Nonequilibrium dynamics of random field Ising spin chains: Exact results via real space renormalization group

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The nonequilibrium dynamics of classical random Ising spin chains with nonconserved magnetization are studied using an asymptotically exact real space renormalization group (RSRG). We focus on random field Ising model (RFIM) spin chains with and without a uniform applied field, as well as on Ising spin glass chains in an applied field. For the RFIM we consider a universal regime where the random field and the temperature are both much smaller than the exchange coupling. In this regime, the Imry-Ma length that sets the scale of the equilibrium correlations is large and the coarsening of domains from random initial conditions (e.g., a quench from high temperature) occurs over a wide range of length scales. The two types of domain walls that occur diffuse in opposite random potentials, of the form studied by Sinai, and domain walls annihilate when they meet. Using the RSRG we compute many universal asymptotic properties of both the nonequilibrium dynamics and the equilibrium limit. We find that the configurations of the domain walls converge rapidly toward a set of system-specific time-dependent positions that are *independent of the initial conditions*. Thus the behavior of this nonequilibrium system is pseudodeterministic at long times because of the broad distributions of barriers that occur on the long length scales involved. Specifically, we obtain the time dependence of the energy, the magnetization, and the distribution of domain sizes (found to be statistically independent). The equilibrium limits agree with known exact results. We obtain the exact scaling form of the two-point equal time correlation function $\overline{\langle S_0(t)S_x(t)\rangle}$ and the two-time autocorrelations $\overline{\langle S_0(t')S_0(t)\rangle}$. We also compute the persistence properties of a single spin, of local magnetization, and of domains. The analogous quantities for the $\pm J$ Ising spin glass in an applied field are obtained from the RFIM via a gauge transformation. In addition to these we compute the two-point two-time correlation function $\overline{\langle S_0(t)S_x(t)\rangle\langle S_0(t')S_x(t')\rangle}$ which can in principle be measured by experiments on spin-glass-like systems. The thermal fluctuations are studied and found to be dominated by rare events; in particular all moments of truncated equal time correlations are computed. Physical properties which are typically measured in aging experiments are also studied, focusing on the response to a small magnetic field which is applied after waiting for the system to equilibrate for a time t_w. The nonequilibrium fluctuation-dissipation ratio $X(t,t_w)$ is computed. We find that for $(t-t_w)\sim t_w^{\hat{\alpha}}$ with $\hat{\alpha}<1$, the ratio equal to its equilibrium value $X=1$, although time translational invariance does not hold in this regime. For $t-t_w\sim t_w$ the ratio exhibits an aging regime with a nontrivial $X=X(t/t_w)\neq 1$, but the behavior is markedly different from mean field theory. Finally the distribution of the total magnetization and of the number of domains is computed for large finite size systems. General issues about convergence toward equilibrium and the possibilities of weakly history-dependent evolution in other random systems are discussed.

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I. INTRODUCTION

In many systems, the development of long range order is controlled by the dynamics of domain walls. The coarsening of domain structures evolving toward equilibrium has been studied extensively in pure systems [1,2], but little is known quantitatively about domain growth in the presence of quenched disorder. In random systems, the nonequilibrium dynamics plays an even more important role than in pure systems, and is most relevant for understanding experiments, since many random systems become glassy at low temperatures, with ultraslow dynamics which prevent full thermal equilibrium from being established within the accessible time scales. This slow dynamics is due, at least in cases in which it is qualitatively understood, to the large free energy barriers that must be overcome for order to be established on long length scales. Some of the best studied cases both experimentally and theoretically are a variety of random magnetic systems, particularly spin glasses and random field magnets [3–5]. Both of these systems have engendered a great deal of controversy about their equilibrium behavior, and the resolution of these controversies has been greatly hampered by the inability of experiments to reach equilibrium. One might well argue, however, that the most interesting properties of such random systems are, in fact, not their equilibrium properties, but the nonequilibrium dynamics involved in their Sisyphean struggle to reach equilibrium.

Because of the dominance of the behavior of so many random systems by the interplay between equilibrium and nonequilibrium dynamic effects, it is important to find models with quenched randomness for which solid results about dynamics can be obtained. In particular, one would like to be able to analyze the effects of activated dynamics caused by large barriers, and compare results to predictions which come from either phenomenolgical scaling approaches to the dynamics—often known somewhat misleadingly as ‘‘droplet models’’ [5], or from mean field approaches [6]. Many of the interesting phenomena go under the general name of *aging*—the dependence of measurable properties such as correlations and responses on the history of the system, in particular on how long it has equilibrated: its ‘‘age.’’ For example, one would like to compute quantities which probe the violations of the fluctuation dissipation relations caused by nonequilibrium effects, and compare them with results obtained in mean field models [6]; this was done previously for coarsening of pure models [7–9].

One of the simplest nontrivial random models, and one in which coarsening occurs, is the random field Ising model (RFIM) in one dimension. Although this system does not have true long-range order, for weak randomness it exhibits a wide range of scales over which the dynamics is qualitatively like those of other random systems, especially those like two-dimensional random field magnets and two-dimensional spin glasses, which do not have phase transitions but exhibit much dynamic behavior qualitatively similar to their three-dimensional counterparts which do have phase transitions. In particular, the weakly random one-dimensional (1D) RFIM has a wide range of length scales over which the typical size of ordered domains grows logarithmically with time at low temperatures. In one-dimensional the RFIM is equivalent to a spin glass in an applied magnetic field; this, or some analogous 1D systems, should be conducive to experimental investigation. For a recent review on the RFIM, see e.g. Ref. [10].

The *equilibrium* properties of the 1D RFIM and of the 1D spin glass in a field were extensively studied [10–24]. Several thermodynamic quantities such as the energy, entropy, and magnetization were computed exactly at low temperature for a binary distribution of the randomness in Refs. [13–15] and for continuous distributions in Refs. [22–24]. Results are also available for distributions of bonds with anomalous weight near the origin [16]. The free energy distribution was studied in Ref. [17]. Equilibrium correlation functions are harder to obtain, and only a few *explicit* exact results exist. For the binary distribution some results are presented in Ref. [20]. Certain special limits have been solved, such as the infinite field strength limit which is related to percolation [24], but for, e.g., higher cumulants of averages of truncated correlations, only the general structure has been discussed [21].

In this paper we derive a host of exact results for the *nonequilibrium dynamics* of the RFIM and for the 1D Ising spin glass in a field, and, as a side benefit, also obtain many equilibrium results, some to our knowledge new. Those of our results for equilibrium quantities that can be compared with previously known ones are found to agree with these.

Here we focus on the universal regime of the RFIM in which both the random field and the temperature are much smaller than the exchange coupling, the length and time scales are sufficiently long, and the random field dominates the dynamics. With the random field being weak and the temperature low, the equilbrium correlation length is long and essentially the same as that at zero temperature—the Imry-Ma length. The coarsening of domains starting from initial conditions with only short distance correlations—such as a quench from high temperature—will, after a rapid initial transient, be dominated by the randomness over a wide range of time scales.

The basic tool that we use in this paper is a real space renormalization group (RSRG) method which we have developed recently to obtain exact results for the nonequilibrium dynamics of several 1D disordered systems [25]. Most of our previous results have been for the *Sinai model* that describes the diffusion of a random walker in a 1D *random static force field*, which is equivalent to a random potential that itself has the statistics of a 1D random walk [26]. It can readily be seen that individual domain walls in the classical RFIM diffuse in a random potential of exactly this Sinai form, the complication being that they annihilate upon meeting. As shown in Refs. [25,27] the RSRG model can also be applied to many-domain-wall problems such as that which corresponds to the RFIM. A few of the results of this paper were already presented in a short paper [25]; the aim of the present paper is to show in detail how the RSRG method applies to such disordered spin models, and to explore more of its consequences. Although we will give here a detailed discussion of the RSRG method for the spin model, we will rely on Ref. [28] for many results about the single particle diffusion aspects of the problem; these we will only sketch, referring the reader to Ref. [28] for details.

As for the single particle problem, the RSRG method allows us to compute a great variety of physical quantities, remarkably including even some which are not known for the corresponding pure model (e.g., the domain persistence exponents δ and ψ). This provides another impetus for the study of the random models. Using the RSRG method we also obtain the equilibrium behavior which corresponds to a well defined scale at which the decimation is stopped.

The RSRG method is closely related to that used to study disordered quantum spin chains [29–34]. The crucial feature of the renormalization group (RG) is coarse graining the energy landscape in a way that preserves the long time dynamics. Despite its approximate character, the RSRG method yields asymptotically exact results for many quantities. As in Ref. [31], it works because the width of the distribution of barrier heights grows without bounds on long length scales, consistent with the rigorous results of Ref. [26]. It is interesting to note that an exact RG has also been applied to the problem of coarsening of the *pure 1D* soft-spin Ising Φ^4 model at zero temperature, for which persistence exponents have been computed [35,36]. Extensions to higher dimensions of the RSRG method for equilibrium quantum models were recently obtained [37]; these introduce hope that other dynamical models could be studied in higher dimensions as well.

The outline of this paper is as follows. In Sec. II we define the RFIM and spin glass models and their dynamics, and in Sec. III we summarize some of the main results and the physics involved. In Sec. IV we explain how the RSRG method is applied to the RFIM, give the corresponding equations and fixed point solutions, and discuss the properties of the asymptotic large time state. In Sec. V we derive exact large time results for single time quantities such as the energy, the magnetization, and the domain size distribution. In Sec. VI we obtain the time dependent single-time two-spin correlation functions both with and without an applied field. In Sec. VII we compute the two time spin autocorrelation as well as the two-point two-time correlations. In Sec. VIII we study aging phenomena which necessitate considering rare events which dominate all moments of the thermal (truncated) correlations which we obtain, as well as the response to a small uniform magnetic field applied after a waiting time t_w as in a typical aging experiment. We also compute the fluctuation dissipation ratio $X(t,t_w)$. In Sec. IX we compute the persistence properties—the probabilities of no changes—of a single spin, of the local magnetization and of domains. Finally, in Sec. X we obtain the distribution of the total magnetization and of the number of domains for large finite size systems. A brief discussion of the possibilities for experimental tests of the predictions and speculation on the applicability of the some of the general features to higher dimensional systems are presented at the end of the paper.

Various technical results are relegated to appendixes. In Appendix A the convergence towards the asymptotic state is shown. In Appendix B the time dependent single-time two-spin correlation functions in an applied field is derived, while in Appendix C, the two-point two-time correlations are computed in detail and in Appendix D some of the finite size properties are analyzed.

II. MODELS AND NOTATION

A. Random field Ising model

1. Statics

We consider the random field Ising chain consisting of N spins $\{S_i=\pm 1\}_{i=1,N}$ with Hamiltonian

$$\mathcal{H}=-J\sum_{i=1}^{n=N-1} S_iS_{i+1}-\sum_{i=1}^{i=N} h_iS_i, \tag{1}$$

with independent random fields $\{\{h_i\}\}$ with identical distribution whose important moments are

$$H\equiv\overline{h_i}, \tag{2}$$

$$g\equiv\overline{h_i^2}-H^2, \tag{3}$$

so that H can be considered as a uniform applied field, and g is the mean-square disorder strength; here and henceforth we denote averages over the quenched randomness—usually equivalent to averages over different parts of the system—by overbars.

The statics and dynamics of the random field Ising model can be studied in terms of domain walls living on the dual lattice: on the bonds $\{(i,i+1)\}$ will be indexed by their left-hand site i. A configuration of spins $\{S_i\}$ can be represented as a series of ''particles'' A corresponding to domain walls of type $(+|-)$ at positions $a_1,\dots,a_{N_A}$, and ''particles'' B corresponding to domain walls of type $(-|+)$ at positions $b_1,\dots,b_{N_B}$; these must of course occur in an alternating sequence $ABABAB\dots$. [The relation between the numbers N_A and N_B of domain walls depends on the boundary conditions: for instance, in a system with periodic boundary conditions, $N_A=N_B$, while in a system with free boundary conditions $|N_A-N_B|\leqslant 1$.]

It is very useful to introduce the *potential* ''felt'' by the domain walls,

$$V(x)=-2\sum_{i=1}^{x} h_i, \tag{4}$$

and rewrite the energy of a configuration $\{S_i\}$ in terms of the positions of the set of domain walls as

$$\mathcal{H}=\mathcal{H}_{ref}+2J(N_A+N_B)+\sum_{\alpha=1}^{N_A} V(a_\alpha)-\sum_{\alpha=1}^{N_B} V(b_\alpha), \tag{5}$$

where $\mathcal{H}_{ref}$ is the energy of the reference configuration where all spins are $(+)$:

$$\mathcal{H}_{ref}=-J(N-1)-\sum_{i=1}^{n=N} h_i. \tag{6}$$

Each domain wall costs an energy $2J$; the domain walls A feel the random potential $+V(x)$; whereas the B walls feel the potential $-V(x)$, i.e., the two types of domain walls feel *opposite* random potentials.

Let us first recall some known features of the statics. In the absence of an applied field $(H=0)$ the system is disordered at $T=0$, and contains domains with typical size given by the Imry-Ma length

$$L_{IM}\approx\frac{4J^2}{g}, \tag{7}$$

obtained from the following simple argument: the creation of a single pair of domain walls (A,B) a distance L apart costs exchange energy $4J$ independent of L. But the random potential has typical variations $|V(x)-V(y)|_{\rm typ}\sim\sqrt{4g|x-y|}$, and thus the typical energy that the system can gain on a length L by using a favorable configuration of the random potential is of order $\sqrt{4gL}$. The two energies become comparable for $L\sim L_{IM}$ and for $L\geqslant L_{IM}$, it becomes favorable for the system to create domain walls. Thus the ground state will contain domains of typical size L_{IM}. One should note the difference between the case of, e.g., bimodal distributions (for which the ground state can be degenerate) and continuous distributions for which it is nondegenerate; we will generally consider continuous distributions for simplicity.

At positive temperature without a random field, the thermal correlation length $L_T\sim\exp(2J/T)$ gives the typical size of domains in the system in equilibrium at temperature T.

But in the presence of the random field the equilibrium state at finite temperature is dominated by the random fields and still given by the Imry-Ma picture provided that

$$1 \ll L_{IM} \ll L_T, \tag{8}$$

which is the regime studied in the present paper. In the presence of a uniform applied field $H>0$, the system at low temperature will contain domains of both orientations but with different typical sizes leading to a finite magnetization per spin $m(g,H)<1$.

2. Nonequilibrium dynamics

In this paper we will study the magnetization nonconserving dynamics of the RFIM model [Eq. (1)] starting from a random initial condition at time $t=0$ corresponding to a quench from a high temperature state. Although our results are independent of the details of the dynamics (provided it satisfies detailed balance and is nonconserving), for definiteness let us consider the Glauber dynamics, where the transition rate from a configuration $\mathcal{C}$ to the configuration $\mathcal{C}_j$, obtained from $\mathcal{C}$ by a flip of the spin j, is

$$W(\mathcal{C}\to\mathcal{C}_j)=\frac{e^{-\beta\Delta E}}{e^{\beta\Delta E}+e^{-\beta\Delta E}}=\frac{1}{2}\left[1-\tanh\left(\frac{\beta\Delta E}{2}\right)\right], \tag{9}$$

which satisfies the detailed balance condition and where $\Delta E=2JS_j(S_{j-1}+S_{j+1})+2h_jS_j$ is the energy difference between the two configurations, which takes the following possible values in terms of domain walls:

$$\Delta E \ \ \{\text{creation of two domain walls}\}=4J\pm 2h_j, \tag{10}$$

$$\Delta E \ \ \{\text{diffusion of one domain wall}\}=\pm 2h_j, \tag{11}$$

$$\Delta E \ \ \{\text{annihilation of two domain walls}\}=-4J\pm 2h_j. \tag{12}$$

In this paper we focus on the regimes

$$\{h_i\}\ll J, \tag{13}$$

$$T\ll J, \tag{14}$$

in which Eq. (8) holds. The randomness will dominate on length scales that are sufficiently large so that the cumulative effect of the random field energies is greater than T, i.e., for scales

$$l \gg \overline{h^2}/T^2. \tag{15}$$

So that there will be only one basic length and time scale, it is simplest to consider $\{h_i\}\sim T$.

There will be a wide range of time scales during which the domains grow from initial sizes of $O(1)$ to sizes of order L_{IM} by which time the system will be close to equilibrium. Since the Imry-Ma length L_{IM} is very large, we expect the universality of the long time dynamics, as we indeed find.

B. Ising spin glass in a magnetic field

In this paper we also consider the 1D Ising spin glass in a uniform field h:

$$\mathcal{H}=-\sum_{i=1}^{i=N-1} J_i\sigma_i\sigma_{i+1}-\sum_{i=1}^{N} h\sigma_i. \tag{16}$$

As is well known, in the case of a bimodal $(\pm J)$ distribution with equal probabilities for either sign, 1D Ising spin glasses in a field are equivalent via a gauge transformation to random field ferromagnets. More precisely, setting $J_i=J\epsilon_i=\pm 1$ for $i=1,\dots,N-1$, and defining $\sigma_1=S_1$ and $\sigma_i=\epsilon_1\cdots\epsilon_{i-1}S_i$ for $i=2,\dots,N$, the gauge transformation gives the new Hamiltonian

$$\mathcal{H}=-\sum_{i=1}^{i=N-1} JS_iS_{i+1}-\sum_{i=1}^{N} h_iS_i, \tag{17}$$

where $h_i=h\epsilon_1\cdots\epsilon_{i-1}$. Since the ϵ_i are independent random variables taking the values ± 1 with probability 1/2, the fields h_i are also independent random variables taking the values $\pm h$ with probability 1/2. Hence the Hamiltonian describes a $(\pm h)$ random field Ising model with $H=0$ and $g=h^2$ in Eq. (1).

The physical interpretation is as follows. The spin glass chain in zero field $(h=0)$ has two ground states, given by $\pm\sigma_i^{(0)}$, where $\sigma_i^{(0)}=\epsilon_1\cdots\epsilon_{i-1}$, which correspond, via the gauge tranformation, to the pure Ising ground states $S_i^{(0)}=+1$ and $S_i^{(0)}=-1$. In the presence of a field $h>0$, the ground state of the spin glass is made out of domains of either zero field ground states. Their typical size is thus given by the Imry-Ma length $L_{IM}=4J^2/h^2$. These domains correspond to the intervals between frustrated bonds since at the position of a domain wall one has $J_i\sigma_i\sigma_{i+1}=JS_iS_{i+1}<0$. Similarly each domain has magnetization:

$$\left|\sum_{i\in\text{ domain}}\sigma_i^{(0)}\right|=\frac{1}{h}\left|\sum_{i\in\text{ domain}}h_i\right|=\frac{1}{2h}|V(a_\alpha)-V(b_\alpha)|, \tag{18}$$

which is thus proportional to the absolute value of the corresponding barrier of the Sinai random potential, a property which will be used below.

Via the gauge transformation, the nonequilibrium dynamics (e.g., the Glauber dynamics) of the spin glass in a field starting either from (i) random initial conditions (corresponding to a quench from high temperature) or from (ii) the pure ferromagnetic state (obtained by applying a large magnetic field that is quickly reduced to be $h\ll J$) corresponds to the nonequilibrium dynamics (e.g., Glauber) of the RFIM, also starting from random initial conditions. Thus in the regime $h\sim T\ll J$, many of the universal results obtained for the RFIM will hold directly for the spin glass in a field. Thus we will study the two models in parallel in the present paper.

An important complication in applying the results to 1D spin glasses is that the strengths of the exchange couplings will in general be random. This provides an extra random potential for the domain walls which has, however, only short range correlations. The conditions for there to be a wide regime of validity of the universal coarsening behavior found here is that the distribution of exchanges be *narrow*,

$$|J|_{max}-|J|_{min}\ll|J|_{min}, \tag{19}$$

ideally with $|J|_{max}-|J|_{min}\sim h$ or smaller. Note that the equilbrium correlation length will be set by $|J|_{min}$ for small h as the domain walls can easily find positions with $|J|\cong|J|_{min}$ that are near to extrema of the potential caused by the random fields. Having defined the models, before turning to the details of the calculations we now briefly decribe the important physics and summarize our main results.

III. QUALITATIVE BEHAVIOR AND SUMMARY OF RESULTS

As for any Ising system, the static and dynamic properties of random spin chains can be fully described in terms of domain walls between ''up'' and ''down'' domains. The crucial simplification in one dimension is that the domain walls, which come in two types $A\equiv(+|-)$ and $B\equiv(-|+)$, are point objects. As discussed above, a random field induces a potential that the walls feel which has the statistics of a random walk so that its variations on a length scale L are of order $\sqrt{L}$. The A walls tend to minima of the potential, while the B walls tend to maxima. If they meet they annihilate but, on the time scales of interest here, the probability that a pair is spontaneously created, $e^{-4J/T}$ can be ignored.

As shown by Sinai [26], the motion of a single domain wall in such a random potential is completely dominated by the barriers that have to be surmounted to find low energy extrema. Since the time to surmount a barrier of height b is of order $e^{b/T}$, and to find a minima a distance L away a barrier of order $b\sim h\sqrt{L}$ will have to be overcome, the typical distance a wall moves in time t is only of order $l(t)\sim\ln^2 t$. Although this motion is controlled by rare thermal fluctuations that take the wall over a large barrier, the position of a wall that started at a known point can, at long times, be predicted with surprising accuracy. This is because the width of the distribution of barriers to go distances of order L is as broad as the magnitude of the lowest barrier which enabled the wall to move that distance. Thus at long times, when the barriers become very high, the probability of going first over other than the lowest surmountable barrier that delimits the region in which the wall is currently, is extremely small. The position of a single wall at long times is thus determined by its initial position and by the height of the maximum barrier, which it *could* surmount up to that time; this quantity, which we denote

$$\Gamma\equiv T\ln t, \tag{20}$$

yields a well defined region that the wall can explore up to time t. The boundaries of the appropriate region that encompasses the wall's initial position are determined by $\Gamma(t)$. The wall will be in local equilbrium in this ''valley'' on time scales of order t, and thus tend to spend most of its time near the bottom of the valley. This behavior, as proved by Golosov [38], implies that the long time dynamics of a single wall is pseudodeterministic: rescaled by the typical distance it has gone, $\ln^2 t$, the wall's position is *asymptotically deterministic* at long times.

In a random field Ising chain, the two types of domain walls move in random potentials which are identical except for their sign. When walls meet, they annihilate. Not surprisingly, since each wall can move a distance of order $\ln^2 t$ in time t and they cannot pass through each other or occupy the same position, the density of domain walls remaining at time t is simply of order $1/\ln^2 t$ so that the correlation length starting from random (or short-range correlated) initial conditions grows as

$$\xi(t)\sim\Gamma^2/g\sim\ln^2 t. \tag{21}$$

This is in sharp contrast to the much faster power-law growth of the correlation length in most nonrandom systems, for example the $\xi(t)\sim\sqrt{t}$ for Ising systems with a nonconserved order parameter.

The time-dependent correlation length can be more precisely defined from the average nonequilibrium correlation function. This, and most other nonequilibrium properties, are scaling functions of the ratio of lengths to powers of log-times, as is characteristic of ''activated dynamic scaling'' [39] which occurs in many random systems. We find that in zero applied field the average equal-time correlation function behaves as

$$\overline{\langle S_0(t)S_x(t)\rangle}=\sum_{n=-\infty}^{\infty}\frac{48+64(2n+1)^2\pi^2 g\dfrac{x}{\Gamma^2}}{(2n+1)^4\pi^4}\times e^{-(2n+1)^2\pi^2 2g(x/\Gamma^2)} \tag{22}$$

whose Fourier transform is, at large time,

$$\sum_{x=-\infty}^{+\infty}e^{iqx}\overline{\langle S_0(t)S_x(t)\rangle}\sim\frac{8}{\pi^2 q^2\xi(t)}\,\mathrm{Re}[\tanh^2(\pi\sqrt{iq\xi(t)}/2)], \tag{23}$$

with Re denoting the real part. These results obtain until the nonequilibrium correlation length

$$\xi(t)=\frac{T^2\ln^2 t}{2g\pi^2} \tag{24}$$

reaches the equilibrium correlation length

$$\xi_{eq}=\frac{8J^2}{\pi^2 g}=\frac{2}{\pi^2}L_{IM}. \tag{25}$$

It is intriguing that the *form* of the nonequilibrium correlations in the universal scaling limit are *identical* to those of the equilibrium correlations, the only difference being the correlation length.

