It might thus be assumed that in this case the same rigidity transition would be found by having only stiff fibers in the system right from the beginning. It is, however, unclear whether connectivity percolation of stiff fibers occurs at the same point as the rigidity percolation. This question is further motivated by noticing that the elasticity properties of random networks are different for stiff-fiber and for fixed-angle (angular forces are present, angles between crossing fibers are fixed) networks [21]. If the angles are not fixed, the network has far lower elastic constants at a given low density. The question then is whether the rigidity transition threshold is also the same for stiff-fiber and for fixed-angle networks. There is also the question of whether or not stiff fibers require multiple connectivity for the rigidity percolation. Constraint counting seems to indicate different percolation thresholds for the two cases. We will show below that the percolation thresholds are indeed different and that the connectivity and rigidity percolations for stiff fibers definitely belong to different universality classes.

A system is rigid if it cannot be deformed without cost of energy, i.e., if any small deformation of the system has a nonzero response. A system is nonrigid or floppy if it can be continuously deformed without loss of energy. The number of (linearly) independent motions that do not cost energy is called the number of floppy modes of the system.

We use the idea of generic rigidity [9,10] to study the rigidity transition in the low-density fiber network. A random network is inherently generic because the random construction takes care of the geometrical singularities, i.e., the probability of geometrical singularities is zero. Replacing the fiber segments in the network by Hookean springs will lead to a situation where the network is never rigid [20]. Additional constraints are then required [15] for rigidity. We consider here a model in which the crossing fibers are connected in such a way that the positions of the crossing points are fixed but the angles between the fibers can vary. The fibers are stiff so that they cannot be bent without cost of energy. This means that, unlike in [15], there is only one independent angle at each crossing point. There is a transition from a floppy to a rigid structure in this model when the density of fibers is increased, and we analyze the correlation length and the fractal dimension of the system at this transition. The transition density is found using finite-size scaling.

## II. METHODS

In the analysis of rigidity in 2D random fiber networks we use here a matching algorithm [22], more specifically the pebble game by Jacobs and Thorpe [9]. This algorithm maps the overconstrained areas and determines the number of floppy modes in the system. It basically represents the degrees of freedom in a system with pebbles. Once a degree of freedom is bound, a pebble is bound, and hence one can keep track of rigidity in a recursive fashion.

There are several ways by which rigidity can be introduced in a random spring network. Of the possible mechanisms one should choose those which are relevant for physical applications. We use stiff fibers whose positions are fixed by the crossing points between the fibers. A stiff fiber cannot bend but can rotate if there is only one crossing point. An alternative strategy would be to weld some crossing points, i.e., to fix the angles between the crossing fibers. The first of these strategies corresponds to a situation in which the cohesion inside (or equivalently on the surface of) the fiber is larger than the forces between the fibers. The welding strategy corresponds to a situation in which two bonded fibers cannot move relative to one another but can still bend. The formation of a paper web in the paper-making process is a combination of these two mechanisms but typically the orientational (i.e., welding type) mechanism is dominant. Other random networks could have a stronger tendency for stiffness. It is known that in (completely) welded networks the connectivity and rigidity transition densities are the same; in this work we show that for stiff fibers this is not the case.

We generate a random network by randomly placing $N_f$ fibers of length $l$ in an area of $L\times L$. We use as the control parameter the density of fibers $q=N_f/L^2$, and denote by $q_c$ the density at the connectivity-percolation threshold [23].

We first map all the crossing points in a network we know to be rigid with high probability for stiff fibers [15]. This can be accomplished, e.g., by using networks of density $q=2q_c$. We then begin to add fibers in a random fashion, and use the pebble-game algorithm to check the rigidity of the system. Adding a fiber means adding a constraint between all neighboring crossing points on that fiber (connectedness) and between all second-nearest crossing points on that fiber (stiffness). We connect the left and the right sides of the network to rigid bars and add a fictitious bond between these bars [24]. Once this fictitious bond becomes redundant, i.e., overconstrained, we know to have created a rigid (stress-bearing) structure that spans over the system. We then record the density of fibers, and from that get the fractal dimension of the stress-bearing backbone at the transition point, and indirectly the correlation length exponent $\nu$.