A remarkable property of the coarsening in the RFIM chain was conjectured in Ref. [25]: the positions of the *set* of domain walls and hence all of the correlations in the nonequilibrium state are *asymptotically deterministic* at long times. This means that while the evolution of the domain wall structure is only logarithmic in time, the domain wall configurations of two runs following quenches using the same sample converge to each other much more rapidly as a power of time. More precisely, the probability that the spin configurations of the two runs measured at the same time after the quench differ substantially in a region of size of order $\xi(t)$ decays to zero as a nonuniversal power of time, becoming negligible well before the system is anywhere near equilibrium. [Strictly speaking, because in a rare region, as discussed later, fluctuations of the position of an individual or neighboring pair of domain walls will exist, one will need to do a certain amount of time averaging for this pseudodeterminism to become most evident.]

The asymptotic determinism will occur even if the initial conditions were macroscopically distinct in the two runs: for example, if one was quenched from a high temperature state in zero applied field, and the other from a high temperature state with a small net magnetization caused by a uniform applied field. This pseudodeterminism in a system with many degrees of freedom is a dramatic effect; it implies that the history dependence can, even under strongly nonequilibrium conditions, be weak; one might thereby be fooled into thinking that such history independence implies equilibrium.

While equal-time correlations during coarsening converge to an equilibriumlike form, two-time quantities show an interesting history dependence, generally depending on *both* times rather than just the time difference as they would do in equilibrium. There are typically three regimes: the later time t, much longer than the earlier, time t', the two times of the same order, and the *difference* between the two times ($t-t'$) much smaller than either time. The scaling variables in these three regimes are $\ln t'/\ln t$, t'/t, and $\ln(t-t')/\ln t$, respectively. In the first and third of these, the condition for asymptopia, that the scaling variable is small, is very difficult to attain in practice; thus knowing the full form of the scaling functions is essential for analyzing experimental or numerical data. Note that only in the last of these regimes should one expect to find an equilibriumlike behavior characteristic of the local equilibrium that is being probed.

In this paper we compute a variety of correlations and response functions that illustrate some of the interesting nonequilibrium behaviors. The simplest of these is the temporal autocorrelations of a single spin. With no applied field, the average autocorrelation function is found to decay as a power of the ratio of the correlation lengths at two times,

$$\overline{\langle S_x(t)S_x(t')\rangle} \sim \left(\frac{\xi(t')}{\xi(t)}\right)^{\lambda}, \qquad (26)$$

for $t>t'$ with the exponent $\lambda=1/2$. In a three-dimensional spin glass, the autocorrelation function determines the nonequilibrium decay of the magnetization after a large uniform field is turned off below the transition temperature [5]. There are complications here due to the fact that a small applied field needs to be left on to provide the randomness, but the basic physics is similar.

The single spin autocorrelation function provides some information on how much memory the system has of its earlier history. More information is provided by the two-spin two-time correlation function $\overline{\langle S_x(t)S_y(t)\rangle\langle S_x(t')S_y(t')\rangle}$ which converges to a scaling function of $(x-y)/\xi(t)$ and $\xi(t')/\xi(t)$, whose Fourier transform in $(x-y)$, $D(q,t,t')$ we compute exactly; to our knowledge, this is the first such computation for a physical model pure or random. For an ideal spin glass in which there are no correlations between the positions of positive and negative exchanges and the distribution of the exchange strengths is symmetric in $J\to -J$, this correlation function D is the average over the randomness—or equivalently a positional average over the regions being probed by the scattering—of the product of the magnetic scattering intensity at time t and wave vector k and that at time t' and wave vector $q-k$.

Other properties associated with ''aging'' can also be computed exactly; we study the thermal fluctuations around the configurations at two different times, and the dynamic linear response to a uniform magnetic field that is turned on after waiting for some time for the system to equilibrate. In equilibrium, these are related by the fluctuation dissipation theorem but we find, as expected, that this relation generally fails except in the limit that the time difference is much less than the waiting time. The behavior we find is, however, rather different from that found in mean-field models [40], and we discuss the contrasts between these results.

In random systems controlled by zero temperature fixed points, such as is the case in the regime of scales studied here, thermal fluctuations and linear response functions are both dominated by rare spatially isolated regions of the system—although which regions dominate depends on the time scales and properties of interest. The study of these thus involves the *corrections* to the deterministic approximation to the dynamics that led, as discussed above, to exact asymptotic results for many other quantities. The dominant events are rare by a factor of, typically, $1/\ln t$, but nevertheless still lead to universal results.

''Persistence'' properties provide another probe of how much memory a system retains of its initial configuration, for example, what the probability is that a spin has never flipped. Surprisingly, this and other related quantities—including some which are not known in pure systems—can also be computed exactly for the RFIM chains.

IV. REAL-SPACE RENORMALIZATION METHOD

The coarsening process taking place in nonequilibrium dynamics of the RFIM starting from random initial conditions can be thought of as a reaction-diffusion process in a 1D random environment (Sinai landscape) for the domain walls. Indeed, in the regime $T\ll J$ considered in this paper, the creation of pairs of domain walls is highly suppressed. The dynamics of the domain walls is thus dominated by the random field as follows. The domains walls A quickly fall to the local minima of the random potential $V(i)\equiv V_i$, whereas

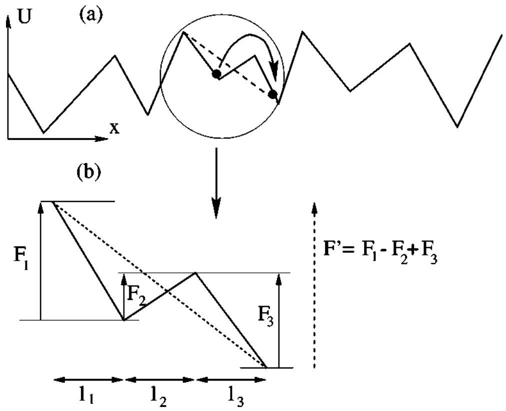

FIG. 1. (a) Energy landscape in the Sinai model (b) decimation method: the bond with the smallest barrier $F_{min}=F_2$ is eliminated, resulting in three bonds being grouped into one.

the B walls quickly move to the local maxima of V_i. Then they slowly diffuse by going over barriers in *opposite* Sinai potentials $\pm V_i$. When a A and B walls meet, they annihilate preserving the alternating $ABAB\dots$ sequence.

We can thus use the real-space renormalisation procedure introduced in Ref. [28] to study the Sinai diffusion of a single particle, and extend it to take into account the annihilation processes. In the single particle case this procedure was shown to be asymptotically exact at long length and time scales. Here it is also expected to be asymptotically exact at large time, since, as in the Sinai case, we find that the the effective distribution of the random barrier heights become infinitely broad in the limit of large scales. Thus the motion over the barriers becomes more and more deterministic at long times.

A. Definition of the real space renormalization procedure

We briefly outline the renormalization procedure, detailed in Refs. [25,28], for the diffusion of a particle in the Sinai landscape V_i. Grouping segments with the same sign of the random field, one can start with no loss of generality from a ‘‘zigzag’’ potential V_i where each segment (‘‘bond’’) is characterized by an energy barrier $F_i=|V_i-V_{i+1}|$ and a length l_i. From the independence of the random fields on each site, the pairs of bond variables (F,l) are independent from bond to bond and are chosen from a distribution $P(F,l)$ normalized to unity.

The RG procedure which captures the long time behavior in a given energy landscape is illustrated in Fig. 1. It consists of the iterative decimation of the bond with the *smallest barrier*, and hence the shortest time scale for domain walls to overcome it [25,28]. This smallest barrier, say, F_2, together with its neighbors, the two bonds 1 and 3, are replaced by a single renormalized bond with barrier

$$F'=F_1-F_2+F_3 \tag{27}$$

and length

$$l'=l_1+l_2+l_3. \tag{28}$$

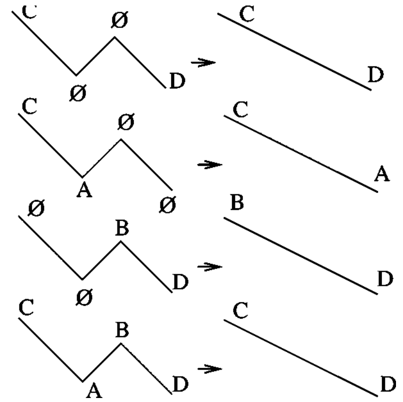

FIG. 2. The four possible cases of evolution under the decimation of the middle bond of the domain walls, given the constraint of alternating sequence of domains. A and B denote the domain walls $(+|-)$ and $(-|+)$, respectively, $\varnothing$ denotes no domain wall present at the top or bottom of the renormalized bond, while C represents either B or $\varnothing$ and D represents either A or $\varnothing$.

This defines a renormalized landscape at scale Γ, where all barriers smaller than Γ have been eliminated.

Since the distribution of barriers is found to become broader and broader [31] an Arrhenius argument implies that the diffusion of a particle becomes better and better approximated at large time by the following ‘‘effective dynamics’’ [25,28]. The position $x(t)$ of a particle that started at x_0 at $t=0$ coincides with—or is at least very close to—the bottom of the renormalized bond at scale

$$\Gamma = T\ln t, \tag{29}$$

which contains x_0. Note that we choose time units so as to set the microscopic (nonuniversal) inverse attempt frequency to unity [25,28]. This RG procedure is thus essentially decimation in *time*. Processes that are faster than a given time scale are decimated away and assumed to be in local equilibrium.

In the presence of domain walls of types A and B, we must keep track of both the diffusion and the possible reactions of the domain walls that occur during the decimation. Upon the decimation of bond 2 (see Fig. 1), there are four possible cases, illustrated in Fig. 2, according to whether or not there is a type A wall at the bottom of bond (2), and whether or not there is a type B wall at the top of bond (2). If (i) there are no type A or B walls, then one simply renormalizes the bond and nothing happens to the domain walls. If (ii) there is an A wall but no B wall, then the A wall goes to the bottom of the new renormalized bond. If (iii) there is a B wall and no A wall then the B wall goes to the top of the new renormalized bond. And if (iv) there is both an A wall and a B wall, then the two domain walls meet and annihilate upon decimation. This annihilation will occur at a time of order $e^{\Gamma/T}$ determined by the barrier of the decimated bond: by assumption, Γ at this scale.

The only truly new process compared to the single particle diffusion is the case (iv). The above RG rule is consistent with the large time dynamics in this case for the following reason. Since A and B domains diffuse with independent thermal noises, the time it takes for them to meet is again $T\ln t = F$ (accurate on a log scale). Indeed the probability that they meet at a point at potential V is $\sim \exp(-(F-V)/T)\exp(-V/T) = \exp(-F/T)$ [as can be seen from considering the equivalent $2d$ diffusion problem (of the pair of wall positions) with an absorbing wall at $x+y=l$].

The second difference from the procedure in the case of the single particle, is that it must be stopped when the renormalisation scale Γ reaches the energy cost of creation of a pair of domain walls:

$$\Gamma = \Gamma_J = 4J. \tag{30}$$

Indeed beyond this scale $T\ln t > \Gamma_J$ one must take into account creation of pairs of domain walls. As will be shown below, the state *at* $\Gamma = \Gamma_J$ gives the final equilibrium state.

B. RG equations and statistical fixed point of the landscape

Independently of whether the bonds contain domain walls (A or B) or not, which is studied in Sec. V, one can study the evolution under RG of the landscape. Since the RG rules preserve the statistical independence of the variables (F,l) from bonds to bonds, it is possible to write closed RG equations for the landscape, i.e., for $P_\Gamma^+(\zeta = F-\Gamma, l)$ and $P_\Gamma^-(\zeta = F-\Gamma, l)$ which denote the probabilities that a $\pm$ renormalized bond at scale Γ has a barrier

$$F = \Gamma + \zeta > \Gamma \tag{31}$$

and a length l, each normalized by $\int_0^\infty d\zeta \int_0^\infty dl P_\Gamma^\pm(\zeta,l) = 1$,

$$(\partial_\Gamma - \partial_\zeta) P_\Gamma^\pm(\zeta,l) = P_\Gamma^\mp(0,.) *_l P_\Gamma^\pm(.,.) *_{\zeta,l} P_\Gamma^\pm(.,.) \tag{32}$$

$$+ P_\Gamma^\pm(\zeta,l) \int_0^\infty dl' (P_\Gamma^\pm(0,l') - P_\Gamma^\mp(0,l')), \tag{33}$$

where $*_l$ denotes a convolution with respect to l only and $*_{\zeta,l}$ with respect to both ζ and l with the variables to be convoluted denoted by dots. As discussed in Refs. [31,28], the solutions of these RG equations depend on an assymetry parameter δ defined as the nonvanishing root of the equation

$$\overline{e^{-4\delta h}} = 1, \tag{34}$$

which reduces in the limit of weak bias H to

$$\delta \simeq \frac{H}{2g}, \tag{35}$$

with $\delta = 0$ in the absence of a uniform applied field. Our results are valid for long times as long as

$$\delta T \ll 1. \tag{36}$$

For large Γ, the Laplace transform of the distributions $P_\Gamma^\pm$ take the following form, in the scaling regime of small δ and small p with $\delta\Gamma$ fixed and $p\Gamma^2$ fixed [31]:

$$\int_0^\infty dl P_\Gamma^\pm(\zeta,l) e^{-ql} = U_\Gamma^\pm\left(\frac{q}{2g}\right) e^{-\zeta u_\Gamma^\pm(q/2g)}$$

$$u_\Gamma^\pm(p) = \sqrt{p+\delta^2}\coth[\Gamma\sqrt{p+\delta^2}] \mp \delta$$

$$U_\Gamma^\pm(p) = \frac{\sqrt{p+\delta^2}}{\sinh[\Gamma\sqrt{p+\delta^2}]} e^{\mp\delta\Gamma}. \tag{37}$$

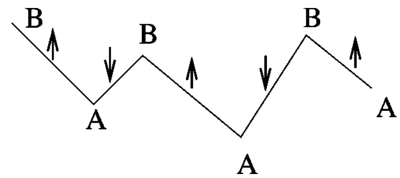

FIG. 3. Full state in the renormalized landscape; as we show, this corresponds to the state at large time (see the text). The top and bottom of the bonds are occupied by B and A domain walls, respectively. The correponding spin orientations are also indicated for the RFIM. Decimation under an increase of Γ corresponds to an annihilation of the domain with the smallest barrier. For the spin glass the position of the domain walls corresponds to the frustrated bonds.

This exact knowledge of the renormalized landscape will be used below to extract physical quantities for the spin models.

C. Convergence toward ''full'' states: Asymptotic determinism

Even armed with the statistical properties of the renormalized landscape, we still have to determine the long time distribution of the occupation of the extrema of this landscape by A and B domain walls. At first sight this seems a difficult problem. Indeed, the positions of A and B walls can be correlated over many bonds of the renormalized landscape since there are *a priori* empty maxima and minima and the domain wall positions must respect the alternating constraint $ABABAB$. However, the RG analysis becomes simple if the system reaches at some stage a ''full'' state which has one A wall at each minimum and one B wall at each maximum of the renormalized landscape, as illustrated in Fig. 3. It is easy to see that such a ''full'' state is preserved by the RG procedure. Also note that this ''full'' state would be obtained from the beginning if, for instance, the initial condition were completly antiferromagnetic.

Generally, we consider random initial conditions and thus the initial state is not a ''full'' state. However, one can show that the system converges *exponentially* in Γ toward a ''full'' state, as we now discuss.

The renormalization procedure for the coarsening process of the RFIM has the following important property: the configuration for the spins at scale Γ depends only on the *renormalized* landscape at Γ and on the *initial* configuration of the spins (e.g., equilibrium at high temperature before the quench at $t=0$). In particular, it does not depend on the initial landscape or—except occasionally at early times before the barrier distribution becomes very broad—on the whole history of the reaction-diffusion processes of domain walls.

Let us first consider one ascending bond with extremities (x,y) of the renormalized landscape and assume that there were initially n domain walls in the interval (x,y). Neglecting for the time being the influence of the two neighboring bonds, it is easy to see that the state at Γ is determined only by n and does not depend on the order in which the reactions between domain walls have occured: indeed, it is determined by the *parity* of n and by the nature of the domain wall closest to the bottom end. There are several cases.

(i) If $n=0$, then the final state of the occupation of the end points of the bond is $(\varnothing,\varnothing)$.

(ii) If n is odd, and if the domain wall closest to the bottom is of type A, then the final state is $(A,\varnothing)$

(iii) If n is odd, and if the domain wall closest to the bottom is of type B, then the final state is $(\varnothing,B)$

(iv) If n is even with $n\geqslant 2$ and the domain wall closest to the bottom is of type A, then the final state is (A,B)

(v) If n is even with $n\geqslant 2$ and the domain wall closest to the bottom wall is of type B, then the final state is $(\varnothing,\varnothing)$.

Of course, to obtain the real occupation of the top and the bottom ends of one renormalized bond, one also needs to consider what happens on the two neighboring renormalized bonds, and to compute the probabilities of the various states, taking properly care of the alternating constraint $ABAB$. This is done in Appendix A. It is found that the crucial feature is that in order for a bond not to have both ends occupied by domain walls, either it or one of its neighboring bonds must have had *no* domain walls on it initially. Since the bonds tend to become progressively longer with time, the chance of this occuring drops rapidly with increasing Γ. This will be true even if the positions of the domain walls have some local correlations, in particular for the case of an initial state that corresponds to a high temperature configuration in a small magnetic field so that there is an initial magnetization and the typical distances between a neighboring A and B wall will depend on which of the two is on the left.

We thus find that the system converges towards the ''full'' state of the renormalized landscape exponentially fast in Γ, with a nonuniversal coefficient that depends on the initial concentration of domain walls, on the inital magnetization, and on the strength of the randomness. This corresponds to a *power law* decay in time, as $t^{-\eta}$ with η nonuniversal, of the probability that a maximum or minimum of the renormalized landscape at time t is unoccupied. The positions of the full set of domain walls at long times are thus asymptotically deterministic and independent of the initial conditions provided these have only short range correlations. Concretely, this asymptotic determinism can be characterized by the typical mean distance between ''missing'' domain walls, i.e., deviations from the deterministic full state. The distance between these ''errors'' will grow exponentially in Γ and hence as a power of time. The only exceptions to this are associated with fluctuations in the (nominally) *full* states: because of the rare times or configurations in which domain walls are either about to annihilate or are spending substantial fractions of their time in more than one valley; these fluctuation induced ''errors'' we will study in detail later; they decay far more slowly with time than any persistent initial condition induced differences between runs.

Since all the universal quantities that we will study have length scales that grow much more slowly—typically logarithmically—in time, or probabilites of occurring which decay much more slowly with time—typically as powers of $1/\Gamma\sim 1/\ln^2 t$—we restrict consideration in what follows to the analysis of full states. Note that our results for the biased case in which there is a small uniform field applied in addition to the random field do yield a power law growth of the correlation length with time; however, in the regime of validity of these results, the power law is very small, and again the effects of ''missing'' domain walls are negligible at long times.

D. Convergence toward equilibrium

As mentioned earlier, the RG procedure must be stopped at

$$\Gamma=\Gamma_J=4J, \tag{38}$$

since at this scale one must start to take into account the energetic benfits of creation of pairs of domain walls. At a scale $\Gamma=\Gamma_J$ the typical domain size is the Imry-Ma length $L_{IM}=4J^2/g$, and the energy cannot be lowered further by *any* process in the full state. Indeed, moving the domains walls without changing their number cannot lower the total energy, since domain walls already occupy all tops and bottoms of the renormalized potential. Decreasing the number of domain walls by two also cannot lower the energy, since the gain is $4J$ while the loss due to the random field for walls separated by a bond of barrier F is $F>\Gamma=4J$. Similarly, to add the two walls the cost is $4J$ and the gain is $F<\Gamma=4J$, since the only positions they can occupy are by definition separated by a barrier $<\Gamma$ which has already been decimated. Thus if the renormalization is stopped at Γ_J, in the small field, low T scaling limit the configuration of the walls corresponds precisely to the ground state and, up to negligible thermal fluctuations, to the thermal *equilibrium state*. Thus we are able to compute equilibrium properties straightforwardly from the renormalization group analysis.

Thus the RG approach allows one to study the approach to equilibrium starting from any initial condition characterized by typical domain sizes $L_0e^{2J/T_0}\ll L_{IM}$, where T_0 represents the temperature before the quench. As explained above, under these conditions the relaxation towards equilibrium always takes place by diffusion and annihilation of domain walls before any domain creation can occur.

E. Approach to equilibrium from more ordered initial conditions

A qualitatively different type of evolution towards equilibrium takes place in the other limit (not studied here),

$$L_0\sim e^{2J/T_0}\gg L_{IM}=\frac{4J^2}{g}, \tag{39}$$

still imposing $h\ll T_0$. This is a regime of initial temperatures low enough that the initial density of domains is very small. In this case a very different relaxation process toward the *same equilibrium state* discussed above takes place: for $\Gamma<\Gamma_J=4J$ well separated domain walls diffuse independently with very rare annihilations. When Γ reaches Γ_J many domain walls are ''suddenly''—within a factor of 2 or so in time—created, and the large initial domains break into many smaller ones of size L_{IM}. Thus, the relaxation is more abrupt in this case than in the more interesting one studied in the present paper.

V. ENERGY, MAGNETIZATION, AND DOMAIN SIZE DISTRIBUTION

From the knowledge of the fixed point for renormalized landscape [Eq. (37)] and the fact that the system reaches the full state (exponentially fast in Γ) where each top is occupied by a B domain and each bottom by a A domain, we can immediately compute several simple quantities. We will compare these results with the existing exact results known in the statics.

A. RFIM without applied field

Specializing Eq. (37) to the case $\delta=0$ in the absence of an applied field, the fixed point of the RG equation is [30,31]

$$\tilde{P}^*(\eta,\lambda)=LT^{-1}_{s\to\lambda}\left(\frac{\sqrt{s}}{\sinh\sqrt{s}}e^{-\eta\sqrt{s}\coth\sqrt{s}}\right), \tag{40}$$

where we have introduced the dimensionless rescaled variables for barriers,

$$\eta=(F-\Gamma)/\Gamma, \tag{41}$$

and for bond lengths,

$$\lambda=2g\frac{l}{\Gamma^2}. \tag{42}$$

1. Number of domain walls per unit length

Thus for large $\Gamma=T\ln t$, the average bond length $\overline{l_\Gamma}$, equal to the average distance between two domain walls, behaves as

$$\overline{l_\Gamma}=\frac{1}{4g}\Gamma^2=\frac{1}{4g}T^2\ln^2 t. \tag{43}$$

The number of domain walls per unit length decays as

$$n(t)=\frac{4g}{T^2\ln^2 t}, \tag{44}$$

up to time $t_{eq}\sim\exp(4J/T)$ at which equilibrium is reached, and

$$n(t_{eq})=n_{eq}=\frac{1}{L_{IM}}. \tag{45}$$

2. Energy density

The energy per spin as a function of time (i.e., of $\Gamma=T\ln t$) is simply given by

$$E_\Gamma\simeq-J+n_\Gamma(2J-\tfrac{1}{2}\langle F\rangle_\Gamma) \tag{46}$$

$$=-J+\frac{4g}{(T\ln t)^2}(2J-T\ln t), \tag{47}$$

where $\langle F\rangle_\Gamma$ denotes the averaged barrier at scale Γ. This formula holds up to time $t_{eq}\sim\exp(4J/T)$ where the ground state energy at equilibrium is reached:

$$E_{gs}\simeq-J-\frac{g}{2J}. \tag{48}$$

Since we consider the regime $g\ll J$, this result is expected to be exact to first order in g/J. It does indeed agrees with the exact result of Ref. [19] concerning the bimodal distribution, expanded to first order in g. To obtain higher orders in the expansion in g/J one would need to compute within the RG higher orders in a $1/\Gamma$ expansion.

Note that the entropy per spin at $T=0$ computed in Ref. [19] for $\pm h$ distributions originates from degenerate configurations occuring from short scales and is thus nonuniversal. If the distribution is continuous we expect $S\sim T$ from short scales, also nonuniversal.