We have checked that the concepts of generic rigidity apply. The strategy we use to make the fibers stiff [i.e., adding second-nearest-neighbor (SNN) springs] would leave the network shaky [15] (i.e., not first-order rigid). If we use generic fibers in which the crossing points are deviated slightly from their original positions then generic rigidity models apply. A different way to view this method is to say that these SNN bonds just simulate stiffness, because they will always produce the same number of degrees of freedom as stiff fibers.

## III. RESULTS

To check rigidity one basically needs to check whether the number of independent constraints exceeds the number of degrees of freedom in the system. If one ignores the fact that constraints are not necessarily independent of one another, the calculations simplify significantly. This approach is due to Maxwell [25] and is called Maxwell counting.

Consider first the connectivity percolation in which the number of degrees of freedom per fiber is one, i.e., fiber is either connected or not. Each crossing point binds now one degree of freedom, i.e., connects two fibers. The density of fibers is $q=N_f/A$, where $N_f$ is the number of fibers and $A$ is the area of the system (notice that area here is dimensionless and measured as a function of the unit area $A_u=l\times l$, where $l$ the length of a fiber). The number of crossing points is $N_c=N_f q/\pi$ [15,23]. We get an estimate for the transition threshold $q_c$ by equalling the number of constraints with the number of degrees of freedom,

$$N_f=N_c=\frac{N_f q_c}{\pi}\rightarrow q_c=\pi. \qquad (1)$$

This estimate underestimates $q_c$ because it supposes that all constraints are independent, and overestimates it because it supposes that all the degrees of freedom need to be bound for spanning a connected network. Which of these effects is stronger depends on the topology of the problem in question, and for random networks one finds that in fact $q_c=5.71$ [23].

For rigidity percolation in 2D the number of degrees of freedom per fiber is three, i.e., the fiber has two translational and one rotational degree of freedom. Each crossing between fibers binds two degrees of freedom (the relative translations). Hence we get

$$3N_f=2N_c=\frac{2N_f q_{\min}}{\pi}\rightarrow q_{\min}=\frac{3}{2}\pi, \qquad (2)$$

where $q_{\min}$ is the rigidity transition threshold. Notice in particular that this estimate is $\frac{3}{2}$ times the similar estimate for the connectivity percolation threshold, i.e., $q_{\min}=1.5q_c$.

We generated 100 to 1000 networks of sizes $10\times 10$ to $300\times 300$. To find the transition threshold and the exponents associated with the rigidity transition, we first determined the probability of having a spanning rigid cluster as a function of density and system size. This was done by adding stiff fibers at random and by checking the redundancy of the fictitious bound that connected the left and the right sides of the network. The probability of finding a spanning cluster at given $q$ and $L$ was approximated by the number of spanning configurations divided by the total number of configurations. For each network we used 1000 different ways to add the stiff fibers.

In Fig. 2 we plot the probability $\pi(q,L)$ of finding a spanning rigid cluster as a function of density $q$ and linear size $L$. All these curves can be collapsed to one curve via [1,11]

$$\pi(q-q_{\min},L)=\phi([q-q_{\min}]L^{1/\nu}), \qquad (3)$$

in which $\nu$ is the correlation length exponent. We look for the best data collapse when $q_{\min}$ and $\nu$ are let to vary, and find in this way that $q_{\min}=1.168q_c$ and $1/\nu=0.85$, which gives $\nu=1.18$.