3. Distribution of lengths of domains

Since the bond lengths in the renormalized landscape are uncorrelated, we obtain the result that the lengths of the domains in the RFIM (both in the long time dynamics and at equilibrium) are *independent random variables*. (Note that this is different from the exact result for the dynamics of the *pure* Ising chain obtained in Ref. [41].)

Moreover, during the coarsening process, the probability distribution of the rescaled length $\lambda=2gl/\Gamma^2=2gl/T^2\ln^2 t$ is obtained as

$$P^*(\lambda)=LT^{-1}_{p\to\lambda}\left(\frac{1}{\cosh(\sqrt{p})}\right) \tag{49}$$

$$=\sum_{n=-\infty}^{\infty}\left(n+\frac{1}{2}\right)\pi(-1)^n e^{-\pi^2\lambda[n+(1/2)]^2}$$

$$=\frac{1}{\sqrt{\pi}\lambda^{3/2}}\sum_{m=-\infty}^{\infty}(-1)^m\left(m+\frac{1}{2}\right)\times e^{-(1/\lambda)[m+(1/2)]^2}. \tag{50}$$

The distribution of the length of domains at equilibrium is also given by $P(\lambda_{eq})$, where

$$\lambda_{eq}=\frac{2gl}{\Gamma_J^2}=\frac{l}{2L_{IM}}. \tag{51}$$

B. Spin glass in a field

Using the gauge transformation described in Sec. II B, the above results readily apply also to the spin glass in a field. Let us recall that a domain in the RFIM corresponds in the SG to an interval between two frustrated bonds. Using the above expressions for the zero applied field RFIM and replacing $g \to h^2$, we thus obtain the averaged size of these domains from Eq. (43), their number per unit length from Eqs. (44) and (45), and their distribution of lengths from Eqs. (49) and (51).

The distribution $P_\Gamma(M)$ of the magnetization $M = |\Sigma_{i \in \text{domain}} \sigma_i|$ of each domain is obtained from the distribution of barriers as

$$P_\Gamma(M) = \frac{1}{M_\Gamma} \exp\left(- \frac{M - M_\Gamma}{M_\Gamma} \right) \theta(M - M_\Gamma), \qquad (52)$$

$$M_\Gamma = \frac{\Gamma}{2h}, \qquad (53)$$

with $\Gamma = T \ln t$. In equilibrium the same result holds with $M_\Gamma \to 2J/h$. Note that since this variable is proportional to the barrier, there are no domains of magnetization smaller than M_Γ. Similarly the joint distribution of magnetization and length is given by Eq. (40) in Laplace transform and rescaled variables $\eta = (M - M_\Gamma)/M_\Gamma$ and $\lambda = 2h^2 l/\Gamma^2$, with again the property of statistical independence of domains. Finally, the energy per spin is given by expression (47), replacing $g \to h^2$.

Note that the results here are, strictly speaking, restricted to the case in which the mapping to a random field Ising model is exact: the case in which all of the exchange interactions have the same magnitude and only differ in sign. Nevertheless, in the more general case with a distribution of $|J|$'s, the universal aspects of the nonequilibrium behavior will be the same as long as there is a nonzero lower bound to this distribution, $|J|_{min}$. However, at times longer than $T \ln t = |J|_{min}$, domain walls will no longer necessarily be annihilated; whether they are or not will depend on the local J as well as on the renormalized potential. This will, of course, also affect the equilibrium positions of domain walls, but because there will always tend to be weak exchanges near to the extrema of the potential caused by the random fields, the changes in the positions of the walls will be negligible on the scale of the correlation length.

C. RFIM in an applied field

In a similar manner to the above analysis, we obtain results when a small uniform field H is applied. Solutions (37) of the RG equations now depend on the parameter δ defined as the nonvanishing root of Eq. (34), equal to $\delta = H/(2g)$ in the small field limit for which our results will be asymptotically exact.

1. Number of domain walls and magnetization

The averaged sizes of domains ($\mp$) (with spins oriented respectively against and along the field) are found both to grow with time as

$$\overline{l_\Gamma^-} = \Gamma^2 \frac{1}{4\gamma g} \left(1 - \frac{\sinh(\gamma)}{\gamma} e^{-\gamma} \right), \qquad (54)$$

$$\overline{l_\Gamma^+} = \Gamma^2 \frac{1}{4\gamma g} \left(e^{\gamma} \frac{\sinh(\gamma)}{\gamma} - 1 \right), \qquad (55)$$

where $\gamma = \Gamma\delta = \delta T \ln t$ and $\Gamma = T \ln t$. In the long time limit it is these are

$$\overline{l_\Gamma^-} \approx \frac{T \ln t}{2H}, \qquad (56)$$

$$\overline{l_\Gamma^+} \approx \frac{g}{2H^2} t^{TH/g}. \qquad (57)$$

Note that the fact that the length of domains with spins in the opposite direction from the applied field still grow, on average, is simply due to the fact that the smallest ones (with barriers smaller than $T \ln t$) keep being eliminated.

The number of domain walls per unit length thus decays as

$$n(t) \simeq \frac{2}{\overline{l_\Gamma^+} + \overline{l_\Gamma^-}} = 4g \frac{\delta^2}{\sinh^2(\delta T \ln t)}, \qquad (58)$$

and the magnetization per spin grows as

$$m(t) = \frac{\overline{l_\Gamma^+} - \overline{l_\Gamma^-}}{\overline{l_\Gamma^+} + \overline{l_\Gamma^-}} = \mathcal{M}\left[\frac{H}{2g} T \ln t \right], \qquad (59)$$

$$\mathcal{M}[\gamma] = \coth(\gamma) - \frac{\gamma}{\sinh(\gamma)^2}. \qquad (60)$$

The function $\mathcal{M}[\gamma]$ starts as $\mathcal{M}[\gamma] \sim \frac{2}{3}\gamma$ for small γ, and goes exponentially to $\mathcal{M} = 1$ for large γ.

These results hold up to time $t_{eq} \sim \exp(4J/T)$ at which equilibrium is reached. The number of domain walls per unit length in the equilibrium state is thus

$$n_{eq} \simeq 4g \frac{\delta^2}{\sinh^2(4\delta J)}, \qquad (61)$$

with averaged sizes

$$\overline{l_{eq}^-} \approx \frac{2J}{H}, \qquad (62)$$

$$\overline{l_{eq}^+} \approx \frac{g}{2H^2} e^{4JH/g}. \qquad (63)$$

The equilibrium magnetization per spin m_{eq} is

$$m_{eq} = m(t_{eq}) = \mathcal{M}\left[\frac{2JH}{g} \right]. \qquad (64)$$

2. Energy per spin

The energy per spin as a function of time (i.e., of $\Gamma = T \ln t$) is simply

$$E_\Gamma = -J - H + n_\Gamma \left(2J - \frac{1}{2} \langle F \rangle_- \right) \qquad (65)$$

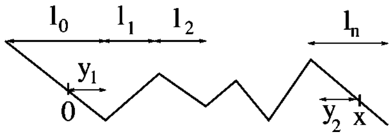

FIG. 4. Renormalized landscape at scale $\Gamma = T\ln t$, indicating sites 0 and x of two spins and the bonds between them.

$$= -J - H + \frac{4\delta^2 g}{\sinh^2(\delta\Gamma)}\left[2J - \frac{1}{2}\left(\Gamma + \frac{1}{2\delta}(1 - e^{-2\Gamma\delta})\right)\right], \tag{66}$$

where $\langle F\rangle_-$ denotes the averaged barrier of bonds against the field at scale Γ. This obtains up to scale $\Gamma_{eq} = 4J$, at which equilibrium is reached with ground state energy

$$E_{gs} = -J - H - \frac{\delta g}{\sinh^2(4J\delta)}(1 - e^{-8J\delta}). \tag{67}$$

This result is compatible with the result Eq. (80) of Derrida-Hilhorst [18] obtained from studying products of random matrices, as can be checked with the correspondance $\alpha_{DH} = 2\delta$. In addition, here we obtain the explicit scaling form in the small δ limit with δJ fixed. We have checked that this scaling form is also consistent with the exact result [Eq. (8) in Ref. [19]] for the bimodal distribution at leading order in δ.

3. Distribution of domain lengths $P^{\pm}_{\Gamma}(l)$

As in the zero field case, the lengths of the domains are independent random variables. Their probability distributions can be obtained from Laplace inversion of Eq. (37). For $\pm$ domains they read

$$P^{\pm}(l) = \sum_{n=0}^{+\infty} c_n^{\pm}(\gamma) s_n^{\pm}(\gamma) e^{-l s_n^{\pm}(\gamma)}, \tag{68}$$

where $\gamma = \delta T\ln t$, and the functions $c_n^{\pm}(\gamma)$ and $s_n^{\pm}(\gamma)$ are given in Eqs. (50)–(53) of Ref. [28].

VI. EQUAL-TIME TWO-SPIN CORRELATION FUNCTION $\overline{\langle S_0(T)S_x(T)\rangle}$ IN THE RFIM

We can compute the disorder averaged two spin correlation function by noting that, in a given environment, the equal-time two-spin thermal correlation $\langle S_0(t)S_x(t)\rangle$ is equal to $+1$ if the points 0 and x are at scale $\Gamma = T\ln t$ on renormalized bonds of the same orientation (i.e. both ascending or both descending), and is equal to -1 otherwise.

A. Zero applied field

The average over the environments is, in zero field $H = 0$,

$$\overline{\langle S_0(t)S_x(t)\rangle} = \sum_{n=0}^{\infty} (-1)^n Q_\Gamma^{(n)}(x), \tag{69}$$

where $Q_\Gamma^{(n)}(x)$ is defined as the probability that point x belongs to the bond n given that the point 0 belongs to the bond 0 of the renormalized landscape (see Fig. 4). We have the normalization $\Sigma_{n=0}^{\infty} Q_\Gamma^{(n)}(x) = 1$. For $n = 0$, the probability that x is on the same renormalized bond as 0 is

$$Q_\Gamma^{(0)}(x) = \frac{1}{\overline{l_\Gamma}}\int_0^\infty dl_0 P_\Gamma(l_0)\int_0^{l_0} dy_1 \int_{y_1}^{l_0} dy_2\, \delta(x - (y_2 - y_1)), \tag{70}$$

and thus, after rescaling the variables, we find

$$Q_\Gamma^{(0)}(x) = q_0\left(X = 2g\frac{x}{\Gamma^2}\right), \tag{71}$$

where

$$q_0(X) = 2\int_X^\infty d\lambda_0 (\lambda_0 - X) P^*(\lambda_0) \tag{72}$$

in terms of the fixed point solution $P^*(\lambda)$ given in Eq. (49) for the distribution of rescaled lengths $\lambda = 2gl/\Gamma^2$. For $n \geq 1$, we have

$$Q_\Gamma^{(n)}(x) = q_n\left(X = 2g\frac{x}{\Gamma^2}\right), \tag{73}$$

with

$$q_n(X) = 2\int_{y_1, y_2, \lambda_i > 0} P^*(\lambda_0)P^*(\lambda_1)\cdots P^*(\lambda_{n-1})P^*(\lambda_n) \tag{74}$$

$$\delta(X - (y_1 + \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + y_2))\theta(\lambda_0 - y_1) \times \theta(\lambda_n - y_2). \tag{75}$$

The Laplace transforms read

$$q_0(s) = \int_0^\infty dX e^{-sX} q_0(X) = \frac{2}{s^2}\left(P^*(s) - 1 + \frac{s}{2}\right), \tag{76}$$

$$q_n(s) = \int_0^\infty dX e^{-sX} q_n(X) = 2\left(\frac{1 - P^*(s)}{s}\right)^2 [P^*(s)]^{n-1} \quad \text{for} \quad n \geq 1, \tag{77}$$

and thus

$$\sum_{n=0}^{\infty} (-1)^n q_n(s) = \frac{1}{s} - \frac{4}{s^2}\left(\frac{1 - P^*(s)}{1 + P^*(s)}\right). \tag{78}$$

Now using the explicit solution $P^*(s) = 1/\cosh(\sqrt{s})$, we finally obtain

$$\overline{\langle S_0(t)S_x(t)\rangle}=LT^{-1}_{s\to X=2g\,(|x|/\Gamma^2)}\left[\frac{1}{s}-\frac{4}{s^2}\tanh^2\left(\frac{\sqrt{s}}{2}\right)\right] \quad (79)$$

$$=\sum_{n=-\infty}^{\infty}\frac{48+64(2n+1)^2\pi^2 g\frac{|x|}{\Gamma^2}}{(2n+1)^4\pi^4}\times e^{-(2n+1)^2\pi^2 2g(|x|/\Gamma^2)} \quad (80)$$

with $\Gamma=T\ln t$. The leading long-distance behavior of the equal time correlation function is proportional to $xe^{-x/\xi(t)}$ rather than a simple exponential. The Fourier transform of the spin-spin correlation function is simply

$$\sum_{x=-\infty}^{+\infty}e^{iqx}\overline{\langle S_0(t)S_x(t)\rangle}=\frac{8}{\pi^2q^2\xi(t)}\,\mathrm{Re}[\tanh^2(\pi\sqrt{iq\xi(t)}/2)], \quad (81)$$

with Re denoting the real part. As explained previously, the renormalization procedure has to be stopped at $\Gamma\simeq\Gamma_J=4J$, i.e., $t\sim t_{eq}\sim\exp(4J/T)$ where the equilibrium state has been reached. Thus Eq. (80) with $\Gamma=\Gamma_J$ gives the mean equilibrium spin correlation function. The correlation length $\xi(t)$ is given by the decay of the $(n=0)$ term which dominates at large distances

$$\xi(t)=\frac{T^2\ln^2 t}{2g\pi^2} \quad (82)$$

up to scale $T\ln t_{eq}=\Gamma_J=4J$, and we obtain the correlation length at equilibrium:

$$\xi_{eq}=\xi(t_{eq})=\frac{8J^2}{\pi^2 g}=\frac{2}{\pi^2}L_{IM}. \quad (83)$$

This formula is in agreement with the limit $h\ll J$ of the exact result for the equilibrium correlation of Ref. [20] in the case of a bimodal distribution $(\pm h)$. Finally, note that the RFIM two-point correlation function $\overline{\langle S_i(t)S_{i+x}(t)\rangle}$ also corresponds for the spin-glass to the following correlation function involving the zero-field $T=0$ ground state $\sigma_i^{(0)}$:

$$\overline{\langle S_i(t)S_{i+x}(t)\rangle}=\overline{\sigma_i^{(0)}\sigma_{i+x}^{(0)}\langle\sigma_i(t)\sigma_{i+x}(t)\rangle}. \quad (84)$$

B. Nonzero applied field

In the presence of an applied field $H>0$, one defines $C_t^{\epsilon_1,\epsilon_2}(x)$ as the probability that $\mathrm{sgn}[S_0(t)]=\epsilon_1$ and $\mathrm{sgn}[S_x(t)]=\epsilon_2$. One has

$$\overline{\langle S_0(t)S_x(t)\rangle}=C_t^{++}(x)+C_t^{--}(x)-2C_t^{+-}(x), \quad (85)$$

We know already that

$$C_t^{++}(x)+C_t^{+-}(x)=\mathrm{Prob}\{\mathrm{sgn}[S_0(t)]=+1\}=\frac{\overline{l_\Gamma^+}}{\overline{l_\Gamma^+}+\overline{l_\Gamma^-}}, \quad (86)$$

$$C_t^{--}(x)+C_t^{+-}(x)=\mathrm{Prob}\{\mathrm{sgn}[S_0(t)]=-1\}=\frac{\overline{l_\Gamma^-}}{\overline{l_\Gamma^+}+\overline{l_\Gamma^-}}, \quad (87)$$

and we can thus write the correlation function as

$$\overline{\langle S_0(t)S_x(t)\rangle}-\overline{\langle S_0(t)\rangle}\;\;\overline{\langle S_x(t)\rangle}=1-4C_t^{+-}(x)-\left(\frac{\overline{l_\Gamma^+}-\overline{l_\Gamma^-}}{\overline{l_\Gamma^+}+\overline{l_\Gamma^-}}\right)^2. \quad (88)$$

Performing an analysis similar to the case of zero field, we obtain the Laplace transform

$$\int_0^\infty dx e^{-qx}C_t^{+-}(x)=\frac{(1-P_\Gamma^+(q))(1-P_\Gamma^-(q))}{(\overline{l_\Gamma^+}+\overline{l_\Gamma^-})q^2(1-P_\Gamma^+(q)P_\Gamma^-(q))}. \quad (89)$$

The Laplace inversion can be performed as in Ref. [28]. The correlation decays as a sum of exponentials, and the term which dominates the asymptotic decay gives a correlation length

$$\xi(t)=\frac{\Gamma^2}{2gs_0^+(\gamma)}, \quad (90)$$

where $\Gamma=T\ln t$, $\gamma=\delta T\ln t$, and the function $s_0^+(\gamma)$ is defined by Eqs. (50) and (52) in Ref. [28]. In particular the asymptotic behavior for large γ is

$$\xi(t)\sim\overline{l_\Gamma^+}\sim\frac{g}{2H^2}t^{HT/g}. \quad (91)$$

Note that even for the equilibrium case $(\gamma=\delta\Gamma_J=4\delta J)$, this correlation length

$$\xi_{eq}\sim\frac{g}{2H^2}e^{4HJ/g} \quad (92)$$

does not seem to have been obtained previously; it is very different from the correlation length of the *truncated* correlations—i.e., that of the thermal fluctuations—computed for bimodal distribution in [20] and discussed here in Sec. VIII A.

VII. AGING AND TWO-TIME CORRELATIONS

Some of the most interesting properties of random systems involve ‘‘aging,’’ the dependence of measured quantities on the history of the system, particularly on how long it has been equilibrated for. In this section we study one of the fundamental properties which show the effects of aging: two-time nonequilibrium correlations. As before, the system is quenched from a random initial condition at time $t=0$, and we study the aging dynamics at late times between time $t'=t_w$—the waiting time—and t, both t' and $t>t'$ being large.

We first consider the autocorrelation function of a given spin, from which one can extract the autocorrelation exponent λ. Note that since this is a single site quantity it applies directly to both the random field and spin glass problems.

A. Spin autocorrelations in zero applied field

We consider the autocorrelations of the random field Ising model in zero applied field, $H=0$:

$$C(t,t')=\overline{\langle S_i(t)S_i(t')\rangle}. \tag{93}$$

Except at short times the system is in the ''full'' state, and hence $C(t,t')$ is simply the probability that the site i—which we take to be the origin—belongs at both time t' and time t to renormalized bonds with the same orientation. Thus this quantity can in principle be obtained from the result in Ref. [28] for the probability $\overline{P(x,t;x't'|0,0)}$ that a particle diffusing in a Sinai landscape starting at 0 at time $t=0$ is at x' at t' and x at t. Indeed $C(t,t')$ (since it is computed in the full state) is simply related to the probability that a particles postitions $X(t)$ and $X(t')$ have the same sign. However, it can also be obtained through a much simpler direct computation, which we now present.

We define $P^{++}_{\Gamma,\Gamma'}(\zeta)$. ($P^{-+}_{\Gamma,\Gamma'}(\zeta)$) as the probability that the origin is on a descending bond at Γ', and is on a descending (ascending) bond of strength ζ at a later stage, Γ. The RG equations read

$$\begin{aligned}(\partial_\Gamma-\partial_\zeta)P^{\pm+}_{\Gamma,\Gamma'}(\zeta)=&-2P^{\mp}_{\Gamma}(0)P^{\pm+}_{\Gamma,\Gamma'}(\zeta)\\&+2P^{\mp}_{\Gamma}(0)P^{\mp}_{\Gamma}(\cdot)*_\zeta P^{\pm+}_{\Gamma,\Gamma'}(\cdot)\\&+P^{\pm}_{\Gamma}(\cdot)*_\zeta P^{\pm}_{\Gamma}(\cdot)P^{\mp+}_{\Gamma,\Gamma'}(0)\end{aligned} \tag{94}$$

together with the initial conditions

$$P^{++}_{\Gamma',\Gamma'}(\zeta)=\frac{1}{2}\int_0^\infty dl\,\frac{lP_{\Gamma'}(\zeta,l)}{\overline{l}_{\Gamma'}}, \tag{95}$$

$$P^{-+}_{\Gamma',\Gamma'}(\zeta)=0. \tag{96}$$

We introduce the scaling variable

$$\alpha\equiv\frac{\Gamma}{\Gamma'}=\frac{\ln t}{\ln t'}. \tag{97}$$

Since for large Γ', $P_{\Gamma'}$ has reached its fixed point value [Eq. (40)], in terms of the rescaled variable $\eta=\zeta/\Gamma$ one has

$$\begin{aligned}&(\alpha\partial_\alpha-(1+\eta)\partial_\eta+1)P^{\pm+}_{\alpha}(\eta)\\&=2\int_0^\eta d\eta' e^{-(\eta-\eta')}P^{\pm+}_{\alpha}(\eta')+\eta e^{-\eta}P^{\mp+}_{\alpha}(0),\end{aligned} \tag{98}$$

together with initial conditions at $\alpha=1$,

$$P^{++}_{\alpha=1}(\eta)=\frac{1}{2}\,\frac{\int_0^\infty d\lambda\,\lambda P^*(\eta,\lambda)}{\overline{\lambda}}=\frac{1}{6}(1+2\eta)e^{-\eta}, \tag{99}$$

and $P^{-+}_{\alpha=1}(\eta)=0$. The solutions are

$$P^{\pm+}_{\alpha}(\eta)=\frac{1}{2}(A^{\pm+}_{\alpha}+\eta B^{\pm+}_{\alpha})e^{-\eta}, \tag{100}$$

with

$$A^{\pm+}_{\alpha}=\frac{1}{6}\pm\frac{1}{3\alpha}\mp\frac{1}{6\alpha^2}, \tag{101}$$

$$B^{\pm+}_{\alpha}=\frac{1}{3}\pm\frac{1}{3\alpha}, \tag{102}$$

which obey the normalization condition $\int_0^\infty d\eta(P^{++}_{\alpha}(\eta)+P^{-+}_{\alpha}(\eta))=\frac{1}{2}$. Since with $H=0$ we have $P^{\pm-}_{\alpha}(\eta)=P^{\mp+}_{\alpha}(\eta)$, we obtain

$$C(t,t')=\int_0^\infty d\eta(P^{++}_{\alpha}(\eta)+P^{+-}_{\alpha}(\eta)-P^{+-}_{\alpha}(\eta)-P^{-+}_{\alpha}(\eta)) \tag{103}$$

$$=A^{2+}_{\alpha}+B^{2+}_{\alpha}-A^{-+}_{\alpha}-B^{-+}_{\alpha}=\frac{4}{3\alpha}-\frac{1}{3\alpha^2}, \tag{104}$$

and thus the autocorrelation function of the RFIM in zero applied field at large times $t\geqslant t'$ is

$$\overline{\langle S_i(t)S_i(t')\rangle}=\frac{4}{3}\left(\frac{\ln t'}{\ln t}\right)-\frac{1}{3}\left(\frac{\ln t'}{\ln t}\right)^2. \tag{105}$$

In particular the asymptotic behavior for fixed t' is

$$\overline{\langle S_i(t)S_i(t')\rangle}\propto\left(\frac{\overline{l}(t')}{\overline{l}(t)}\right)^\lambda, \tag{106}$$

where $\overline{l}(t)\sim\ln^2 t$ is the characteristic length of the coarsening in the RFIM, and where the *autocorrelation exponent* is

$$\lambda=\tfrac{1}{2}. \tag{107}$$

The autocorrelation being invariant under gauge transformations, for the spin glass we immediately obtain

$$\overline{\langle\sigma_i(t)\sigma_i(t')\rangle}=\overline{\langle S_i(t)S_i(t')\rangle}\sim\left(\frac{\overline{l}(t')}{\overline{l}(t)}\right)^\lambda\quad\text{with}\quad\lambda=\frac{1}{2}. \tag{108}$$

Note that this value of λ saturates the lower bound of $d/2$ in contrast to the pure 1D Ising case which saturates the upper bound of $\lambda=d$ [5].