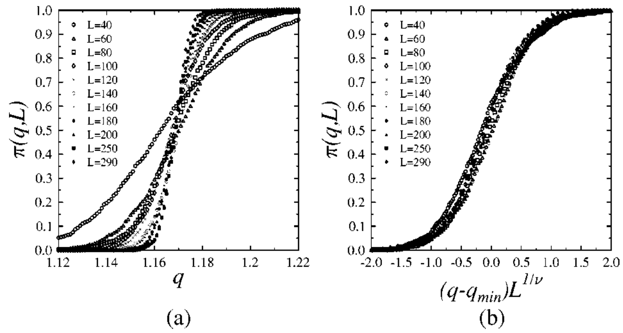


FIG. 2. (a) Probability of finding a spanning rigid cluster in a network of size $L\times L$ and of density $q_0$. (b) Probability of finding a spanning cluster plotted as a function of the scaled variable $(q-q_{\min})L^{1/\nu}$. We use here $q_{\min}=1.168$ and $1/\nu=0.85$, which give the best data collapse. Data collapse is satisfied within the estimated error.

We can also use finite-size scaling to find another estimate for the correlation length exponent. This can be done when the (linear) size of the system is less than the correlation length $\xi$. This happens close to the transition threshold. Since $\pi=\phi([q-q_{\min}]L^{1/\nu})$ for $\xi>L\gg 1$, i.e., for $q$ close to $q_{\min}$, we get

$$d\pi/dq=L^{1/\nu}\phi'([q-q_{\min}]L^{1/\nu}). \qquad (4)$$

So close to $q=q_{\min}$, $d\pi/dq$ diverges as $L^{1/\nu}$. This means that the standard deviation

$$\Delta q_{\min}=\langle\sqrt{(q_{est}-q_{av})^2}\rangle \qquad (5)$$

scales as $L^{-1/\nu}$. Here $q_{est}$ is the density of spanning cluster for each of the configurations and $q_{av}$ is the average of these densities. We plot the standard deviation in Fig. 3. Fitting a straight line to this log-log plot gives $\nu=1.17\pm 0.02$, which is definitely in the universality class of the 2D central-force rigidity percolation [11,14,15]. The previous data collapse confirms this observation.

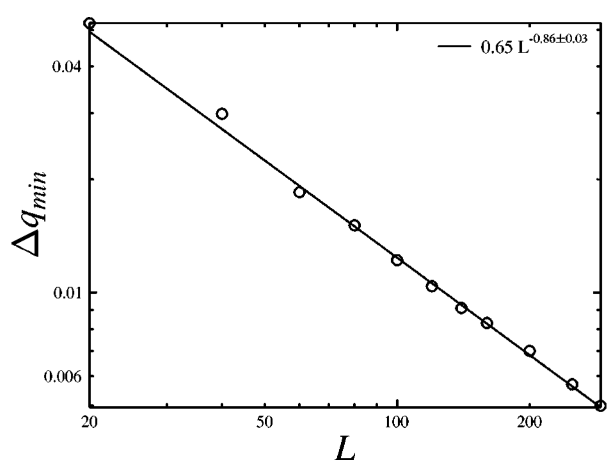


FIG. 3. The standard divergence of $q_{\min}$, i.e., $\Delta q_{\min}$, plotted as a function of system size $L$. The slope of the line is $-1/\nu$ and will hence give the $\nu$ exponent.

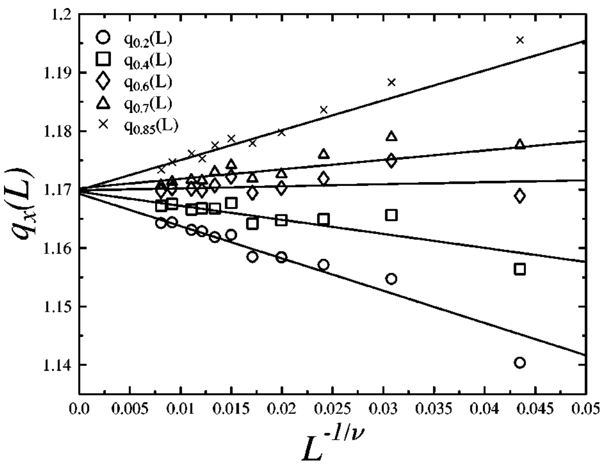


FIG. 4. Estimation of the transition threshold. The fitted lines intersect for $L\rightarrow\infty$ at $q_{\min}=1.1698\pm0.0004$.

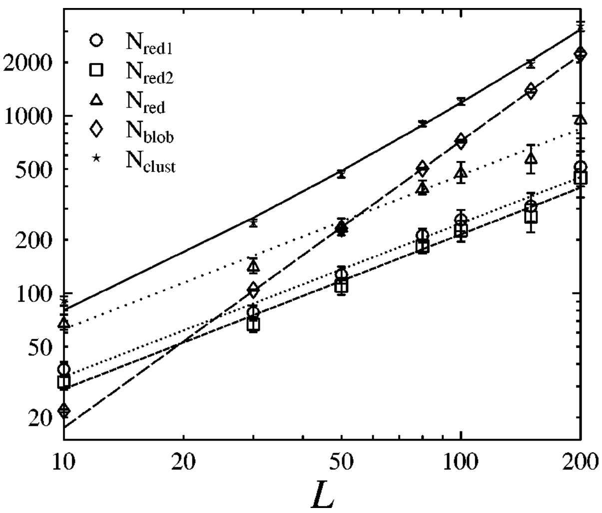


FIG. 5. Breaking of the rigid backbone when one critical bond is removed. The backbone always breaks into $N_{\rm red}+N_{\rm blob}$ stress-bearing clusters, in which $N_{\rm red}$ is the number of critical bonds and $N_{\rm blob}$ the number of overconstrained regions. The circles denote the number of critical nearest neighbor connections, the squares the number of critical second nearest neighbor connections, the triangles the number of critical bonds, the diamonds the number of overconstrained regions, and the asterisks the total number of stress-bearing clusters. The lines are linear fits to the data points, from which we find that $N_{\rm red}\sim L^{0.86\pm0.02}=L^{1/\nu}$ and $N_{\rm blob}\sim L^{1.61\pm0.04}$.

How does one identify $q_{\min}$ from a finite sample? Since the sample is finite there is a finite probability of finding a spanning cluster at any $q>q_0=1/L$. Now that we know the correlation length exponent $\nu$, we can proceed as follows. The transition threshold $q_{\min}$ can be evaluated by plotting $q_x(L)$, where $\pi(q_x(L),L)=x$ for some fixed $x<1$. The $q_x$ can now be called the effective threshold value since at $q_x$, on the average, a fraction $x$ of the networks have a spanning rigid cluster. Expanding now $\pi(q,L)=\phi([q-q_{\min}]L^{1/\nu})$ around $q=q_{\min}$, we find that

$$q_x(L)=\text{const}\cdot L^{-1/\nu}+q_{\min}, \tag{6}$$

where the constant depends on $x$ and is positive for $x>\pi(q_{\min})$ and negative for $x<\pi(q_{\min})$. More exactly this holds only asymptotically for $L\rightarrow\infty$. If $q_x(L)$ is plotted as a function of $L^{-1/\nu}$, we get a series of (asymptotically) straight lines that intersect at $q=q_{\min}$. This plot is shown in Fig. 4, and we find from it that $q_{\min}=(1.1698\pm0.0004)q_c$.

There is one more test that we are able to do, and this is the fractal dimension of the percolation cluster. In the case of rigidity the fractal dimension of the rigid backbone is much easier to find than the fractal dimension of the total percolation cluster (because of the dangling ends) [19]. We map the number of fibers in the backbone at the transition threshold as a function of $L$. This quantity scales like $N_{bb}\sim L^{d_{bb}}$. We find $d_{bb}=1.79\pm0.04$, which also agrees with the central-force universality class.