B. Autocorrelations for the RFIM in an applied field

In the presence of an applied field $H>0$ the calculation is similar to the above; it is detailed in Appendix B. Using the scaling variables

$$\gamma=\delta\Gamma\approx\frac{H}{2g}T\ln t, \tag{109}$$

$$\gamma'=\delta\Gamma'\approx\frac{H}{2g}T\ln t' \tag{110}$$

for small H, and the magnetization per spin,

$$\overline{\langle S_i(t)\rangle}=m(t)=\mathcal{M}(\gamma)=\coth\gamma-\frac{\gamma}{\sinh^2\gamma}, \tag{111}$$

the result for the autocorrelation function is

$$\overline{\langle S_i(t')S_i(t)\rangle}-\overline{\langle S_i(t)\rangle}\quad\overline{\langle S_i(t')\rangle}=\frac{1}{\sinh^2\gamma}((2\gamma-\gamma')\mathcal{M}(\gamma')+\gamma'\coth\gamma'-1). \tag{112}$$

The long time asymptotic behavior is

$$\overline{\langle S_i(t')S_i(t)\rangle}-\overline{\langle S_i(t)\rangle}\quad\overline{\langle S_i(t')\rangle}=\frac{4HT\ln t}{gt^{HT/g}}m(t'). \tag{113}$$

This result is valid, strictly, to leading order in δT. In general the exponent for the power law decay in time which occurs here and for other quantities in the presence of a uniform applied field will have $O(\delta^2T^2)$ corrections.

C. Two-point two-time correlation function for the spin glass

In order to characterize the spatial aspects of aging dynamics in the spin glass we have computed the correlation function

$$\overline{\langle S_0(t)S_x(t)\rangle\langle S_0(t')S_x(t')\rangle}=F\left(\frac{x}{\Gamma'^2};\frac{\Gamma}{\Gamma'}\right), \tag{114}$$

with $t>t'$; this becomes, at long times, a scaling function of

$$X\equiv\frac{x}{T^2\ln^2 t'} \tag{115}$$

and

$$\alpha=\ln t/\ln t'\geqslant 1. \tag{116}$$

We compute the scaling function $F[X,\alpha]$ in Appendix C. For simplicity we have set $2g=1$. In Laplace transform variables we have

$$\hat{F}[p,\alpha]=\int_0^\infty dXe^{-pX}F(X;\alpha)=\frac{1}{p}-\frac{4}{p^2}\tanh^2\left(\frac{\sqrt{p}}{2}\right)+\frac{2}{\alpha^2p^3\sinh^2(\sqrt{p})}\coth^2\left(\frac{\alpha\sqrt{p}}{2}\right)[8+3p-16\cosh(\sqrt{p})+8\sqrt{p}\sinh(\sqrt{p})+(8+5p)\cosh(2\sqrt{p})-12\sqrt{p}\sinh(2\sqrt{p})]-\frac{16}{\alpha^2p^2}\frac{\coth\left(\frac{\alpha\sqrt{p}}{2}\right)}{\sqrt{p}}\times(1-\sqrt{p}\coth(\sqrt{p}))\left(\frac{2}{\sqrt{p}}\tanh\left(\frac{\sqrt{p}}{2}\right)-1\right)-\frac{16}{\alpha^2p^3\sinh^2\left(\frac{\alpha\sqrt{p}}{2}\right)}(1-\sqrt{p}\coth(\sqrt{p}))^2. \tag{117}$$

Because the change of correlations between two points is caused only by passing of domain walls through one of the end points, the two-time correlation function decays at large x to the square of the autocorrelation function (105), i.e., $\lim_{p->0}p\hat{F}[p,\alpha]=(4\alpha-1)^2/(9\alpha^4)$. The spatial decay to this constant value is determined by the closest poles to the imaginary axis in the complex p plane. This yields an exponential decay with a characteristic length which is the maximum of $\xi(t')=(\Gamma'/\pi)^2$ and $\xi(t)/4=(\Gamma/2\pi)^2$. In the regime $\alpha\sim 1$, corresponding to $\ln t\approx\ln t'$, we have the following expansion to order $O(\alpha-1)$:

$$\int_0^\infty dXe^{-pX}F(X;\alpha)=\frac{1}{p}-(\alpha-1)\frac{8}{p^2}\left(1-\frac{\sqrt{p}}{\sinh(\sqrt{p})}\right). \tag{118}$$

Note that in experiments one could in principle measure

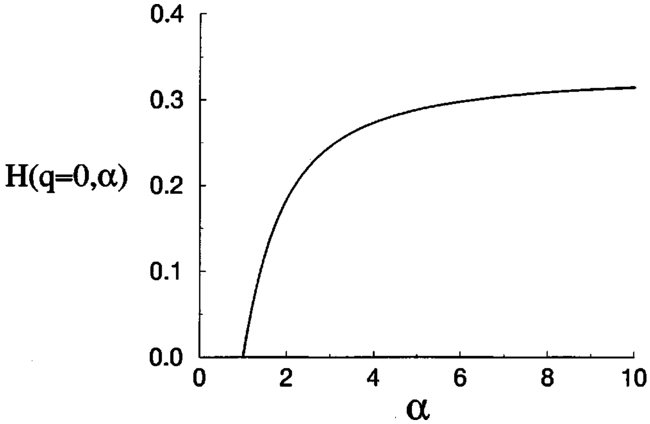

FIG. 5. Value at $q=0$ of the Fourier transform $\hat{H}(q=0,\alpha)$ (y axis) as a function of $\alpha = \ln t/\ln t'$ (x axis).

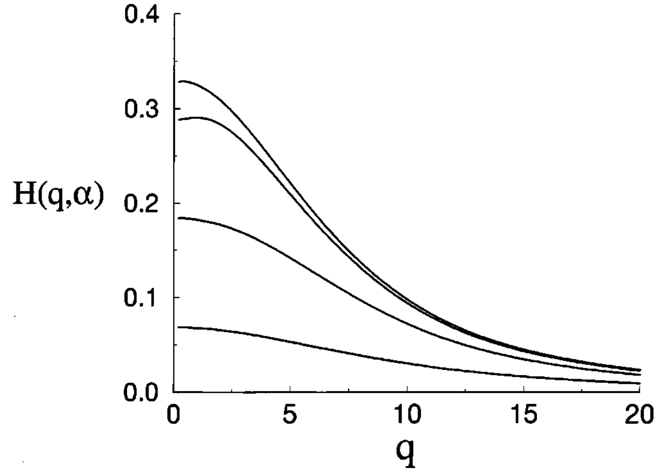

FIG. 6. Fourier transform $\hat{H}(q,\alpha)$ (y axis) of the scaling function for the two-point two-time spin glass correlation as a function of q (x axis) for four different values of $\alpha = 1.25$, 2, 5, and 20.

the Fourier transform which is related to the cross-correlations of the scattering speckle patterns at two different times. It reads

$$\sum_{x=-\infty}^{+\infty} \overline{\langle S_0(t)S_x(t)\rangle\langle S_0(t')S_x(t')\rangle} e^{iQx} = (T\ln t')^2 \hat{H}\left(q = Q(T\ln t')^2, \alpha = \frac{\ln t}{\ln t'} \right), \tag{119}$$

$$\hat{H}(q,\alpha) = 2\mathcal{R}(\hat{F}[p,\alpha] - (4\alpha-1)^2/(9\alpha^4 p))|_{p=iq}, \quad q\neq 0, \tag{120}$$

where $\mathcal{R}$ denotes the real part and the $1/p$ part has been substracted to get rid of the $\delta(q)$ part. The value of the Fourier transform as $q\to 0$ is

$$\hat{H}(q=0,\alpha) = \frac{-6+40\alpha-59\alpha^2-20\alpha^3+45\alpha^4}{135\alpha^4}, \tag{121}$$

which is plotted in Fig. 5. The Fourier transform is plotted in Fig. 6 for several values of α. Note the maximum at $q>0$ which develops for large α and which is related to the non-monotonic behavior of the correlation as a function of x. The fact that the above correlation indeed reaches its limit by below can be seen as follows for large α. For $x \ll \Gamma^2$, 0 and x belong to the same domain, and thus the above two points two time correlation is approximately equal to $\overline{\langle S_0(t')S_x(t')\rangle}$. However, this decays exponentially with x/Γ'^2 (and thus exponentially in α^2 if one chooses x/Γ^2 fixed but very small) while the asymptotic value [Eq. (105)] decays only algebraically.

VIII. RARE EVENTS, TRUNCATED CORRELATIONS AND RESPONSE TO A FIELD

In this section we compute time dependent thermal (truncated) correlations as well as the response to a uniform field applied at time t_w. For this one needs to go beyond the effective dynamics, which just places each wall at a specific location at each time. Indeed, in the effective dynamics, the local magnetization (i.e., the thermal average) $\langle S_x(t)\rangle$ at a given point x is given by the orientation of the renormalized bond containing the point x at scale Γ, and is thus either $+1$ or -1. Thus truncated correlations are zero to leading order, and to estimate them one needs to consider the rare events in which a domain wall can be found with substantial probabilities at two different positions. Such events occur with a probability $1/\Gamma$, and the two positions of the domain wall when they do occur are typically separated by distance of order Γ^2. For example, for the single point Edwards-Anderson order parameter these lead to corrections to the zero temperature value of unity of order

$$1 - \overline{(\langle S_x(t)\rangle)^2} \propto \frac{1}{\ln t}. \tag{122}$$

In this section, to simplify the notation somewhat, we set $2g=1$.

A. Description of the important rare events

The rare events that are important for the RFIM turn out to be the same (as far as the energy landscape is concerned) as the ones that we considered in our previous study of the aging properties of the Sinai model [28], with a slightly different physical interpretation and different observables to be computed. There are two types [42] of such rare events that occur with probability $1/\Gamma$, denoted (a) and (c) in Ref. [28]. We now describe them.

Events (a) concern bonds which contain two almost degenerate extrema (see Fig. 7). In the following, we will use the probability $D^{(a)}_{\Gamma,\Gamma'}(x)$ that two points 0 and $x>0$ belong to a renormalized bond at Γ which has two degenerate extrema separated by a barrier smaller than Γ' such that the two points are both located between the two degenerate extrema. This is, using the calculations of Ref. [28],

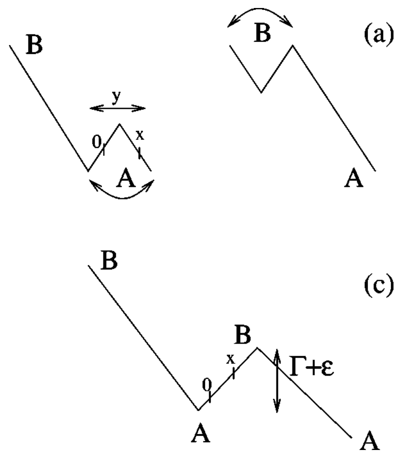

FIG. 7. Rare events (a) and (c) discussed in the text

$$D^{(a)}_{\Gamma,\Gamma'}(x)=\frac{4}{\Gamma^2}\int_{l>y,y>x}P_\Gamma(l-y)(y-x)\int_0^{\Gamma'}d\Gamma_0\frac{1}{\Gamma_0^4}\hat{r}\left(\frac{y}{\Gamma_0^2}\right) \tag{123}$$

$$=\frac{4}{\Gamma^2}\int_{y>x}(y-x)\frac{1}{\Gamma'^3}G\left(\frac{y}{\Gamma'^2}\right)$$

$$=\frac{16\Gamma'}{\Gamma^2}\sum_{k=1}^{+\infty}\frac{1}{k^2\pi^2}e^{-(x/\Gamma'^2)k^2\pi^2}, \tag{124}$$

where the functions $\hat{r}(Y)$ and $G(Y)$ were defined in Eqs. (E5-8) and (174) of Ref. [28].

In event (a), the corresponding domain wall (A or B) will fluctuate thermally between the two extrema (see Fig. 7). At equilibrium, the thermal probabilities of finding the domain wall in the two positions are p and $(1-p)$ with $p=1/(1+e^{-u/T})$ and u representing the energy difference between the two minima. The random variable u is distributed uniformly around $u=0$. In the following, we will need

$$c_n(T)=\overline{(4p(1-p))^n}=\frac{1}{2}T\,4^n\int_0^{+\infty}dz\frac{z^{n-1}}{(1+z)^{2n}}$$

$$=T\sqrt{\pi}\,\frac{\Gamma[n]}{\Gamma\left[n+\frac{1}{2}\right]}, \tag{125}$$

the factor 1/2 arises as the integral is only [43] over $u>0$.

Events (c) correspond to bonds about to be decimated, with a barrier $\Gamma+\epsilon\approx\Gamma$. The probability $D^{(c)}_\Gamma(x)$ that the segment $[0,x]$, $x>0$ belongs at scale Γ to a bond of barrier Γ (i.e., $\zeta=0$):

$$D^{(c)}_\Gamma(x)=\frac{2}{\Gamma^2}\int_{l>x}P_\Gamma(\zeta=0,l)(l-x)$$

$$=\frac{2}{\Gamma}\int_0^{+\infty}d\lambda\,\lambda P^*\left(\eta=0,\lambda+\frac{x}{\Gamma^2}\right) \tag{126}$$

$$=\frac{4}{\Gamma}\sum_{k=1}^{+\infty}(-1)^{k+1}\frac{1}{k^2\pi^2}e^{-(x/\Gamma^2)k^2\pi^2}. \tag{127}$$

For events (c) associated with a barrier $\Gamma+\epsilon$, the probability p_c that the two corresponding domain walls have not yet annihilated at Γ is given by $p_c=\exp(-e^{-\epsilon/T})$. In the following, we will need

$$d_n(T)=\overline{(4p_c(1-p_c))^n}=T\,4^n\int_0^{+\infty}\frac{dz}{z}e^{-nz}(1-e^{-z})^n$$

$$=T2^{2n}\sum_{k=0}^{n-1}C^k_{n-1}\ln\left(1+\frac{1}{k+n}\right), \tag{128}$$

where we have used that the distribution in ϵ is uniform around $\epsilon=0$.

B. Two point truncated equal time correlations

1. Nonequilibrium behavior

We consider the various moments of the truncated equal time correlation function defined as

$$C_n(x,t)=\overline{[\langle S_0(t)S_x(t)\rangle-\langle S_0(t)\rangle\langle S_x(t)\rangle]^n}; \tag{129}$$

$C_n(x,t)$ is the sum of two contributions. The first contribution $C^{(a)}_n(x,t)$ originates from events (a) described above, where we have $\langle S_0(t)S_x(t)\rangle=+1$, while $\langle S_0(t)\rangle=\langle S_x(t)\rangle=\pm(1-2p)$ because the domain wall fluctuates between the two extrema (see Fig. 7). We thus obtain the first contribution as

$$C^{(a)}_n(x,t)=c_n(T)D^a_{\Gamma,\Gamma}(x), \tag{130}$$

with $\Gamma=T\ln t$.

The second contribution $C^{(c)}_n(x,t)$ originates from events (c) described above, where we have $\langle S_0(t)S_x(t)\rangle=+1$ while $\langle S_0(t)\rangle=\langle S_x(t)\rangle=\pm(1-2p_c)$. The second contribution is thus given by

$$C^{(c)}_n(x,t)=d_n(T)D^c_\Gamma(x). \tag{131}$$

The final result for the moments of the truncated equal time correlations is thus

$$C_n(x,t)=\frac{4}{T\ln t}\sum_{k=1}^{+\infty}\frac{(4c_n(T)+(-1)^{k+1}d_n(T))}{k^2\pi^2}\times e^{-|x|k^2\pi^2/(T\ln t)^2}, \tag{132}$$

where $c_n(T)$ and $d_n(T)$ are given in Eqs. (125) and (128). Note that to this order in $1/\Gamma$ the n dependence is only contained in the prefactor and in particular the correlation length $\xi_{th}(t)=(T\ln t)^2/\pi^2$ extracted from Eq. (132) does *not* depend on n and is equal to result (82). For the spin glass, the above formula gives the *even* moments $C_{2n}(x,t)$. For the following sections, we will need

$$C_1(t)=\sum_{x=-\infty}^{x=+\infty} C_1(x,t)=\left(\frac{32}{45}+\frac{14}{45}\ln 2\right)T^2 \ln t. \quad (133)$$

2. Equilibrium truncated correlations

To compute the equilibrium truncated correlations the method is very similar. One must stop the RG at scale $\Gamma_J=4J$, and again consider events (a) (which give the same contribution as above replacing Γ by Γ_J) and events (c) but with a different interpretation and result since there are now at equilibrium. Indeed in the renormalized landscape at scale Γ_J the only thermal fluctuations [apart from the (a) events] come from barriers $\Gamma=\Gamma_J+\epsilon\approx\Gamma_J$. The barriers much larger than Γ_J are occupied by a pair of domains with a probability of almost 1, while the barriers well below Γ_J (which have been decimated at previous stages) are occupied with a probability of almost 0. The barriers with $\Gamma=\Gamma_J+\epsilon\approx\Gamma_J$ are occupied with a probability $p=1/(1+e^{-\epsilon/T})$. Thus we now have the equilibrium truncated correlations:

$$\begin{aligned}C_n^{(eq)}(x)&=c_n(T)(D^a_{\Gamma_J,\Gamma_J}(x)+D^c_{\Gamma_J}(x))\\&=\frac{1}{J}c_n(T)\sum_{k=1}^{+\infty}\frac{(4+(-1)^{k+1})}{k^2\pi^2}e^{-|x|k^2\pi^2/(4J)^2}.\end{aligned} \quad (134)$$

Here, as above, the correlation length $\xi_{th}^{eq}=16J^2/\pi^2$ extracted from Eq. (134) to this order in Γ_J does *not* depend on n. Since it was argued in Ref. [21] that the correlation lengths of the $C_n^{(eq)}(x)$ generically depend on n, our results suggest that here this dependence is subleading in Γ_J. Our result for the correlation length of $C_1^{(eq)}(t)$ coincides with the result of Ref. [20] and with result (83) with $2g=1$ [44]. However, the detailed form of the functions $C_n^{(eq)}(x)$ obtained here depends explicitly on n.

3. Approach to equilibrium for truncated correlations

Since the equal time result for $\Gamma<\Gamma_J=4J$ and the equilibrium result differ only by substituting $d_n(T)$ by $c_n(T)$ in the (c) events, there should be a nontrivial crossover near $\Gamma=\Gamma_J$ toward equilibrium controlled by events (c), which we now analyze. Let us consider a bond with barrier $F=\Gamma_J+\epsilon$. When $\Gamma=T\ln t$ is close to Γ_J this bond can be either occupied [with probability $p(t)$] or empty [with probability $1-p(t)$]. One has

$$\frac{dp}{dt}=\frac{1}{\tau_1}(1-p(t))-\frac{1}{\tau_2}p(t), \quad (135)$$

where $\tau_1=e^{\Gamma_J/T}$ is the inverse rate of creation of a pair of domain walls (which immediately migrate to the end points of the bond) and $\tau_2=e^{(\Gamma_J+\epsilon)/T}$ is the inverse rate of annihilation of the pair of domain walls located at the end points of the bond. Thus, substituting $t=e^{\Gamma/T}$, one finds

$$p(t)=\frac{1}{1+e^{-\epsilon/T}}+\frac{e^{-\epsilon/T}}{1+e^{-\epsilon/T}}\exp\left(-(1+e^{-\epsilon/T})\frac{t}{t_{eq}}\right), \quad (136)$$

where $\ln t_{eq}=\Gamma_J/T$, and one has $t/t_{eq}=e^{-(\Gamma_J-\Gamma)/T}$. Integrating over ϵ, one obtains that the crossover to equilibrium for $t\sim t_{eq}$ is described by

$$C_n(x,t)=C_n^{(a)}(x)+C_n^{(c)}(x), \quad (137)$$

$$C_n^{(a)}(x)=c_n(T)D^{(a)}_{\Gamma_J,\Gamma_J}(x), \quad (138)$$

$$C_n^{(c)}(x)=e_n(T,t)D^{(c)}_{\Gamma_J}(x), \quad (139)$$

with

$$\begin{aligned}e_n(T,t)&=\overline{4p(t)(1-p(t))}\\&=4^nT\int_0^{+\infty}dz\,\frac{z^{n-1}}{(1+z)^{2n}}\\&\quad\times(1-e^{-(1+z)t/t_{eq}})^n(1+ze^{-(1+z)t/t_{eq}})^n\end{aligned} \quad (140)$$

$$\begin{aligned}&=4^nT\sum_{k=0}^{n}\sum_{p=0}^{n}(-1)^kC_n^kC_n^p\Gamma[n+p]\\&\quad\times U\left(n+p,1+p-n,(k+p)\frac{t}{t_{eq}}\right)\\&\quad\times e^{-(k+p)(t/t_{eq})},\end{aligned} \quad (141)$$

where $\ln t_{eq}=4J/T$. This expression crosses over from $e_n(T,t/t_{eq}\ll 1)\to d_n(T)$ and $e_n(T,t/t_{eq}\gg 1)\to c_n(T)$.

C. Two-point two-time truncated correlations

We now consider the truncated two-point two-time correlations

$$C_n(x,t,t_w)=\overline{(\langle S_0(t)S_x(t_w)\rangle-\langle S_0(t)\rangle\langle S_x(t_w)\rangle)^n}. \quad (142)$$

The calculation is very similar to the equal-time truncated correlations. We first consider the events of type (a), where we have now to keep track of the barrier Γ_0 between the two almost degenerate extrema. There is a nonvanishing contribution if the barrier Γ_0 is smaller than Γ_w but larger than $\hat{\Gamma}=T\ln(t-t_w)$, so that equilibration cannot take place between t_w and t. In that case we have $\langle S_0(t_w)S_x(t)\rangle=+1$ while $\langle S_0(t_w)\rangle=\langle S_x(t)\rangle=\pm(1-2p)$, where $p=1/(1+e^{-u/T})$ as introduced above in the description of events of type (a). These events lead to the contribution

$$C_n^{(a)}(x,t,t_w)=c_n(T)\frac{4}{\Gamma^2}\int_{l>y,y>x}P_\Gamma(l-y)(y-x)\times\int_{\hat\Gamma}^{\Gamma_w}d\Gamma_0\frac{1}{\Gamma_0^4}\hat r\left(\frac{x}{\Gamma_0^2}\right) \quad (143)$$

$$=c_n(T)(D^a_{\Gamma,\Gamma_w}(x)-D^a_{\Gamma,\hat\Gamma}(x)), \quad (144)$$

where $c_n(T)$ and $D^a_{\Gamma,\Gamma_w}(x)$ are given above. Note that the contribution of these events vanishes when $\hat\Gamma$ becomes equal to Γ_w.

We now consider events of type (c), which give different contributions and must be examined separately, in the scaling regime $\hat\Gamma = T\ln(t-t_w)<T\ln t_w=\Gamma_w$ and the scaling regime $t-t_w\sim t_w$, i.e., $\hat\Gamma=\Gamma_w$.

We first consider $\hat\Gamma<\Gamma_w$. In that regime the events of type (c) should be considered at scale Γ_w where we have $\langle S_0(t_w)S_x(t)\rangle=+1$ while $\langle S_0(t_w)\rangle=\langle S_x(t)\rangle=\pm(1-2p)$ which, together with the (a) events, gives the total contribution

$$C_n(x,t,t_w)=C_n^{(a)}(x,t,t_w)+d_n(T)D^c_{\Gamma_w}(x); \quad (145)$$

this formula holds for $\hat\Gamma=T\ln(t-t_w)<T\ln t_w=\Gamma_w$.

In the regime $t-t_w\sim t_w$ i.e., $\hat\Gamma=\Gamma_w$, the (c) events also start to equilibriats, which we now study. Let $p_c(t_w)=\exp(e^{-\epsilon/T})$ the probability that the domain walls separated by the barrier $\Gamma_w+\epsilon$ have not yet annihilated at t_w. Let

$$p_c(t)=p_c(t_w)\exp\left(-\frac{t-t_w}{t_w}e^{-\epsilon/T}\right)=\exp\left(-\frac{t}{t_w}e^{-\epsilon/T}\right)$$

be the probability that they also have not yet annihilated at t. One has (with x and 0 belonging to the bond being decimated)

$$\langle S_x(t_w)\rangle=1-2p_c(t_w), \quad (146)$$

$$\langle S_x(t)\rangle=1-2p_c(t), \quad (147)$$

$$\langle S_0(t)S_x(t_w)\rangle=1-2p_c(t_w)+2p_c(t), \quad (148)$$

$$\langle S_0(t)S_x(t_w)\rangle-\langle S_0(t)\rangle\langle S_x(t_w)\rangle=4p_c(t)(1-p_c(t_w)). \quad (149)$$

Thus in the regime $t-t_w\sim t_w$, one obtains the total contributions

$$C_n(x,t,t_w)=d_n(T,t,t_w)D^c_{\Gamma_w}(x), \quad (150)$$

$$d_n(T,t,t_w))=\overline{(4p_c(t)1-p_c(t_w))^n}=T\,4^n\int_0^{+\infty}\frac{dz}{z}e^{-nt/t_wz}(1-e^{-z})^n. \quad (151)$$

In the limit $t\gg t_w$ these truncated correlations decay, for fixed $x/(T\ln t_w)^2$, as $C_n(x,t,t_w)\sim(t_w/t)^n$. They decay to zero as there are no other contributions for later time t. In the following we will need

$$C_1(t,t_w)=\sum_{x=-\infty}^{x=+\infty}C_1(x,t,t_w) \quad (152)$$

$$=\frac{32}{45}T^2\left(\ln t_w-\frac{(\ln(t-t_w))^3}{(\ln t_w)^2}\right)+\frac{14}{45}T^2\ln 2\ln t_w$$

$$\text{for}\quad 0<\frac{\ln(t-t_w)}{\ln t_w}<1 \quad (153)$$

$$=\frac{14}{45}T^2\ln t_w\ln\left(1+\frac{t_w}{t}\right)\quad\text{for}\quad\frac{\ln(t-t_w)}{\ln t_w}\sim 1 \quad (154)$$

$$=\frac{14}{45}T^2\frac{t_w}{t}\ln t_w\quad\text{for}\quad\frac{\ln(t-t_w)}{\ln t_w}>1. \quad (155)$$

D. Response to an applied field

In order to compare with typical aging experiments, we will consider the following two histories for the system and compare them.

(i) Apply $H>0$ starting from $t=0$: the magnetization per spin $m(t)$ will then grow in time as computed in (60) up to time $t_{eq}\sim e^{4J/T}$, where m_{eq} is reached.

(ii) Keep $H=0$ between $t=0$ and t_w, and then apply $H>0$ for $t>t_w$: in this case the magnetization per spin $m(t,t_w)$ remains 0 up to time t_w, and then grows to again reach m_{eq} in the large time limit.

We now estimate $m(t,t_w)$ in the case (ii) in the ''small applied field regime,'' where $H\sim 1/\Gamma_w^2$. It is convenient to define

$$\hat t\equiv t-t_w, \quad (156)$$

and introduce the ratio between

$$\hat\Gamma\equiv T\ln\hat t=T\ln(t-t_w) \quad (157)$$

and $\Gamma_w=T\ln t_w$:

$$\hat\alpha=\frac{\hat\Gamma}{\Gamma_w}=\frac{\ln(t-t_w)}{\ln t_w}. \quad (158)$$

We separately discuss the three regimes $0<\hat\alpha<1$, $\hat\alpha\sim1$, and $\hat\alpha>1$.

1. Response at early times $0<\hat\alpha<1$ from degenerate wells

We first study the scaling limit of small H and large Γ_w with $H\Gamma_w^2$ fixed and $0<\hat\alpha=(\ln\hat t/\ln t_w)<1$ fixed. In this regime the dominant contributions come from bonds with near degenerate extrema, i.e., the rare events of type (a) described in Sec. VIII A.

Let us consider an ascending bond with a secondary minimum separated by a distance y and a barrier Γ_0 at a potential $u>0$ above the minimum. Using the results and notations of Ref. [28] we find that the probability that an ascending bond of this type at Γ_w (before the field is turned on) is

$$r_{\Gamma_w}(y,\Gamma_0)d\Gamma_0 dy = 2\theta(\Gamma_w-\Gamma_0)P_{\Gamma_0}(0,.)*_y P_{\Gamma_0}(0,.)d\Gamma_0 dy. \tag{159}$$

Just before Γ_w, there is thermal equilibrium between the two extrema, and thus the probability that the primary extremum is occupied by an A domain wall is $p_{eq}(u)=1/(1+e^{-u/T})$. The thermally averaged total magnetization of the segment y is then $\langle M\rangle = y(1-2p_{eq}(u))$. One now turns the field on at Γ_w, and the thermally averaged magnetization of the segment remains the same (up to negligible probability) until the time $\hat{\Gamma}=\Gamma_0$ when a new equilibrium is attained (here we can neglect the equilibration time scale). Just after $\hat{\Gamma}=\Gamma_0$ the occupation probability of the left minimum is now $p_{eq}(u')=1/(1+e^{-u'/T})$, where $u'=u-2Hy$. The new magnetization is $\langle M\rangle' = y(1-2p_{eq}(u-2Hy))$. Similarly, the contribution of a descending bond (with y,Γ_0) is given by the same formula with $u<0$. Finally the contribution of degeneracy of hills—fluctuations of the B domains–yields an overall factor of two. The total contribution of all these events to the magnetization per spin is

$$\begin{aligned} m(t,t_w) &= m^{(a)}(t,t_w) \\ &= 2\frac{1}{\Gamma_w^2}\int_0^{\hat{\Gamma}} d\Gamma_0 \int_0^{+\infty} dy \int_0^{+\infty} du\, r_{\Gamma_w}(y,\Gamma_0) y \\ &\quad \times (1-2p_{eq}(u-2Hy)+1-2p_{eq}(-u-2Hy)). \end{aligned} \tag{160}$$

This gives

$$\begin{aligned} m(t,t_w) &= m^{(a)}(t,t_w) \\ &= 4\frac{1}{\Gamma_w}\frac{\hat{\Gamma}}{\Gamma_w}\int_0^{+\infty} dY\, Y G(Y) \int_0^{+\infty} du \left(\frac{1}{1+e^{(u-2H\hat{\Gamma}^2 Y)/T}} \right. \\ &\quad \left. - \frac{1}{1+e^{(u+2H\hat{\Gamma}^2 Y)/T}} \right) \end{aligned} \tag{161}$$

$$= 8(H\hat{\Gamma}^2)\frac{1}{\Gamma_w}\frac{\hat{\Gamma}}{\Gamma_w}\int_0^{+\infty} dY\, Y^2 G(Y) \tag{162}$$

$$= \frac{32}{45} H \frac{\hat{\Gamma}^3}{\Gamma_w^2} = \frac{32}{45}(H\Gamma_w^2)\frac{\hat{\alpha}^3}{\Gamma_w}. \tag{163}$$

In this regime, the magnetization as a scaling function of $H\Gamma_w^2$ is thus *exactly linear*. It can be shown that a nonlinear response in $H\Gamma_w^2$ couples to the curvature of the distribution of difference of potential near zero and is of higher order in $1/\Gamma_w$.

2. Response at times $\hat{\alpha}\simeq 1$

When $t-t_w$ is of order t_w a second effect adds to the one computed above. It corresponds to the events of type (c) described in Sec. VIII A where the barrier of a bond at Γ_w is equal to $\Gamma_w+\epsilon$ where $\epsilon = O(1)$ (of arbitrary sign). In the absence of the field the pair of domains at the endpoints of the bond have not yet annihilated at time t_w, with a probability $p_c(t_w)=\exp(-e^{-\epsilon/T})$. When adding the field at t_w the barriers suddenly either increases (for descending bonds) or decreases (ascending bonds) by $2Hl$ where l is the length of the bond. For $t>t_w$ [and such that $t-t_w \sim O(t_w)$] the probability $p_c(t)$ that the domain has not yet annihilated depends on H and is

$$p_c(t) = p_c(t_w)\exp\left[-\left(\frac{t-t_w}{t_w}\right)e^{(-\epsilon\mp 2Hl)/T}\right] \tag{164}$$

for descending and ascending bonds respectively. Events (c) thus result in a difference in magnetization compared to the zero field case equal to

$$\begin{aligned} m^{(c)}(t,t_w) &= \frac{2}{\Gamma_w^2}\int_0^\infty dl \int_{-\infty}^{+\infty} d\epsilon P^*(0,l)\, l \exp(-e^{-\epsilon/T}) \\ &\quad \times \left(\exp\left(-\frac{t-t_w}{t_w}e^{(1/T)(-\epsilon-2Hl)}\right) \right. \\ &\quad \left. - \exp\left(-\frac{t-t_w}{t_w}e^{(1/T)(-\epsilon+2Hl)}\right)\right) \end{aligned} \tag{165}$$

$$\begin{aligned} &= \frac{2T}{\Gamma_w}\int_0^\infty d\lambda\, P^*(\eta=0,\lambda)\lambda \\ &\quad \times \ln\left(\frac{t_w+(t-t_w)e^{2H\lambda\Gamma_w^2/T}}{t_w+(t-t_w)e^{-2H\lambda\Gamma_w^2/T}}\right). \end{aligned} \tag{166}$$

The total magnetization is now

$$m(t,t_w) = m^{(a)}(t,t_w) + m^{(c)}(t,t_w) = \frac{32}{45}HT\ln t_w + m^{(c)}(t,t_w). \tag{167}$$

Note that in the present regime the magnetization as a scaling function of $H\Gamma_w^2$ is complicated and nonlinear.

In the limit where $H\Gamma_w^2$ is small, one has

$$\begin{aligned} m^{(c)}(t,t_w) &= \frac{8}{\Gamma_w}H\Gamma_w^2\frac{t-t_w}{t}\int_0^\infty d\lambda\, P^*(\eta=0,\lambda)\lambda^2 \\ &\quad + O((H\Gamma_w^2)^2) \\ &= \frac{14}{45}H\Gamma_w\frac{t-t_w}{t} + O((H\Gamma_w^2)^2), \end{aligned} \tag{168}$$

where we have used $\int_0^\infty d\lambda\, P^*(\eta=0,\lambda)\lambda^2 = 7/180$.

Although the above function [Eq. (166)] is complicated, at the special time such that $t-t_w=t_w$ it takes the simple value

$$m^{(c)}(2t_w,t_w)=\frac{4}{\Gamma_w}H\Gamma_w^2\int_0^\infty d\lambda P^*(\eta=0,\lambda)\lambda^2$$
$$=\frac{7}{45}H\Gamma_w. \quad (169)$$

3. Response at times $\alpha=\hat{\alpha}>1$

For time differences $t-t_w \gg\gg t_w$, corresponding to $\hat{\alpha}=\alpha>1$, the response of the RFIM chain to an applied field will be dominated by the effective dynamics described by the RSRG procedure. When the field $H>0$ is turned on, the descending bonds with (F,l) become $(F+2Hl,l)$ and the ascending bonds become $(F-2Hl,l)$ except if $\Gamma_w<F<\Gamma_w+2Hl$, since in this case they must be immediately decimated. Technically, from the point of view of the landscape, it is more convenient to symmetrize the initial condition at $\hat{\Gamma}=\Gamma_w$ which amounts to artificially reintroducing the descending bonds $\Gamma_w-2Hl<F<\Gamma_w$ (these bonds, being redecimated immediately, do not introduce any errors for $\alpha>1$). This corresponds to the following initial distributions at $\hat{\Gamma}=\Gamma_w$:

$$P^+_{\Gamma_w}(F,l)=P_{\Gamma_w}(F,l)-2Hl\partial_F P_{\Gamma_w}(F,l)$$
$$+2HP_{\Gamma_w}*_{F,l}P_{\Gamma_w}*_l(lP_{\Gamma_w}(0,l))$$
$$-4HP_{\Gamma_w}(F,l)\int_0^\infty dl'l'P_{\Gamma_w}(0,l'), \quad (170)$$

$$P^-_{\Gamma_w}(F,l)=P_{\Gamma_w}(F,l)+2Hl\partial_F P_{\Gamma_w}(F,l)$$
$$-2HP_{\Gamma_w}*_{F,l}P_{\Gamma_w}*_l(lP_{\Gamma_w}(0,l))$$
$$+4HP_{\Gamma_w}(F,l)\int_0^\infty dl'l'P_{\Gamma_w}(0,l'). \quad (171)$$

We now check that the magnetization corresponding to this initial condition is the one at the end of the $\hat{\alpha}=1$ regime [i.e., the $t\to+\infty$ limit of Eq. (168)]. It is, to first order in H,

$$m_{eff}(\hat{\Gamma}=\Gamma_w,\Gamma_w)=\frac{\int_0^{+\infty}dl l\int_0^\infty d\zeta[P^+_{\Gamma_w}(\zeta,l)-P^-_{\Gamma_w}(\zeta,l)]}{\int_0^{+\infty}dl l\int_0^\infty d\zeta[P^+_{\Gamma_w}(\zeta,l)+P^-_{\Gamma_w}(\zeta,l)]} \quad (172)$$

$$\simeq\frac{8H}{\Gamma_w^2}\int_0^\infty dl l P_{\Gamma_w}(\Gamma_w,l)=\frac{14}{45}H\Gamma_w. \quad (173)$$

We write, to linear order $P^{\pm}_{\hat{\Gamma}}(F,l)=P_{\hat{\Gamma}}(F,l)+2HQ^{\pm}_{\hat{\Gamma}}(F,l)$, with the initial condition in Laplace transform variables at $\hat{\Gamma}=\Gamma_w$,

$$Q^+_{\Gamma_w}(\zeta,p)=-\partial_p(U_{\Gamma_w}(p)u_{\Gamma_w}(p)e^{-\zeta u_{\Gamma_w}(p)})$$
$$-U'_{\Gamma_w}(0)U_{\Gamma_w}(p)^2\zeta e^{-\zeta u_{\Gamma_w}(p)}$$
$$+2U'_{\Gamma_w}(0)U_{\Gamma_w}(p)e^{-\zeta u_{\Gamma_w}(p)}, \quad (174)$$

$$Q^-_{\Gamma_w}(\zeta,p)=-Q^+_{\Gamma_w}(\zeta,p). \quad (175)$$

To compute the magnetization we are interested only in $P^+_{\hat{\Gamma}}(F,l)-P^-_{\hat{\Gamma}}(F,l)=2HQ_{\hat{\Gamma}}(F,l)$, where $Q_{\hat{\Gamma}}(F,l)=Q^+_{\hat{\Gamma}}(F,l)-Q^-_{\hat{\Gamma}}(F,l)$ satisifes the linearized RG equation

$$(\partial_{\hat{\Gamma}}-\partial_\zeta)Q_{\hat{\Gamma}}(\zeta,p)=-Q_{\hat{\Gamma}}(0,p)P_{\hat{\Gamma}}(.,p)*_{\zeta l}P_{\hat{\Gamma}}(.,p)$$
$$+2P_{\hat{\Gamma}}(0,p)Q_{\hat{\Gamma}}(.,p)*_\zeta P_{\hat{\Gamma}}(.,p)$$
$$+2Q_{\hat{\Gamma}}(0,0)P_{\hat{\Gamma}}(\zeta,p). \quad (176)$$

The solution therefore has the form

$$Q_{\hat{\Gamma}}(\zeta,p)=(A_{\hat{\Gamma}}(p)+\zeta B_{\hat{\Gamma}}(p))e^{-\zeta u_{\hat{\Gamma}}(p)}, \quad (177)$$

where the coefficients $A_{\hat{\Gamma}}(p)$ and $B_{\hat{\Gamma}}(p)$ satisfy the RG equations

$$\partial_{\hat{\Gamma}}A_{\hat{\Gamma}}(p)=-u_{\hat{\Gamma}}(p)A_{\hat{\Gamma}}(p)+B_{\hat{\Gamma}}(p)+2U_{\hat{\Gamma}}(p)A_{\hat{\Gamma}}(0), \quad (178)$$

$$\partial_{\hat{\Gamma}}B_{\hat{\Gamma}}(p)=-u_{\hat{\Gamma}}(p)B_{\hat{\Gamma}}(p), \quad (179)$$

with initial condition at $\hat{\Gamma}=\Gamma_w$:

$$A_{\Gamma_w}(p)=-2\partial_p(U_{\Gamma_w}(p)u_{\Gamma_w}(p))+4U_{\Gamma_w}(p)U'_{\Gamma_w}(0), \quad (180)$$

$$B_{\Gamma_w}(p)=2U_{\Gamma_w}(p)u_{\Gamma_w}(p)u'_{\Gamma_w}(p)-2U_{\Gamma_w}(p)^2U'_{\Gamma_w}(p). \quad (181)$$

The solutions are

$$B_{\hat{\Gamma}}(p)=B_{\Gamma_w}(p)\frac{\sinh(\Gamma_w\sqrt{p})}{\sinh(\hat{\Gamma}\sqrt{p})}, \quad (182)$$

$$A_{\hat{\Gamma}}(p)=(A_{\Gamma_w}(p)+(\hat{\Gamma}-\Gamma_w)B_{\Gamma_w}(p))\frac{\sinh(\Gamma_w\sqrt{p})}{\sinh(\hat{\Gamma}\sqrt{p})}$$
$$-2(\hat{\Gamma}-\Gamma_w)\frac{\sqrt{p}}{\sinh(\hat{\Gamma}\sqrt{p})}. \quad (183)$$

In the limit $\hat{\Gamma}\gg\Gamma_w$, we have

$$B_{\hat{\Gamma}}(p)\simeq\frac{\sqrt{p}}{\sinh(\hat{\Gamma}\sqrt{p})}, \quad (184)$$

$$A_{\hat{\Gamma}}(p) \simeq -\hat{\Gamma}\frac{\sqrt{p}}{\sinh(\hat{\Gamma}\sqrt{p})}, \tag{185}$$

and recover the first order linearization in δ of the biased fixed point solutions [Eqs. (37)]:

$$P^{\pm}_{\Gamma}(\zeta,p) = U^{\pm}_{\Gamma}(p)e^{-\zeta u^{\pm}_{\Gamma}(p)} = P_{\Gamma}(\zeta,p)(1 \mp \delta\Gamma)(1 \pm \delta\zeta)$$
$$= P_{\Gamma}(\zeta,p) + 2(\delta\zeta - \delta\Gamma)\frac{\sqrt{p}}{\sinh(\Gamma\sqrt{p})}e^{-\zeta\sqrt{p}\coth(\Gamma\sqrt{p})}. \tag{186}$$

The magnetization at first order in H is given by

$$m_{eff}(\hat{\Gamma}) = \frac{\int_0^{+\infty} dl l \int_0^{\infty} d\zeta [P^{+}_{\hat{\Gamma}}(\zeta,l) - P^{-}_{\hat{\Gamma}}(\zeta,l)]}{\int_0^{+\infty} dl l \int_0^{\infty} d\zeta [P^{+}_{\hat{\Gamma}}(\zeta,l) + P^{-}_{\hat{\Gamma}}(\zeta,l)]} \tag{187}$$

$$\simeq \frac{2H}{\hat{\Gamma}^2}\int_0^{\infty} dl l \int_0^{\infty} d\zeta Q_{\hat{\Gamma}}(\zeta,l)$$
$$= \frac{2H}{\hat{\Gamma}^2}\left[-\partial_p \left(\frac{A_{\hat{\Gamma}}(p)}{u_{\hat{\Gamma}}(p)} + \frac{B_{\hat{\Gamma}}(p)}{u^2_{\hat{\Gamma}}(p)} \right) \right]\Bigg|_{p=0} \tag{188}$$
$$= H\hat{\Gamma}\left[\frac{2}{3} - \frac{16}{45}\left(\frac{\Gamma_w}{\hat{\Gamma}} \right)^3 \right]. \tag{189}$$

It grows from

$$m_{eff}(\hat{\Gamma} = \Gamma_w) = \frac{14}{45}H\Gamma_w \tag{190}$$

to the asymptotic regime for large $\hat{\Gamma} \gg \Gamma_w$,

$$m_{eff}(\hat{\Gamma} \gg \Gamma_w) = \frac{2}{3}H\hat{\Gamma} \tag{191}$$

that corresponds to the behavior of the magnetization of the biased case at first order in δ [Eq. (60)].

We now summarize our results for the magnetization in the regimes $\hat{\alpha} < 1$, $\hat{\alpha} \simeq 1$, and $\hat{\alpha} > 1$:

$$m(\hat{\Gamma},\Gamma_w) = \frac{32}{45}H\Gamma_w\hat{\alpha}^3 \quad \text{for} \quad \hat{\alpha} = \frac{\hat{\Gamma}}{\Gamma_w} < 1, \tag{192}$$

$$m(\hat{\Gamma},\Gamma_w) = \frac{32}{45}H\Gamma_w + \frac{14}{45}H\Gamma_w\frac{t-t_w}{t} \quad \text{for} \quad \hat{\alpha} = \frac{\hat{\Gamma}}{\Gamma_w} \simeq 1, \tag{193}$$

$$m(\hat{\Gamma},\Gamma_w) = H\hat{\Gamma}\left[\frac{2}{3} - \frac{16}{45\hat{\alpha}^3} \right] + \frac{32}{45}H\Gamma_w \quad \text{for} \quad \hat{\alpha} = \frac{\hat{\Gamma}}{\Gamma_w} > 1. \tag{194}$$

E. Fluctuation-dissipation violation ratio

Having computed truncated correlations and the response to an applied field, we now discuss the fluctuation-dissipation violation ratio, a measure of the nonequilibrium behavior of the system. For two observables A and B, the fluctuation-dissipation-theorem (FDT) violation ratio X is defined as [6,40]

$$TR_{A,B}(t,t_w) = X_{A,B}(t,t_w)\partial_{t_w}C_{A,B}(t,t_w), \tag{195}$$

where $C_{A,B}(t,t_w)$ represents the truncated correlation

$$C_{A,B}(t,t_w) = \langle A(t)B(t_w)\rangle - \langle A(t)\rangle\langle B(t_w)\rangle, \tag{196}$$

and $R_{A,B}(t,t_w)$ represents the response in the observable A at time t to a field H_B linearly coupled to the observable B in the Hamiltonian through a term of the form $-H_B B$:

$$R_{A,B}(t,t_w) = \frac{\delta\langle A(t)\rangle}{\delta H_B(t_w)}\bigg|_{H_B=0}. \tag{197}$$

Here we have computed the magnetization resulting from an uniform magnetic field H, so that the observables A and B are given by $A = (1/L)\Sigma_{i=1}^{L}S_i$ and $B = \Sigma_{i=1}^{L}S_i$, respectively. From the magnetization

$$m(t,t_w) = \frac{1}{L}\sum_i \langle S_i(t)\rangle = \overline{\langle S_0(t)\rangle} = H\int_{t_w}^{t} du R(t,u) \tag{198}$$

and the truncated correlation

$$C_1(t,t_w) = \frac{1}{L}\sum_{i=1}^{L}\sum_{j=1}^{L} [\langle S_i(t)S_j(t_w)\rangle - \langle S_i(t)\rangle\langle S_j(t_w)\rangle]$$
$$= \sum_x \overline{\langle S_0(t)S_x(t_w)\rangle - \langle S_0(t)\rangle\langle S_x(t_w)\rangle}, \tag{199}$$

one obtains the fluctuation-dissipation ratio $X(t,t_w)$ as

$$X(t,t_w) = -T\frac{\partial_{t_w}m(t,t_w)/H}{\partial_{t_w}C_1(t,t_w)}. \tag{200}$$

Note that we have used the infinite size limit to replace translational averages by disorder averages.