In view of the possible factors that determine the nature of rigidity transition, we also studied what happens to the rigid backbone when one of the critical bonds (red bonds) is removed. The main interest here was to see into how many rigid clusters the backbone breaks. We first constructed a 2D network that was at the rigidity threshold. The red fibers created at the last addition of a fiber, the fibers that belonged to the rigid backbone, and the red bonds were determined. Then we used the pebble game to create only the rigid backbone without the red bonds. The red bonds were thereafter added so that each time one of them was left out. We kept track of the recognized overconstrained regions that the pebble game mapped, and thereby constructed a list of stress-bearing clusters. Each bond belonged to exactly one rigid cluster. The number of these rigid clusters was determined. All of this was done to networks of sizes $10\times10$ to $200\times200$. When a red bond was removed the backbone was always broken into many rigid clusters. These clusters could be classified as rigid blobs in the backbone (i.e., overconstrained regions) or red bonds. Red bonds form rigid clusters of size one, and the sizes of the rigid blobs increase as the system size is increased. It did not matter which red bond was removed, the rigid backbone always broke into the same number of rigid blobs (the same blobs) and red bonds (the same red bonds). Of course in different configurations different bonds are critical and hence the number of rigid clusters that the backbone breaks into is different. We checked how this number of clusters increases as the system size is increased. We noticed that both the number of blobs and the number of red bonds increases as the linear size of the system raised to some power. The power law for red bonds was $N_{red}\sim L^{1/\nu}=L^{0.86\pm0.04}$ and for rigid blobs $N_{\rm blob}\sim L^{1.61\pm0.04}$. Notice the higher exponent in the latter quantity. The latter power law would also imply that the average size of the rigid blobs which will form the rigid backbone scales just below transition threshold like $L^{d_{bb}-1.61}=L^{0.18\pm0.04}$. The results are shown in Fig. 5.

## IV. CONCLUSIONS

The main motivation for this work was to analyze the rigidity percolation for stiff fibers at low densities. The density at which rigidity percolates in this case must be at least that of connectivity percolation, but is it higher? Our results show that it definitely is higher but only slightly.

In an earlier work [15] we analyzed the rigidity percolation in random networks of nonstiff fibers, in which case the percolation threshold was approached by, e.g., randomly adding next-nearest-neighbor constraints along individual fibers, i.e., by making them stiff. Rigidity percolation was found in this case to be continuous and to belong to the 2D central-force universality class independent of the stiffening mechanism. We therefore expected the rigidity percolation in the random network of stiff fibers, reached by increasing the density of fibers, also to be continuous and in the same universality class. This turned out to be the case. As the transition occurred now at a lower density of fibers, we were in fact able to consider larger systems than before, and found therefore more reliable estimates for the scaling exponents.

We also determined the number of rigid clusters into which the rigid backbone at percolation threshold is broken if one of the critical bonds is removed. This number diverges rapidly with system size and is determined by rigid blobs, whose number increases much faster than the number of rigid clusters formed by just one fiber. So there appears to be in this system a similar house-of-cards effect as in the Cayley tree problem, which, however, displays a first-order rigidity transition.

The scaling with system size of the number of single-fiber clusters was found to be $\sim L^{1/\nu}$, in agreement with previous results [10]. On the other hand, the number of rigid blobs $N_{\rm blob}$ has not been considered before, and its scaling with system size, $N_{\rm blob}\sim L^{1.61}$, defines a new exponent $\delta=1.61$, whose possible relation to the previously determined scaling exponents of 2D central-force rigidity percolation is not known at the moment. Notice also that the results reported here indicate that $q_{\rm min}\approx \nu q_c$, a result which appears a coincidence as we have not found any reason for why the correlation length exponent should determine the transition density.

As for possible future work we only note here that it would be instructive to consider the relation above the rigidity transition of the elastic properties of the system and its topological rigidity properties. If such a relation would exist, it would provide new insight into the formation of macroscopic elasticity. Also, topological tools are quite efficient for large systems, which are difficult to analyze by *ab initio* numerical methods.

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