We have computed $C_1(t,t_w)$ [Eqs. (153)–(155)] and $m(t,t_w)$ [Eqs. (192)–(194)] in the three regimes $0 < \hat{\alpha} < 1$, $\hat{\alpha} \sim 1$, and $\hat{\alpha} > 1$, and obtained the following expressions for the fluctuation dissipation violation ratio at large times $t,t_w \gg 1$:

$$X(t,t_w) = 1 \qquad \text{for} \quad 0 < \hat{\alpha} = \frac{\ln(t-t_w)}{\ln t_w} < 1, \tag{201}$$

$$X(t,t_w) = \frac{t+t_w}{t} \qquad \text{for} \quad \frac{t-t_w}{t_w} \quad \text{a fixed number} \quad (\hat{\alpha} = 1), \tag{202}$$

$$X(t,t_w)=\frac{t}{t_w \ln t_w}\left(1+\frac{24}{7}\frac{\ln^2 t_w}{\ln^2 t}\right) \quad \text{for}$$

$$\hat{\alpha}=\frac{\ln(t-t_w)}{\ln t_w}>1. \tag{203}$$

The behavior of X upon increasing the time difference $t-t_w$ is as follows. First, of course, one expects, when $t-t_w$ is a fixed number, a nonuniversal equilibrium regime (not studied here) where time translational invariance and the FDT is obeyed. Increasing the time difference as $t-t_w \sim t_w^{\hat{\alpha}}$, in the regime $0<\hat{\alpha}<1$, we find that *the FDT theorem is still obeyed* to leading order, which is quite interesting since time translational invariance does *not* hold in that regime. Next, in the regime $\hat{\alpha}=1$, t/t_w fixed, X becomes a nontrivial scaling function of $(t-t_w)/t_w$, which interpolates between $X=2$ for $(t-t_w)/t_w \to 0$ and $X=1$ for $(t-t_w)/t_w \to +\infty$. In order to match the value $X=1$, there is thus a nontrivial crossover regime between the end of the quasiequilibrium regime $\hat{\alpha}<1$ and the beginning of the $\hat{\alpha}=1$ regime. This crossover occurs for $(t-t_w)\sim t_w/\ln t_w$, and is given by

$$X(t,t_w)=F_1((t-t_w)\ln t_w/t_w), \tag{204}$$

$$F_1(y)=\frac{14y+96}{7y+96}. \tag{205}$$

Finally, for very separated times, in the regime $\hat{\alpha}>1$, we find that X grows toward $+\infty$. This occurs again after a crossover between the end of the $\hat{\alpha}=1$ and the $\hat{\alpha}>1$ regime which occurs on time scale $t\sim t_w \ln t_w$, where X is given by

$$X(t,t_w)=1+\frac{31}{7}\frac{t}{t_w \ln t_w}, \tag{206}$$

which matches both the required limits.

We can now compare with the mean field models [45]. As in the mean field [6,40], here we find an aging regime where X is nontrivial, and for t/t_w fixed it is a function of this scaling variable. On the other hand, contrary to mean field models, the ratio X here is never a function of only $C_1(t,t_w)$, in the regime $\hat{\alpha}=1$ because of the extra power of $\ln t$ in $C_1(t,t_w)$, and in general because of the presence of both scales (t,t_w) and $(\ln t, \ln t_w)$. In addition, X here is found to be nonmonotonic, and the values taken by X are not within the interval $[0,1]$. In particular X tends to $+\infty$ in the asymptotic regime $t\sim t_w^{\alpha}$ with $\alpha>1$ since truncated correlations become very small compared to the response in that this regime. Since the ratio X has an interpretation in some contexts as an inverse effective temperature $X=1/T_{eff}$ [40], one would find here that $T_{eff}\to 0$ at large time separations, in contrast to the result that $T_{eff}\to +\infty$ in mean field models. Although this might appear surprising at first sight, one should remember that in finite dimensions many of the properties of the RFIM are controlled—in the the renormalization group sense—by a zero temperature fixed point; this includes the intermediate regime of length scales studied here for a system—the 1D RFIM—with no phase transition.

It is also interesting to compare our results with the fluctuation-dissipation ratio X in pure systems presenting domain growth, typically ferromagnets below their critical temperature $T<T_c$ [7–9] or at $T=T_c$. The ratio X is usually computed by considering the local observables $A=B=S_0$ but to compare with the present study we will translate these results for a spatially uniform applied field.

Let us briefly recall why $X=0$ in the large time scaling regime for $T<T_c$ (X decays to 0 for large t_w). When initial conditions (high temperature $T>T_c$) before the quench are chosen to have zero magnetization $[\Sigma_x\langle S_x(t)\rangle=0]$ the spin autocorrelation coincides with the truncated correlation $\Sigma_x\langle S_x(t)\rangle=0$, and takes a simple scaling form [1]

$$C_1^{pure}(t,t_w)=\sum_x \langle S_0(t)S_x(t_w)\rangle \simeq L(t_w)^d f_1\left(\frac{L(t_w)}{L(t)}\right), \tag{207}$$

where $L(t)$ is the typical size of domains at time t an d the dimension of space. On the other hand, the magnetization when a uniform field has been applied between t_w, and t behaves as [8,9,46]

$$M_{auto}(t,t_w)\sim L(t_w)^{d-a} f_2\left(\frac{L(t_w)}{L(t)}\right), \tag{208}$$

with $a=1$ for Ising order parameter [7–9] and $a=d-2$ ($d>2$) for $O(N)$ model [9]. Note the reduction of M with respect to C_1 by a factor $1/L(t_w)$ in ferromagnets, which immediately implies $X=0$ in the scaling regime $L(t_w)\sim L(t)$ ($X=0$ as soon as $a>0$). As is usually argued, the origin of this factor can be seen by considering the system at $t=t_w$ and focusing on the immediate response to a small applied pulse field [46] (here in $d=1$ for simplicity): each interface responds by a factor $O(1)$ and since they occupy only a portion $1/L(t_w)$ of the system this yield a total response $1/L(t_w)$. In contrast, note that *exactly at criticality*, $T=T_c$, there is a nontrivial $X=X(t/t_w)$ which appears as a dimensionless amplitude ratio, by a different mechanism [9].

For the RFIM case, the *full* correlation function has the same scaling form as Eq. (207) [as can be seen by generalizing the result for the autocorrelation computed in Eq. (105)] but the *truncated* does not since we have obtained [Eqs. (153)–(155)]

$$C_1(t,t_w)=\ln t_w \phi_1\left(\frac{\ln(t-t_w)}{\ln t_w}\right) \quad \text{for} \quad 0<\frac{\ln(t-t_w)}{\ln t_w}<1 \tag{209}$$

$$=\ln t_w \phi_2\left(\frac{t_w}{t}\right) \quad \text{for} \quad \frac{\ln(t-t_w)}{\ln t_w}\geqslant 1, \tag{210}$$

whereas for the RFIM magnetization we have found [Eqs. (192)–(194)]

$$m(\hat{\Gamma},\Gamma_w)=\ln t_w \psi_1\left(\frac{\ln(t-t_w)}{\ln t_w}\right) \qquad \text{for} \quad \hat{\alpha}=\frac{\hat{\Gamma}}{\Gamma_w}<1 \tag{211}$$

$$m(\hat{\Gamma},\Gamma_w)=\ln t_w \psi_2\left(\frac{t_w}{t}\right) \qquad \text{for} \quad \hat{\alpha}=\frac{\hat{\Gamma}}{\Gamma_w}\simeq 1 \tag{212}$$

$$m(\hat{\Gamma},\Gamma_w)=\ln t_w \psi_3\left(\frac{\ln t}{\ln t_w}\right) \qquad \text{for} \quad \hat{\alpha}=\frac{\hat{\Gamma}}{\Gamma_w}>1. \tag{213}$$

These expressions are rather different from that for the ferromagnet [Eq. (207)], first because both t and $\ln t$ appear in these scaling forms. The only domain growthlike nontrivial scaling regime with $L(t)=T\ln^2 t$ occurs for $\alpha>1$. In this regime the magnetization has a form similar to Eq. (208) with a reduction factor $1/\sqrt{L(t_w)}$ [rather than $1/L(t_w)$ in the pure system]. Using the discussion of Sec. VIII D 3 one understands that the origin is quite different from the pure case. In the immediate response to an applied field [Eq. (173)] only a small fraction of domains $1/\Gamma_w \sim 1/\sqrt{L(t_w)}$ responds, but their response is *very large* since the full domain, of length $\sim L(t_w)$ flips. The truncated correlation on the other hand is very small and does not take the scaling form [Eq. (207)]. This yield a value $X=+\infty$ in this regime.

Similarly, the origin of the nontrivial value of X in the regime $t\sim t_w$ ($\alpha=1$) is very different from the pure case. Both the response and correlations originiate from rare events and take the form $(1/\ln t_w)L(t_w)f(t/t_w)$, where now the factor $1/\ln t_w$ is the probability of the rare event and $f(t/t_w)$ its contribution to activated dynamics. Since they are of the same order this yield a nontrivial X.

IX. PERSISTENCE PROPERTIES OF THE RFIM

We now turn to a study of the *persistence* properties of the random field Ising model, which have received substantial attention for other systems evolving towards equilibrium. Two of the primary properties of interest in this context are the time decay of the probability that a *spin has never flipped* up to time t and the time decay of the probability that a *domain has survived* up to time t. The results for the single spin persistence for the RFIM (Secs IX A and IX B below) are also valid for the spin glass. The large time limit of these quantities can be computed from the asymptotic full state (see Fig. 3) on which we focus below.

A. Persistence of a single spin

In zero applied field the probability $\Pi(t)$ that a given *spin* at $x=0$ has never flipped up to time t in a single run, is equal to the probability that neither the nearest domain wall on one side, nor the nearest (opposite type) domain wall on the other side have crossed $x=0$. In Ref. [28], we found that the probability $\Pi_1(t)$ that a given Sinai particle does not cross its starting point up to time t decays as $\Pi_1(t)\sim \overline{l}(t)^{-\theta_1}$ with $\theta_1=1/2$. We thus obtain that $\Pi(t)$ in the zero field RFIM decays as

$$\Pi(t)\sim \overline{l}(t)^{-\theta} \quad \text{with} \quad \theta=2\theta_1=1. \tag{214}$$

This should be compared with the result [47] for the pure Ising model, where $\theta=\frac{3}{4}$ [corresponding there to the characteristic length $l(t)\sim\sqrt{t}$]. This out of equilibrium behavior holds up to time $t=t_{eq}$ corresponding to renormalization scale $\Gamma=\Gamma_J$ at which equilibrium is reached.

In the presence of an applied field $H>0$, we can use our previous result [28] for the biased Sinai diffusion. We have found that the probabilities $\Pi_1^+(t)$ [$\Pi_1^-(t)$] that a given Sinai particle remains on the right (left) of its starting point up to time t, in the case of a drift in the (+) direction, behave as

$$\Pi_1^+(t)\approx \frac{2\delta}{1-e^{-2\delta\Gamma}}, \tag{215}$$

$$\Pi_1^-(t)\approx \frac{2\delta}{e^{2\delta\Gamma}-1}. \tag{216}$$

These give the probabilities $\Pi^{\pm}(t)$ that a given spin in the RFIM keeps the value ($\pm$) up to time t, which in the limit of large $\delta T\ln t \gg 1$ behaves as

$$\Pi^+(t)\approx \frac{H^2}{g^2}, \tag{217}$$

$$\Pi_1^-(t)\approx \frac{H^2}{g^2}t^{-2HT/g}. \tag{218}$$

B. Persistence of the local time-averaged magnetization

In addition to the persistence of a single spin, one can also obtain the statistics of the flips of the thermal average of the local magnetization, i.e., $\langle S_x(t)\rangle$, at a given site x. As explained in Ref. [28] in the context of the Sinai model, quantities averaged over many runs behave very differently than quantities for a single run; in particular while the spin of interest in one run may flip many times, if the same spin is examined in many runs with statistically similar initial conditions, the average over the runs at a given time into the runs will flip far less often.

The local magnetization will successively be equal to 1 and -1 with only very small probability, at large times, of being be equal to an intermadiate value. The sequence of flips is given by the sequence of changes of orientation of a bond during the renormalisation procedure extensively studied in Ref. [28]. From that analysis, we obtain the following results for the RFIM.

In zero field $H=0$, we denote by k the *number of changes of sign* of $\langle S_x(t)\rangle$ at a given point x between 0 and t. The distribution of the rescaled variable

$$\kappa=\frac{k}{\ln\Gamma}=\frac{k}{\ln(T\ln t)} \tag{219}$$

is characterized by the asymptotic decay

$$\mathrm{Prob}(\kappa)\sim \overline{l}(t)^{-\overline{\theta}(\kappa)}, \tag{220}$$

where the generalized exponent $\overline{\theta}(\kappa)$ is

$$\overline{\theta}(\kappa)=\frac{\kappa}{2}\ln\left[2\kappa\left(\kappa+\sqrt{\kappa^2+\frac{5}{4}}\right)\right]+\frac{3}{4}-\frac{\kappa}{2}-\frac{1}{2}\sqrt{\kappa^2+\frac{5}{4}}. \tag{221}$$

The exponent $\overline{\theta}(\kappa)$ is a positive convex function : it decays from $\overline{\theta}(\kappa=0)=(3-\sqrt{5})/4$ to $\overline{\theta}(\frac{1}{3})=0$, and then grows again for $\kappa>1/3$. This implies, in particular, that

$$\kappa=\frac{k}{\ln(T\ln t)}\rightarrow\frac{1}{3}\quad\text{(with probability 1 at large time).} \tag{222}$$

All of the moments of κ will be dominated by the typical behavior; i.e., $\langle\kappa^m\rangle\equiv 3^{-m}$ for all m. The full dependence on κ of the $\overline{\theta}(\kappa)$ function describes the *tails* of the probability distribution of κ, i.e., the large deviations. Note also that the probability that $\langle S_x(t)\rangle$ doesn't flip up to time t decays as $\overline{l}(t)^{-\overline{\theta}}$ with exponent

$$\overline{\theta}=\overline{\theta}(\kappa=0)=\frac{3-\sqrt{5}}{4}, \tag{223}$$

which is significantly smaller than the corresponding decay exponent $\theta=1$ in Eq. (214) for a single spin.

Since the renormalization procedure has to be stopped at $\Gamma=\Gamma_J$ at which the equilibrium is reached, and that at later times no more changes occur in the local magnetization we obtain that the *total* number of flips is

$$k_{tot}\rightarrow\frac{1}{3}\ln(4J), \tag{224}$$

the decay of the tails of the probability distribution of $\kappa=k_{tot}/\ln(4J)$, being described by the same function $\overline{\theta}(\kappa)$ as above in terms of the length L_{IM}.

Another result from Ref. [28] is the characterization of the full sequence of the times $\Gamma_1=T\ln t_1,\ldots,\Gamma_k=T\ln t_k$ where the local magnetization $\langle S_x(t)\rangle$ flips. The sequence of scales $\{\Gamma_k\}$ is *a multiplicative Markovian process* constructed with the simple rule $\Gamma_{k+1}=\alpha_k\Gamma_k$, where $\{\alpha_k\}$ are independent identically distributed random variables of probability distribution $\rho(\alpha)$:

$$\rho(\alpha)=\frac{1}{\alpha}\frac{1}{\lambda_+-\lambda_-}(\alpha^{-\lambda_-}-\alpha^{-\lambda_+})\quad\text{with}\quad\lambda_\pm=\frac{3\pm\sqrt{5}}{2}. \tag{225}$$

As a consequence, $\Gamma_k=\alpha_{k-1}\alpha_{k-2}\cdots\alpha_2\Gamma_1$ is simply the product of random variables; thus we obtain, using the central limit theorem, that

$$\lim_{k\rightarrow\infty}\left(\frac{\ln\Gamma_k}{k}\right)=\langle\ln\alpha\rangle=3, \tag{226}$$

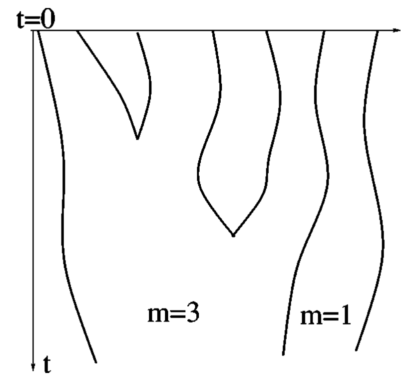

FIG. 8. Temporal evolution of domains in the RFIM. For each surviving domain at time t, m denotes the number of ancestor domains in the initial condition at $t=0$.

and we thus recover that the number k of changes of $\langle S_x(t)\rangle$ grows as $\ln\Gamma=\ln\ln t$ and that the rescaled variable $\kappa=k/\ln(T\ln t)$ is equal to 1/3 with probability 1, as in Eq. (222).

These results can be extended to the RFIM in the presence of an applied field $H>0$. The total number of flips in this case saturates to a finite value given by a scaling function of H and J identical to the one given in Sec. IV E in Ref. [28] for the total number of returns to the origin in the Sinai model.

C. Domain persistence

Persistence can also be defined for larger scale patterns. In the RFIM the simplest pattern (beyond a single spin) is a domain. When a domain of, e.g., consecutive + spins disappears, the two nearest domains of − spins merge. Thus domains can either grow by merging with other domains, or die, and this leads to interesting survival properties, that were studied by Krapivsky and Ben Naim [48] for the pure Ising model. Here we slightly change their definitions of the exponents to adapt to the logarithmic characteristic length scale $\overline{l}(t)=\overline{l}_\Gamma\sim(T\ln t)^2$ of the RFIM.

Of all the domains which still exist at time t, we define $R_m(t)$ as the number of domains which were formed out of m of the initial domains. This is illustrated in Fig. 8. Then $\Sigma_m R_m(t)=N(t)$, the total number of domains at time t, and the fraction of initial domains which have a descendent still "alive" at t is $S(t)=\Sigma_m mR_m(t)=\langle m\rangle N(t)$. Following Ref. [48], one defines the exponents

$$N(t)\sim\overline{l}(t)^{-1}, \tag{227}$$

$$S(t)\sim\overline{l}(t)^{-\psi}, \tag{228}$$

$$R_1(t)\sim\overline{l}(t)^{-\delta}, \tag{229}$$

in terms of the mean domain length $\overline{l}(t)$. Asymptotically, $R_m(t)$ has a scaling form

$$R_m(t)\approx\overline{l}(t)^{\psi-2}\tilde{R}\left(\frac{m}{\overline{l}(t)^{1-\psi}}\right). \tag{230}$$

The scaling function behaves, in the pure case, as $\tilde{R}(z) \sim z^{\sigma}$ for small z and as an exponential at large z. One finds the exponent relation $\delta = \nu + (\nu - \psi)(1+\sigma)$.

Let us now study the RFIM using decimation. In the asymptotic state (full state), domains coincide with bonds, and when a bond is decimated the domain on this bond dies, and the two neighbors merge into a new domain (the new bond). We thus associate with each bond an auxiliary variable m counting the number of initial domains which are ancestors of the domain supported by the bond. Upon decimation of bond 2, the rule for the variables m is simply

$$m' = m_1 + m_3. \tag{231}$$

It turns out that this rule coincides with the magnetization rule of the RFTIC [31] and thus one has the scaling $m \sim \Gamma^{\Phi}$ with $\phi = (1+\sqrt{5})/2$, leading to

$$\psi = \frac{3-\sqrt{5}}{4} = 0.190983. \tag{232}$$

This should be compared with the value for the pure Ising model, which has only been determined numerically [48] as $\psi \approx 0.252$.

Here it is interesting to note that we have found that $\psi = \overline{\theta}$, i.e., the general exact bound $\psi \leqslant \overline{\theta}$ is saturated [27]. This bound for ψ comes from the fact that a point that has never been crossed by any domain wall up to time t for the effective dynamics, necessarily belongs to a domain that has a descendant still living up to time t. In the effective dynamics in the ''full'' renormalized landscape, we have also the reciprocal property: a domain wall surviving between t' and t necessarily contains points that have never been crossed by any domain wall between t' and t. Note that the equality $\psi = \theta$ also appears in the coarsening of domains for the 1D $T=0$ Landau-Ginzburg ϕ^4 soft-spin Ising model [35] for the same reason. The strict inequality $\psi < \overline{\theta}$ requires the possibility that a domain wall can survive between t' and t even if all points inside it at t' get crossed by a domain wall between t' and t (see Ref. [27] for examples).

We now study the probability $R_1(t)$ that a domain has survived up to time t without merging with any other domain. This requires that the two domain walls A and B living at the boundaries of this domain do not meet any other domain wall up to time t. In the asymptotic full state, these conditions imply that three consecutive bonds cannot be decimated. Thus the domain in the middle cannot grow and the probability that it survives upon decimation thus decays exponentially in Γ.

We thus obtain that $R_1(t)$ decays exponentially in Γ (and thus with a nonuniversal power of time determined by the initial condition which determines the convergence towards the asymptotic state). It thus corresponds to

$$\delta = +\infty. \tag{233}$$

As a consequence, the scaling function $R(z)$, not computed here, has an essential singularity at the origin.

X. FINITE SIZE PROPERTIES

The real-space renormalization procedure can also be used analytically to study large finite size systems [32,28]. We will extensively use the analysis of the Sinai walker with either absorbing or reflecting boundary counditions [28]. For the random field Ising model, one may consider several boundary conditions: (i) Fixed spins at the ends: this corresponds for the Glauber dynamics of the RFIM to domain walls which behave like Sinai walkers with reflecting boundary conditions. (ii) Free boundary conditions: this corresponds to absorbing boundary conditions for the domain walls. Here we give some results for (i), whose derivation is detailed in Appendix D, and discuss (ii) at the end.

A. Fixed spins at both ends

Assume, for definiteness, that spins at both extremities are fixed to the values $S_0 = +1$ and $S_L = -1$, where L is the system size. There is thus an even number of domains in the system, and one can describe its statistics at large time using the following generating function. Let us define $I_L(k;\Gamma)$ as probability that the system of size L at time t (i.e., at scale $\Gamma = T \ln t$) contains exactly $n = 2k+2$ domains, with $k = 0,1,2,\ldots$. One obtains (see Appendix D) the generating function in Laplace transform with respect to the length L as

$$\int_0^{\infty} dL e^{-qL} \left(\sum_{k=0}^{\infty} z^k I_L(k;\Gamma) \right) = \frac{\sinh^2(\Gamma\sqrt{p+\delta^2})}{p\cosh^2(\Gamma\sqrt{p+\delta^2}) + \delta^2 - z(p+\delta^2)}, \tag{234}$$

where $p = q/2g$.

In the case of zero applied field $H=0$ ($\delta=0$) one easily performs the Laplace inversion, and obtains

$$\sum_{k=0}^{\infty} z^k I_L(k;\Gamma) = LT^{-1}_{p\to 2gL}\left(\frac{\sinh^2(\Gamma\sqrt{p})}{p(\cosh^2(\Gamma\sqrt{p}) - z)} \right) = \tan(\alpha) \sum_{n=-\infty}^{+\infty} \frac{e^{-(2gL/\Gamma^2)(\alpha+n\pi)^2}}{\alpha + n\pi}, \tag{235}$$

where $\alpha = \arccos\sqrt{z} \in (0,\pi/2)$ for $z \in (0,1)$. In particular we obtain the average and mean squared total number of domains in the system at time t:

$$\langle n \rangle = 2\langle k \rangle_{L,\Gamma} + 2 = 4g\frac{L}{\Gamma^2} + \frac{4}{3} + o(L^0), \tag{236}$$

$$\langle n^2 \rangle_{L,\Gamma} - (\langle n \rangle_{L,\Gamma})^2 = \frac{8gL}{3\Gamma^2} + O(L^0), \tag{237}$$

with $\Gamma = T \ln t$. Again, the same quantities at equilibrium are simply obtained by setting $\Gamma = 4J$ in the above formulas.

In the presence of an applied field $H>0$ ($\delta>0$), one similarly obtains

$$\langle n\rangle = 2\langle k\rangle_{L,\Gamma}+2 \equiv 4gL\frac{\delta^2}{\sinh^2\gamma}+2\frac{1-\gamma\coth\gamma}{\sinh^2\gamma}+2+o(L^0), \tag{238}$$

$$\langle n^2\rangle_{L,\Gamma}-(\langle n\rangle_{L,\Gamma})^2 = 8gL\frac{\delta^2}{\sinh^2\gamma}\left(1+\frac{2(1-\gamma\coth\gamma)}{\sinh^2\gamma}\right)+O(L^0), \tag{239}$$

with $\gamma=\delta T\ln t$.

We have also obtained the probability distribution $F_L(M;\Gamma)$ of the total magnetization of the system:

$$M_L=\sum_{j=1}^{L}\langle S_j\rangle=\sum_{i=1}^{i=2k+2}(-1)^{i+1}l_i. \tag{240}$$

In a Laplace transform with respect to L and M, it is

$$\int_0^{+\infty}dLe^{-qL}\int_{-L}^{+L}dMe^{-rM}F_L(M;\Gamma) = \overline{l}_\Gamma\frac{E_\Gamma^+(q+r)E_\Gamma^-(q-r)}{1-P_\Gamma^+(q+r)P_\Gamma^-(q-r)}, \tag{241}$$

where

$$E_\Gamma^\pm(q)=\frac{\delta e^{\mp\delta\Gamma}}{\sinh(\delta\Gamma)(\sqrt{p+\delta^2}\coth(\Gamma\sqrt{p+\delta^2})\mp\delta)}, \tag{242}$$

$$P_\Gamma^\pm(q)=\frac{\sqrt{p+\delta^2}e^{\mp\delta\Gamma}}{\sinh(\sqrt{p+\delta^2}\Gamma)(\sqrt{p+\delta^2}\coth(\Gamma\sqrt{p+\delta^2})\mp\delta)}, \tag{243}$$

where $p=q/2g$. Note that $q+r$ and $q-r$ are simply the Laplace variables associated with the total rescaled positive and negative lengths.

B. Free boundary conditions

Free boundary conditions in the RFIM correspond to absorbing boundary conditions for the diffusing domain walls. However, the study is slightly different from the one carried in Ref. [28] because now there are particles (A or B) both at the bottom and the top.

The structure of the renormalized landscape at Γ and the full state near the boundary is now the following (see Fig. 9). There is an absorbing zone of length l_0. Then there is a first bond (barrier F_1 length l_1) of arbitrary sign (contrary to Sinai's case, where the first bond was always descending). The first particle at the common endpoint of the first bond and the second bond and is either an a A particle (descending first bond) or a B particle (ascending first bond). The RG procedure is unchanged in the bulk. The new case is when bond 1 is decimated. Then the absorbing zone becomes of length l_0+l_1, while the domain wall (A or B) leaves the system (as the cluster formed by the absorbing zone and the previously first bond flips). Bond 2 keeps the same barrier and length (and becomes the new bond 1). Let $R^\pm(l)$ be the probability distribution of the length l of the absorbing zone. It satisfies the RG equation

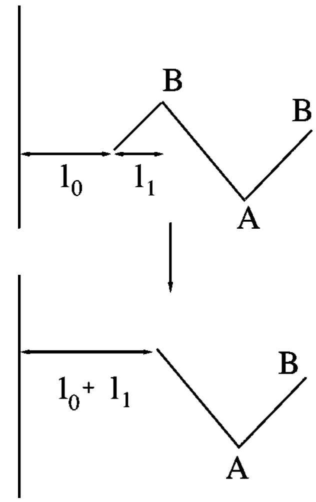

FIG. 9. RG rule near a free boundary, as explained in the text. The same rule holds when the first renormalized bond is descending by exchanging A and B.

$$\partial_\Gamma R_\Gamma^\pm(l)=P_\Gamma^\pm(0,\cdot)*_l R_\Gamma^\mp(\cdot)-R_\Gamma^\pm(l)\int dl' P_\Gamma^\mp(0,l'). \tag{244}$$

In the symmetric case the solution in Laplace transform is

$$R_\Gamma(p)=\frac{2}{\Gamma\sqrt{p}}\tanh\left(\frac{\Gamma\sqrt{p}}{2}\right). \tag{245}$$

Interestingly the shape of the size distribution is the same as for the Sinai particle with absorbing boundaries but with a global length rescaling by a factor 1/4. A similar study can be made in the biased case.

Let us close by noting that the approach to equilibrium will be modified near a free boundary, as compared with the bulk. Indeed, near the free boundary it is possible to create a single domain wall with energy cost 2J. Thus, for times such that $2J<\Gamma=T\ln t<4J$ one must consider different rules for the first renormalized bond.

XI. DISCUSSION AND FUTURE PROSPECTS

In this paper we have shown how a powerful real space dynamical renormalization group method can be used to study the properties of the one dimensional random field Ising model and 1D spin glasses in a field. We have recovered many known results for the long time *equilibrium* behavior, and obtained what we believe are some new ones. But the main advantages of this method is that it enables the computation and physical understanding of many nonequilibrium features of the coarsening process. Although the RSRG method is approximate, it retains all the information needed to obtain exact results for universal long time, large distance quantities. As exemplified by the calculations of the response and truncated correlations, it has the great advantage over many more formal methods by virtue of providing a clear physical interpretation of the origins of the processes (e.g., particular types of rare events) that dominate many properties of interest. In this last section we address two issues: the potential observability of some of the one-dimensional physics that we have studied in detail; and the crucial issue of which features of these one-dimensional systems might apply in higher dimensional random systems.

A. One-dimensional systems

Although random field Ising ferromagnets are not directly realizable experimentally, other systems in the same universality class should be. For random antiferromagnets, an applied magnetic field couples to the antiferromagnetic order parameter as a random field, most simply if the local magnetic moments vary randomly; such systems have the obvious advantage of the strength of the dominant randomness being readily adjustable. Similarly, spin glasses, although unfrustrated in zero field, do exist in quasi-one-dimensional systems. In both these types of systems, rapid quenches could be performed by decreasing the field at low temperatures down to a value of order $T \ll J$ at which coarsening can take place. In practice, other types of randomness in, for example, antiferromagnetic or frustrated two-leg spin-ladder systems in an applied field may have advantages.

There is one physical phenomenon of which one must be wary: in many random systems—especially spin-glasses [5]—the equilibrium states toward which the system strives depend hypersensitively on macroscopic parameters such as the temperature and the magnetic field; this is often referred to—somewhat confusingly—as ''chaos'' [49]. If this were the case here, then changing the temperature or the magnetic field would not correspond simply to speeding up the dynamics, but would instead drive the system toward a new equilibrium which might differ on the scales being probed from the original one. Indeed, this effect is the origin of much of the most interesting aging phenomena seen in three-dimensional spin glasses [5,3]. Fortunately—although less interestingly—this effect does not occur in random field chains: although the effective local random fields could change in a nonuniform way with temperature or with applied field, this will lead only to smooth modifications of the large scale potential and thus, provided the changes are made sufficiently slowly, will not change the universal aspects of the coarsening provided all lengths are scaled appropriately.

The most obvious difficulty for any experiment is one that is ubiquitous in random systems: how does one obtain a reasonable range of length scales if the correlations are growing only logarithmically in time? The situation here is not quite as bad as it might seem as on macroscopic time scales $\xi \sim \ln^2 t$ is not all that short: with a microscopic time τ_0 of the order of picoseconds, 1 min corresponds to coarsening by a factor of 1000 in length scale. But for one-dimensional random-field systems, one can do much better: the correlation length is of order

$$\xi(t) \sim a_o \left(\frac{T \ln(t/\tau_0)}{h} \right)^2, \tag{246}$$

with a_0 a microscopic scale. Thus the correlation length can be controlled by a combination of increasing time, increasing T, and decreasing h. To obtain a wide range of $\xi(t)$, it is beneficial to start with h large enough that the shortest time scales which can be measured still correspond to microscopic lengths and then decrease the field gradually.

While some of the nonequilibrium properties studied here should be amenable to conventional experimental probes in magnetic systems, some of the most interesting predictions will probably not be: how the specific set of domain walls depends on equilibration time and how it varies—or does not vary—from run to run on the same sample. For these one needs either microscopic imaging probes—perhaps with magnetic atomic force or tunnelling microscopy—or scattering data with enough resolution that speckle patterns from a finite spot size can be measured. As mentioned earlier, the spatial Fourier transform of the two-point two-time correlation calculated here is, for spin glasses, related to the correlations between the speckle patterns at the two different times. While measuring this with magnetic x-ray or neutron scattering may not be possible, it should be feasible via light scattering on systems in the same universality class in which the length scales are sufficiently long.

One system on which light scattering measurements might be possible is a nematic liquid crystal in a long thin tubes with a square crossection and heterogeneities on the surfaces which would couple randomly to the two possible symmetry related orientations of the nematic director. Whether this or other systems can be formed in a regime in which the dynamics are sufficiently fast to allow a wide range of length scales to equilibrate is a question whose answer must rely on a quantitative analysis of the physics on the scale of the domain wall structures that should occur.

B. Higher-dimensional systems

Many of the qualitative features of the coarsening process in RFIM chains are also expected to obtain in higher dimensional random systems. These are particularly interesting in systems in which, in contrast to the one-dimensional models, true long-range order should exist in equilibrium. Nevertheless, the characteristics of the coarsening process in one-dimensional models with weak randomness should have much in common with such systems as long as consideration is restricted, as we have primarily done, to time scales such that the full equilibrium correlation length cannot be attained.

Features that are common to many higher dimensional random systems—random exchange ferromagnets, random field magnets and, although more controversially, spin glasses—include the growth of some kind of domain structure by thermal activation over barriers that grow with length scale and are broadly distributed and the existence of aging and other history dependent phenomena. From a renormalization group point of view, these features are general characteristics of systems controlled by random zero-temperature fixed points [5,39]. The notions of local equilibrium within constraints caused by large barriers, and of domination of the dynamics at any given long time scale by rare regions of the sample in which the rate for going over barriers is of order the frequency being probed, are both important. In general, these will, as in the one-dimensional models, result in three different regimes for two-time properties depending on whether the difference between the times is much smaller, of order of, or much larger than, the earlier time.

One of the most intriguing questions concerns the pseudo-determinism found in the one-dimensional random models. Will this exist in some higher dimensional systems in which the domain walls are lines or surfaces whose evolution in time will involve topological changes, rather than points which can simply move or annihilate? That is, will domain walls tend to lie, at long times, in similar sample-and-time-specific positions which are only weakly dependent on the initial conditions? Or will they evolve in different runs in very distinct ways–perhaps by retaining much more memory of the initial conditions?

Two examples of a *single* random walker diffusing in random potentials illustrate some of the difficulties of drawing any definitive conclusions. Consider, first, a random potential which is independently random at each site with a long power law tail ot the distribution of the depth of the potential wells. The deep wells will give rise to a subdiffusive behavior; indeed, simple considerations of the time to find a deep trap and the time to escape it imply that the typical distance the walker will diffuse in time t only grows as a power of $\ln t$. But the statistics of the *set of sites* the walker visits by the time it has made a given number of jumps from one site to another will be identical to that of a normal free random walker; it is just the *time spent* on each site that causes the slow diffusion. Since a pair of random walkers in dimensions higher than 4 have a nonzero probability of never visiting any sites in common, it is clear that the long time behavior is *not* deterministic in high dimensions: two different runs starting from the same point will have a probablity that vanishes at long times of the two walkers being at the same site. In two dimensions, in contrast, the pair of walkers will be very likely to be trapped at the same site at long times. (The three-dimensional case is more subtle but will be more like the high dimensional that the low-dimensional case.)

A second type of random potential yields different behavior: a Gaussian random potential with mean zero and $\overline{[V(\mathbf{x})-V(\mathbf{y})]^2}\sim|\mathbf{x}-\mathbf{y}|^{2\zeta}$ with $\zeta>0$. A pair of random walkers in such a potential will tend to be trapped in the *same* deep valley at long times. This can be argued by considering the borders of valleys which are surrounded by barriers of at least a height Γ. If these typically do not have a lot of fine-scale structure, then they should act as effective traps for all walkers in the vicinity. All the walkers will then tend to eventualy leave such a valley over the same barrier, and hence end up in the same next-larger-scale valley as well. The behavior would then be asymptotically deterministic as in the one-dimensional Sinai model.

The crucial feature that distinguishes these two cases, seems to be the smoothness of the potential: in the first example, the deep potential wells come from very short-wavelength fluctuations, and there is thus no useful notion of a coarse-grained potential. But in the second example, the deep potential wells come, as in one dimension, from long-wavelength slowly varying components; thus coarse graining the potential, to yield approximate dynamics which are asymptotically exact, should be possible. We should emphasize, however, that it is by no means established even for this simple model of a random walk in a smoothly varying Gaussian random potential that the conclusion argued above—that the dynamics will be asymptotically deterministic—will be valid in high dimensions. What happens in three dimensional random magnetic systems in which there are truly many degrees of freedom, we must leave as an important open question.

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APPENDIX A: CONVERGENCE TOWARD THE FULL STATE

In this appendix we analyze the rapid convergence towards the *full state* that was discussed in Sec. IV C. We consider, for simplicity, random initial configurations of the spins, in which there is an independent probability w that each intersite position is occupied by a domain wall, and a probability $1-w$ that it is not. Such random initial configurations describe, for example, initial equilibrium before a quench from a temperature T_0 which is high enough that the random fields are negligible and the system behaves like a pure chain. This corresponds to $T_0=1/\beta_0\sim J\gg\{h_n\}$ and the choice $w=e^{-\beta_0 2J}/(1+e^{-\beta_0 2J})$. In particular, initial conditions where all spins are independent and take value ± 1 with probability 1/2 corresponds to infinite temperature $\beta_0=0$, and $w=1/2$.

For these type of initial configurations, the probability that there exist $n=0,\ldots,l$ domain walls among l consecutive lattice spacings is simply given by the binomial distribution

$$R(n|l)=\frac{l!}{n!(l-n)!}w^n(1-w)^{l-n}. \tag{A1}$$

In the renormalized landscape, the length l of descending (ascending) bonds at scale Γ is distributed with $P_\Gamma^+(l)$ $[P_\Gamma^-(l)]$, whose Laplace transforms are obtained after integration over ζ in Eq. (37). The probability that there had been initially *no* domain walls in the interval occupied by a descending (ascending) bond of the renormalized landscape plays an important role; it is

$$r_{zero}^{\pm}(\Gamma)=\sum_{l=1}^{\infty}P_\Gamma^{\pm}(l)R(0|l)=\sum_{l=1}^{\infty}P_\Gamma^{\pm}(l)(1-w)^l. \tag{A2}$$

The probability that there had been an odd number n is

$$r_{odd}^{\pm}(\Gamma)=\sum_{l=1}^{\infty} P_{\Gamma}^{\pm}(l)\sum_{n\ odd}^{l} R(n|l)=\sum_{l=1}^{\infty} P_{\Gamma}^{\pm}(l)\frac{1-(1-2w)^{l}}{2}, \tag{A3}$$

and the probability that there had been a nonvanishing even number is

$$r_{even}^{\pm}(\Gamma)=1-r_{odd}^{\pm}(\Gamma)-r_{zero}^{\pm}(\Gamma). \tag{A4}$$

To characterize the statistical properties of the spin configurations on the renormalized landscape, we focus on one renormalized bond at scale Γ—call it ''2''—and its local environment which determines whether or not the maximum and minimum at the ends of bond 2 are occupied by domain walls. We introduce the probabilities $V_{\Gamma}^{desc}(\epsilon_2;\{\epsilon_1,\epsilon_3\})$ $[V_{\Gamma}^{asc}(\epsilon_2;\{\epsilon_1,\epsilon_3\})]$, where $\epsilon_i=\pm$, that a descending (resp. ascending) bond of the renormalized landscape at scale Γ is in phase $\epsilon_2=\pm$, with its left neighboring bond in phase ϵ_1 and its right neighboring bond in phase ϵ_3. The normalization conditions are

$$\sum_{\epsilon_1=\pm}\sum_{\epsilon_2=\pm}\sum_{\epsilon_3=\pm} V_{\Gamma}^{desc}(\epsilon_2;\{\epsilon_1,\epsilon_3\})=1, \tag{A5}$$

$$\sum_{\epsilon_1=\pm}\sum_{\epsilon_2=\pm}\sum_{\epsilon_3=\pm} V_{\Gamma}^{asc}(\epsilon_2;\{\epsilon_1,\epsilon_3\})=1. \tag{A6}$$

If $\epsilon_2=-$ on a descending bond, then it is not possible to have $\epsilon_1=+$, since this would correspond to a domain wall of type A at a maxima, and similarly it is not possible to have $\epsilon_3=+$ since this would correspond to a domain wall of type B at a minima; thus we have immediately that

$$V_{\Gamma}^{desc}(-;\{+-\})=V_{\Gamma}^{desc}(-;\{-+\})=V_{\Gamma}^{desc}(-;\{2+\})=0. \tag{A7}$$

Similarly we have

$$V_{\Gamma}^{asc}(+;\{+-\})=V_{\Gamma}^{asc}(-;\{-+\})=V_{\Gamma}^{asc}(-;\{2-\})=0. \tag{A8}$$

If, however, $\epsilon_2=+$ on a descending bond ($\epsilon_2=-$ on a ascending bond), then all four possible neighborhoods of this bond can occur. Since it is a bit lengthy to give the full enumeration of all possible cases with their corresponding probabilities, we give here only the final results

$$V_{\Gamma}^{desc}(-;\{--\})=\frac{r_{zero}^{+}(\Gamma)}{2}, \tag{A9}$$

$$V_{\Gamma}^{asc}(+;\{++\})=\frac{r_{zero}^{-}(\Gamma)}{2}, \tag{A10}$$

and

$$V_{\Gamma}^{desc}(+;\{2-\})=(1-r_{zero}^{-}(\Gamma))\left(1-\frac{r_{zero}^{+}(\Gamma)}{2}-\frac{r_{zero}^{+}(\Gamma)r_{zero}^{-}(\Gamma)}{2}\right)+\frac{(r_{zero}^{-}(\Gamma))^2}{2}r_{even}^{+}(\Gamma), \tag{A11}$$

$$V_{\Gamma}^{desc}(+;\{+-\})=V_{\Gamma}^{desc}(+;\{-+\})=\frac{r_{zero}^{-}(\Gamma)}{2}-\frac{(r_{zero}^{-}(\Gamma))^2}{2}(r_{zero}^{+}(\Gamma)+r_{even}^{+}(\Gamma)), \tag{A12}$$

$$V_{\Gamma}^{desc}(+;\{2+\})=\frac{(r_{zero}^{-}(\Gamma))^2}{2}(r_{zero}^{+}(\Gamma)+r_{even}^{+}(\Gamma)), \tag{A13}$$

and similarly for *ascending* bonds by exchanging $\pm\to\mp$.

The important property of these probabilities is that apart from $V_{\Gamma}^{desc}(+;\{--\})$ and $V_{\Gamma}^{asc}(-;\{++\})$, which correspond to locally full configuration of the domain walls, all the allowed V have $r_{zero}^{\pm}(\Gamma)$ as a factor, i.e., to have a bond 2 that does not have domain walls at both its extremities requires that at least one of the three bonds (1,2, or 3) had exactly zero domain walls in the initial configuration.

Using the fixed point solution [Eq. (37)], we find that

$$r_{zero}^{\pm}(\Gamma)\simeq\int_0^{\infty}dl P_{\Gamma}^{\pm}(l)(1-w)^{l}=\hat{P}_{\Gamma}^{\pm}\left(q=\ln\left(\frac{1}{1-w}\right)\right), \tag{A14}$$

and thus this and the probabilities of non-full bonds converges exponentially in Γ to 0,

$$r_{zero}^{\pm}(\Gamma)\sim e^{-\Gamma(\sqrt{p+\delta^2}\pm\delta)}, \tag{A15}$$

with $p=(1/2g)\ln[1/(1-w)]$.

Thus the system converges towards the ''full'' state of the renormalized landscape exponentially in Γ, with a non universal coefficient depending on the initial concentration of domain walls through the parameter w, and on the strength of the disorder through g. For the symmetric case of no applied field ($\delta=0$), the fraction of the extrema of the renormalized landscape at time t that are not occupied by a domain wall goes to zero as a *power* of time,

$$\text{Prob[missing domain wall]}\sim\frac{1}{t^{\eta_e}}, \tag{A16}$$

with

$$\eta_e=T\sqrt{\frac{\ln 2}{2\overline{h^2}}}. \tag{A17}$$

Note that in the initial state for the landscape, and in the symmetric case $H=0$, h_n is positive or negative with probability 1/2, and thus after grouping the consecutive h_n of the same sign to give the initial zig-zag landscape, we have, for an initial distribution of length,

$$P_0(l)=\frac{1}{2^l} \quad \text{for} \quad l=1,2,\ldots,\infty. \tag{A18}$$

This corresponds to

$$r_{zero}(\Gamma=0)=\frac{1-w}{1+w}. \tag{A19}$$

As an example, for $w=\frac{1}{2}$, we have $r_{zero}(\Gamma=0)=\frac{1}{3}$, and $V^{desc}_{\Gamma=0}(-;\{--\})=\frac{1}{6}$.

APPENDIX B: AUTOCORRELATIONS IN THE RFIM WITH AN APPLIED FIELD

We have now to solve RG equations (94) given in the text with the initial conditions

$$P^{++}_{\Gamma',\Gamma'}(\zeta)=\int_0^\infty dl\,\frac{lP^{+}_{\Gamma'}(\zeta,l)}{\overline{l}_{\Gamma'}}, \tag{B1}$$

$$P^{-+}_{\Gamma',\Gamma'}(\zeta)=0. \tag{B2}$$

Since for large Γ, $P^{\pm}$ have reach their fixed point values [Eq. (37)], we obtain [using the simplified notations $u^{\pm}_{\Gamma}=u^{\pm}_{\Gamma}(p=0)=U^{\pm}_{\Gamma}(p=0)$]

$$\begin{aligned}(\partial_\Gamma-\partial_\zeta)P^{\pm+}_{\Gamma,\Gamma'}(\zeta)&=-2u^{\mp}_{\Gamma}P^{\pm+}_{\Gamma,\Gamma'}(\zeta)\\&\quad+2u^{\mp}_{\Gamma}u^{\pm}_{\Gamma}\int_0^\infty d\zeta' e^{-u^{\pm}_{\Gamma}(\zeta-\zeta')}P^{\pm+}_{\Gamma,\Gamma'}(\zeta')\\&\quad+(u^{\pm}_{\Gamma})^2\zeta e^{-u^{\pm}_{\Gamma}\zeta}P^{\mp+}_{\Gamma,\Gamma'}(0),\end{aligned} \tag{B3}$$

together with the initial conditions

$$\begin{aligned}P^{2+}_{\Gamma',\Gamma'}(\zeta)&=\frac{1}{\overline{l}_{\Gamma'}}e^{-\zeta u^{+}_{\Gamma'}}(-\partial_p U^{+}_{\Gamma}(p)\\&\quad+\zeta U^{+}_{\Gamma}(p)\partial_p u^{+}_{\Gamma}(p))|_{p=0},\end{aligned} \tag{B4}$$

$$P^{-+}_{\Gamma',\Gamma'}(\zeta)=0. \tag{B5}$$

The solutions are thus of the form

$$P^{\pm+}_{\Gamma,\Gamma'}(\zeta)=\frac{1}{\overline{l}_{\Gamma}}e^{-\zeta u^{\pm}_{\Gamma}}u^{\pm}_{\Gamma}(A^{\pm+}_{\Gamma,\Gamma'}+\zeta B^{\pm+}_{\Gamma,\Gamma}), \tag{B6}$$

where the coefficients satisfy

$$B^{\pm+}_{\Gamma,\Gamma'}=\partial_\Gamma A^{\pm+}_{\Gamma,\Gamma'}, \tag{B7}$$

$$\partial^2_\Gamma A^{\pm+}_{\Gamma,\Gamma'}=u^{+}_{\Gamma}u^{-}_{\Gamma}(A^{2+}_{\Gamma,\Gamma'}+A^{-+}_{\Gamma,\Gamma'}). \tag{B8}$$

Introducing the sum $\Sigma^{+}_{\Gamma,\Gamma'}=A^{++}_{\Gamma,\Gamma'}+A^{-+}_{\Gamma,\Gamma'}$ and the difference $\Delta^{+}_{\Gamma,\Gamma'}=A^{2+}_{\Gamma,\Gamma'}-A^{-+}_{\Gamma,\Gamma'}$, we obtain the decoupled equations

$$\partial^2_\Gamma\Delta^{+}_{\Gamma,\Gamma'}=0, \tag{B9}$$

$$\partial^2_\Gamma\Sigma^{+}_{\Gamma,\Gamma'}=2u^{+}_{\Gamma}u^{-}_{\Gamma}\Sigma^{+}_{\Gamma,\Gamma'}, \tag{B10}$$

together with the boundary conditions

$$\Delta^{+}_{\Gamma',\Gamma'}=\Sigma^{+}_{\Gamma',\Gamma'}=A^{2+}_{\Gamma',\Gamma'}=\frac{1}{2\delta^2}(\gamma'\coth\gamma'-1), \tag{B11}$$

$$\begin{aligned}\partial_\Gamma\Delta^{+}_{\Gamma,\Gamma'}|_{\Gamma=\Gamma'}&=\partial_\Gamma\Sigma^{+}_{\Gamma,\Gamma'}|_{\Gamma=\Gamma'}\\&=B^{++}_{\Gamma',\Gamma'}=\frac{1}{2\delta}\left(\coth\gamma'-\frac{\gamma'}{\sinh^2\gamma'}\right),\end{aligned} \tag{B12}$$

where $\gamma'=\delta\Gamma'$ as usual. In terms of the function

$$\rho(\gamma)=\gamma\coth\gamma-1, \tag{B13}$$

and its derivative with respect to γ,

$$\mathcal{M}(\gamma)=\coth\gamma-\frac{\gamma}{\sinh^2\gamma}, \tag{B14}$$

we finally obtain

$$A^{\pm+}_{\Gamma,\Gamma'}=\frac{1}{4\delta^2}(\rho(\gamma)\pm\gamma\mathcal{M}(\gamma')\pm\rho(\gamma')\mp\gamma'\mathcal{M}(\gamma')), \tag{B15}$$

$$B^{\pm+}_{\Gamma,\Gamma'}=\frac{1}{4\delta}(\mathcal{M}(\gamma)\pm\mathcal{M}(\gamma')). \tag{B16}$$

In the same way we obtain the solutions for the $-$ initial condition,

$$P^{\pm-}_{\Gamma,\Gamma'}(\zeta)=\frac{1}{\overline{l}_{\Gamma}}e^{-\zeta u^{\pm}_{\Gamma}}u^{\pm}_{\Gamma}(A^{\pm-}_{\Gamma,\Gamma'}+\zeta B^{\pm-}_{\Gamma,\Gamma}) \tag{B17}$$

with $A^{\pm-}_{\Gamma,\Gamma'}=A^{\mp+}_{\Gamma,\Gamma'}$ and $B^{\pm-}_{\Gamma,\Gamma'}=B^{\mp+}_{\Gamma,\Gamma'}$. We may check the normalizations

$$\int_0^\infty d\zeta(P^{+\pm}_{\Gamma,\Gamma'}(\zeta)+P^{-\pm}_{\Gamma,\Gamma'}(\zeta))=\frac{1}{2}(1\pm\mathcal{M}(\gamma')). \tag{B18}$$

The autocorrelation is

$$\begin{aligned}&\overline{\langle S_i(t')S_i(t)\rangle}\\&\quad=\int_0^\infty d\zeta(P^{2+}_{\Gamma,\Gamma'}(\zeta)+P^{2-}_{\Gamma,\Gamma'}(\zeta)-P^{+-}_{\Gamma,\Gamma'}(\zeta)-P^{-+}_{\Gamma,\Gamma'}(\zeta))\end{aligned} \tag{B19}$$

$$=\frac{1}{l_\Gamma}\left(2(A^{2+}_{\Gamma,\Gamma'}-A^{-+}_{\Gamma,\Gamma'})+\left(\frac{1}{u^+_\Gamma}+\frac{1}{u^-_\Gamma}\right)(B^{2+}_{\Gamma,\Gamma'}-B^{-+}_{\Gamma,\Gamma'})\right) \tag{B20}$$

$$=(\coth\gamma)\mathcal{M}(\gamma')+\frac{1}{\sinh^2\gamma}(\gamma\mathcal{M}(\gamma')+\rho(\gamma') -\gamma'\mathcal{M}(\gamma')). \tag{B21}$$

Note that $\overline{\langle S_i(t')\rangle}=\mathcal{M}(\gamma')$ and $\overline{\langle S_i(t)\rangle}=\mathcal{M}(\gamma)=\coth\gamma-\gamma/\sinh^2\gamma$.

APPENDIX C: TWO-POINT TWO-TIME CORRELATIONS $\overline{\langle S_0(t)S_x(t)\rangle\langle S_0(t')S_x(t')\rangle}$

1. Definitions

Two points at 0 and $x>0$ are given. To compute the two spin two time correlation we introduce the following quantities. Let $\Omega^{2n,+}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n},l_{2n};x)$, $n=0,1,2\ldots$, [and, respectively, $\Omega^{2n,-}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n},l_{2n};x)$] be the probability that at Γ', x, and 0 are on bonds of same orientation (respectively different orientation), and that at Γ, x, and 0 are on bonds of same orientation, with a configuration $(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n},l_{2n})$. One similarly defines $\Omega^{2n+1,+}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n+1},l_{2n+1};x)$, $n=0,1,2\ldots$, [respectively $\Omega^{2n+1,-}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n+1},l_{2n+1};x)$] be the probability that, at Γ', x and 0 are on bonds of same orientation (different orientations) and that, at Γ, x and 0 are on bonds of different orientations, with a configuration $(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n+1},l_{2n+1})$.

Initial condition at $\Gamma=\Gamma'$ (for all $n\geq 0$):

$$\Omega^{2n,+}_{\Gamma',\Gamma'}(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n},l_{2n};x), \tag{C1}$$

$$=P_{\Gamma'}(\zeta_0,l_0)P_{\Gamma'}(\zeta_1,l_1)\ldots P_{\Gamma'}(\zeta_{2n},l_{2n})W_{\Gamma'}\times\left(l_0,l_{2n},\sum_{i=0}^{2n}l_i-x\right), \tag{C2}$$

$$\Omega^{2n+1,-}_{\Gamma',\Gamma'}(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n+1},l_{2n+1};x), \tag{C3}$$

$$=P_{\Gamma'}(\zeta_0,l_0)P_{\Gamma'}(\zeta_1,l_1)\ldots P_{\Gamma'}(\zeta_{2n+1},l_{2n+1})W_{\Gamma'}\times\left(l_0,l_{2n+1},\sum_{i=0}^{2n+1}l_i-x\right), \tag{C4}$$

with the notation

$$W_\Gamma(l_1,l_2,L)=\frac{2}{\Gamma^2}\int_0^{l_1}dy_1\int_0^{l_2}dy_2\,\delta(L-y_1-y_2) \tag{C5}$$

$$=\frac{2}{\Gamma^2}[\min(l_1,L)-\max(0,L-l_2)]\,\theta[\min(l_1,L)-\max(0,L-l_2)]. \tag{C6}$$

We have, of course, $\Omega^{2n,-}_{\Gamma',\Gamma'}=\Omega^{2n+1,+}_{\Gamma',\Gamma'}=0$.

In the end we are interested in the probabilities $P^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(x)$ that $\langle S_0(t')S_x(t')\rangle=\epsilon'$ and $\langle S_0(t)S_x(t)\rangle=\epsilon$. We have the normalization

$$P^{+,+}_{\Gamma,\Gamma'}(x)+P^{-,-}_{\Gamma,\Gamma'}(x)+P^{-,+}_{\Gamma,\Gamma'}(x)+P^{+,-}_{\Gamma,\Gamma'}(x)=1. \tag{C7}$$

The correlation function is

$$\overline{\langle S_0(t)S_x(t)\rangle\langle S_0(t')S_x(t')\rangle} =P^{+,+}_{\Gamma,\Gamma'}(x)+P^{-,-}_{\Gamma,\Gamma'}(x)-P^{-,+}_{\Gamma,\Gamma'}(x)-P^{+,-}_{\Gamma,\Gamma'}(x). \tag{C8}$$

In terms of the functions Ω, we have

$$P^{+,+}_{\Gamma,\Gamma'}(x)=\sum_{n=0}^{\infty}\int_{\zeta_0,l_0,\ldots\zeta_{2n},l_{2n}}\Omega^{2n,+}_{\Gamma,\Gamma'}\times(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n},l_{2n};x), \tag{C9}$$

$$P^{+,-}_{\Gamma,\Gamma'}(x)=\sum_{n=0}^{\infty}\int_{\zeta_0,l_0,\ldots\zeta_{2n},l_{2n}}\Omega^{2n,-}_{\Gamma,\Gamma'}\times(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n},l_{2n};x), \tag{C10}$$

$$P^{-,+}_{\Gamma,\Gamma'}(x)=\sum_{n=0}^{\infty}\int_{\zeta_0,l_0,\ldots\zeta_{2n+1},l_{2n+1}}\times\Omega^{2n+1,+}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n+1},l_{2n+1};x), \tag{C11}$$

$$P^{-,-}_{\Gamma,\Gamma'}(x)=\sum_{n=0}^{\infty}\int_{\zeta_0,l_0,\ldots\zeta_{2n+1},l_{2n+1}}\times\Omega^{2n+1,-}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1,\ldots;\zeta_{2n+1},l_{2n+1};x). \tag{C12}$$

2. RG equations

The RG equations for $\Omega^{m,\epsilon'}_{\Gamma,\Gamma'}$ for $\epsilon'=\pm$ and $m=0,1,2,\ldots$ read

$$\left(\partial_\Gamma-\sum_{k=0}^{m}\partial_{\zeta_k}\right)\Omega^{m,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1;\ldots\zeta_m,l_m;x) =-2P_\Gamma(\zeta=0)\Omega^{m,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1;\ldots\zeta_m,l_m;x) \tag{C13}$$

$$+\sum_{k=0}^{m}\int_{z,l+l'+l''=l_k}\Omega^{m+2,\epsilon'}_{\Gamma,\Gamma'}\times(\zeta_0,l_0;\ldots;\zeta_{k-1},l_{k-1};z,l;0,l';\zeta_k-z,l'';\ldots;\zeta_m,l_m;x) \tag{C14}$$

$$+\int_{z,l+l_0'+l_1'=l_0} P_\Gamma(\zeta_0-z,l)\Omega^{m+1,\epsilon'}_{\Gamma,\Gamma'} \times(0,l_0';z,l_1';\zeta_1,l_1;\dots;\zeta_m,l_m;x) \quad \text{(C15)}$$

$$+\int_{z,l+l_m'+l_{m+1}'=l_m} \Omega^{m+1,\epsilon'}_{\Gamma,\Gamma'} \times(\zeta_0,l_0;\dots;\zeta_{m-1},l_{m-1};z,l_m';0,l_{m+1}';x) \times P_\Gamma(\zeta_m-z,l) \quad \text{(C16)}$$

$$+\int_{z,l+l'+l_0'=l_0} P_\Gamma(z,l)P_\Gamma(0,l')\Omega^{m,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0 -z,l_0';\zeta_1,l_1;\dots;\zeta_m,l_m;x) \quad \text{(C17)}$$

$$+\int_{z,l_m'+l'+l=l_m} \Omega^{m,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;\dots;\zeta_{m-1},l_{m-1}; \times z,l_m';x)P_\Gamma(0,l')P_\Gamma(\zeta_p-z,l). \quad \text{(C18)}$$

In particular, for $m=0$, we have

$$(\partial_\Gamma-\partial_{\zeta_0})\Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;x) =-2P_\Gamma(\zeta=0)\Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;x) \quad \text{(C19)}$$

$$+\int_{z,l+l'+l''=l_0} \Omega^{2,\epsilon'}_{\Gamma,\Gamma'}(z,l;0,l';\zeta_0-z,l'';x) \quad \text{(C20)}$$

$$+\int_{z,l+l_0'+l_1'=l_0} P_\Gamma(\zeta_0-z,l)\Omega^{1,\epsilon'}_{\Gamma,\Gamma'}(0,l_0';z,l_1';x) \quad \text{(C21)}$$

$$+\int_{z,l+l''+l'=l_0} \Omega^{1,\epsilon'}_{\Gamma,\Gamma'}(z,l'';0,l';x)P_\Gamma(\zeta_0-z,l) \quad \text{(C22)}$$

$$+\int_{z,l+l'+l_0'=l_0} P_\Gamma(z,l)P_\Gamma(0,l')\Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0-z,l_0';x) \quad \text{(C23)}$$

$$+\int_{z,l_0'+l'+l=l_0} \Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(z,l_0';x)P_\Gamma(0,l')P_\Gamma(\zeta_0-z,l) \quad \text{(C24)}$$

$$+\int_{z,l_0'+l'+l=l_0} \Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(0,l_0';x)P_\Gamma(z,l')P_\Gamma(\zeta_0-z,l). \quad \text{(C25)}$$

3. Form of solutions

For $n\geq 1$ and $\epsilon'=\pm$, we set

$$\Omega^{2n,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;\dots;\zeta_{2n},l_{2n};x) =E^{+,\epsilon'}_{\Gamma,\Gamma'}\left(\zeta_0,l_0;\zeta_{2n},l_{2n};\sum_{i=0}^{2n} l_i-x\right) \times P_\Gamma(\zeta_1,l_1)\dots P_\Gamma(\zeta_{2n-1},l_{2n-1}), \quad \text{(C26)}$$

$$\Omega^{2n+1,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;\dots;\zeta_{2n+1},l_{2n+1};x) =E^{-,\epsilon'}_{\Gamma,\Gamma'}\left(\zeta_0,l_0;\zeta_{2n+1},l_{2n+1};\sum_{i=0}^{2n+1} l_i-x\right) \times P_\Gamma(\zeta_1,l_1)\dots P_\Gamma(\zeta_{2n},l_{2n}), \quad \text{(C27)}$$

and also

$$\Omega^{1,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1;x)=E^{-,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_1,l_1;l_0+l_1-x), \quad \text{(C28)}$$

where $P_\Gamma(\zeta,l)$ satisfy the bond equation, and where E satisfies the RG equations

$$(\partial_\Gamma-\partial_{\zeta_1}-\partial_{\zeta_2})E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,l_1;\zeta_2,l_2;L) =-2P_\Gamma(\zeta=0)E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,l_1;\zeta_2,l_2;L) \quad \text{(C29)}$$

$$+P_\Gamma(0,.)*_{l_1}P_\Gamma(.,.)*_{\zeta_1,l_1}E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(.,.;\zeta_2,l_2;L) \quad \text{(C30)}$$

$$+P_\Gamma(0,.)*_{l_2}P_\Gamma(.,.)*_{\zeta_2,l_2}E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,l_1;.,.;L) \quad \text{(C31)}$$

$$+\int_{l+l'+l_1'=l_1} E^{\mp,\epsilon'}_{\Gamma,\Gamma'}(0,l_1';\zeta_2,l_2;L-l) \times P_\Gamma(.,l)*_{\zeta_1}P_\Gamma(.,l') \quad \text{(C32)}$$

$$+\int_{l+l'+l_2'=l_2} E^{\mp,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,l_1;0,l_2';L-l) \times P_\Gamma(.,l)*_{\zeta_2}P_\Gamma(.,l') \quad \text{(C33)}$$

$$+\int_{l+l'+l_1'=l_1} E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(.,l_1';\zeta_2,l_2;L-l-l')*_{\zeta_1} \times P_\Gamma(.,l)P_\Gamma(0,l') \quad \text{(C34)}$$

$$+\int_{l+l'+l_2'=l_2} E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,l_1;.,l_2';L-l-l')*_{\zeta_2} \times P_\Gamma(.,l)P_\Gamma(0,l'), \quad \text{(C35)}$$

with the initial conditions

$$E^{\epsilon,\epsilon'}_{\Gamma',\Gamma'}(\zeta_1,l_1;\zeta_2,l_2;L) = \delta_{\epsilon,\epsilon'} P_{\Gamma'}(\zeta_1,l_1) P_{\Gamma'}(\zeta_2,l_2) W_{\Gamma'}(l_1,l_2,L). \tag{C36}$$

The RG equation for $\Omega^{0,\epsilon'}_{\Gamma,\Gamma'}$ becomes

$$(\partial_\Gamma - \partial_{\zeta_0}) \Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;x)$$

$$= -2P_\Gamma(\zeta=0)\Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,l_0;x) \tag{C37}$$

$$+ \int_{z,l+l'+l''=l_0} E^{+,\epsilon'}_{\Gamma,\Gamma'}(z,l;\zeta_0 - z,l'';l_0 - x) P_\Gamma(0,l') \tag{C38}$$

$$+ \int_{l+l'_0+l'_1=l_0} P_\Gamma(.,l) *_{\zeta_0} \Omega^{1,\epsilon'}_{\Gamma,\Gamma'}(0,l'_0;.,l'_1;x) \tag{C39}$$

$$+ \int_{l+l'_0+l'_1=l_0} \Omega^{1,\epsilon'}_{\Gamma,\Gamma'}(.,l'_0;0,l'_1;x) *_{\zeta_0} P_\Gamma(.,l) \tag{C40}$$

$$+ 2P_\Gamma(0,.) *_{l_0} P_\Gamma(.,.) *_{\zeta_0,l_0} \Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(.,.;x) \tag{C41}$$

$$+ P_\Gamma(.,.) *_{\zeta_0,l_0} P_\Gamma(.,.) *_{l_0} \Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(0,.;x), \tag{C42}$$

with the initial condition

$$\Omega^{0,\epsilon'}_{\Gamma',\Gamma'}(\zeta_0,l_0;x) = \delta_{\epsilon',+1} P_{\Gamma'}(\zeta_0,l_0) W_{\Gamma'}(l_0,l_0,l_0 - x). \tag{C43}$$

Using the form of the solutions, we thus obtain

$$P^{+,+}_{\Gamma,\Gamma'}(x) = \int_{\zeta_0,l_0} \Omega^{0,+}_{\Gamma,\Gamma'}(\zeta_0,l_0;x) \tag{C44}$$

$$+ \sum_{n=1}^{\infty} \int_{\zeta_0,\zeta_{2n}} \int_{L,l_0,l_1 \dots l_{2n}} E^{++}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_{2n}, \times l_{2n};L) \delta\left(L - \left(\sum_{i=0}^{2n} l_i - x\right)\right) \times P_\Gamma(l_1) \cdots P_\Gamma(l_{2n-1}) \tag{C45}$$

$$P^{+,-}_{\Gamma,\Gamma'}(x) = \int_{\zeta_0,l_0} \Omega^{0,-}_{\Gamma,\Gamma'}(\zeta_0,l_0;x) \tag{C46}$$

$$+ \sum_{n=1}^{\infty} \int_{\zeta_0,\zeta_{2n}} \int_{L,l_0,l_1 \dots l_{2n}} E^{+-}_{\Gamma,\Gamma'}(\zeta_0,l_0;\zeta_{2n}, \times l_{2n};L) \delta\left(L - \left(\sum_{i=0}^{2n} l_i - x\right)\right) \times P_\Gamma(l_1) \cdots P_\Gamma(l_{2n-1}) \tag{C47}$$

$$P^{-,+}_{\Gamma,\Gamma'}(x) \tag{C48}$$

$$= + \sum_{n=0}^{\infty} \int_{\zeta_0,\zeta_{2n+1}} \int_{L,l_0,l_1,l_{2n+1}} E^{-,+}_{\Gamma,\Gamma'} \times (\zeta_0,l_0;\zeta_{2n+1},l_{2n+1};L) \delta\left(L - \left(\sum_{i=0}^{2n+1} l_i - x\right)\right) \times P_\Gamma(l_1) \cdots P_\Gamma(l_{2n}) \tag{C49}$$

$$P^{-,-}_{\Gamma,\Gamma'}(x) \tag{C50}$$

$$= \sum_{n=0}^{\infty} \int_{\zeta_0,\zeta_{2n+1}} \int_{L,l_0,l_1,l_{2n+1}} E^{-,-}_{\Gamma,\Gamma'} \times (\zeta_0,l_0;\zeta_{2n+1},l_{2n+1};L) \delta \times \left(L - \left(\sum_{i=0}^{2n+1} l_i - x\right)\right) P_\Gamma(l_1) \cdots P_\Gamma(l_{2n}). \tag{C51}$$

4. Laplace transforms

It is convenient to introduce the Laplace transforms

$$E^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,p_1;\zeta_2,p_2;p) = \int_0^\infty dl_1 \int_0^\infty dl_2 \int_0^\infty dL e^{-p_1 l_1 - p_2 l_2 - pL} \times E_{\Gamma,\Gamma'}(\zeta_1,l_1;\zeta_2,l_2;L). \tag{C52}$$

Using the fixed point solution

$$P_\Gamma(\zeta,p) = U_\Gamma(p) e^{-\zeta u_\Gamma(p)}, \tag{C53}$$

the RG equation for E becomes

$$(\partial_\Gamma - \partial_{\zeta_1} - \partial_{\zeta_2}) E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,p_1;\zeta_2,p_2;p)$$

$$= -2U_\Gamma(0) E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,p_1;\zeta_2,p_2;p) \tag{C54}$$

$$+ U^2_\Gamma(p_1) \int_0^{\zeta_1} dz e^{-(\zeta_1 - z) u_\Gamma(p_1)} E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(z,p_1;\zeta_2,p_2;p) \tag{C55}$$

$$+ U^2_\Gamma(p_2) \int_0^{\zeta_2} dz e^{-(\zeta_2 - z) u_\Gamma(p_2)} E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,p_1;z,p_2;p) \tag{C56}$$

$$+ U_\Gamma(p_1) U_\Gamma(p_1 + p) \frac{e^{-\zeta_1 u_\Gamma(p_1+p)} - e^{-\zeta_1 u_\Gamma(p_1)}}{u_\Gamma(p_1) - u_\Gamma(p_1+p)} \times E^{\mp,\epsilon'}_{\Gamma,\Gamma'}(0,p_1;\zeta_2,p_2;p) \tag{C57}$$

$$+ U_\Gamma(p_2) U_\Gamma(p_2 + p) \frac{e^{-\zeta_2 u_\Gamma(p_2+p)} - e^{-\zeta_2 u_\Gamma(p_2)}}{u_\Gamma(p_2) - u_\Gamma(p_2+p)} \times E^{\mp,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,p_1;0,p_2;p) \tag{C58}$$

$$+U_\Gamma^2(p_1+p)\int_0^{\zeta_1}dz e^{-(\zeta_1-z)u_\Gamma(p_1+p)}E_{\Gamma,\Gamma'}^{\pm,\epsilon'}\times(z,p_1;\zeta_2,p_2;p) \tag{C59}$$

$$+U_\Gamma^2(p_2+p)\int_0^{\zeta_2}dz e^{-(\zeta_2-z)u_\Gamma(p_2+p)}E_{\Gamma,\Gamma'}^{\pm,\epsilon'}\times(\zeta_1,p_1;z,p_2;p), \tag{C60}$$

with the initial conditions

$$E_{\Gamma',\Gamma'}^{\epsilon,\epsilon'}(\zeta_1,p_1;\zeta_2,p_2;p)=\delta_{\epsilon,\epsilon'}\frac{2}{\Gamma'^2p^2}[U_{\Gamma'}(p_1)e^{-\zeta_1u_{\Gamma'}(p_1)}-U_{\Gamma'}(p_1+p)e^{-\zeta_1u_{\Gamma'}(p_1+p)}] \tag{C61}$$

$$[U_{\Gamma'}(p_2)e^{-\zeta_2u_{\Gamma'}(p_2)}-U_{\Gamma'}(p_2+p)e^{-\zeta_2u_{\Gamma'}(p_2+p)}]. \tag{C62}$$

Also defining

$$\Omega_{\Gamma',\Gamma'}^{2n=0,\epsilon'}(\zeta_0,p_0;p)=\int_0^\infty dl_0e^{-p_0l_0}\int_0^\infty dxe^{-px}\Omega_{\Gamma',\Gamma'}^{2n=0,\epsilon'}(\zeta_0,l_0;x), \tag{C63}$$

the RG equation becomes

$$(\partial_\Gamma-\partial_{\zeta_0})\Omega_{\Gamma,\Gamma'}^{0,\epsilon'}(\zeta_0,p_0;p)=-2u_\Gamma\Omega_{\Gamma,\Gamma'}^{0,\epsilon'}(\zeta_0,p_0;p) \tag{C64}$$

$$+\int_z E_{\Gamma,\Gamma'}^{+,\epsilon'}(z,p_0+p;\zeta_0-z,p_0+p;-p)P_\Gamma(0,p_0+p) \tag{C65}$$

$$+U_\Gamma(p_0)\int_0^{\zeta_0}dze^{-(\zeta_0-z)u_\Gamma(p_0)}\Omega_{\Gamma,\Gamma'}^{1,\epsilon'}(0,p_0;z,p_0;p) \tag{C66}$$

$$+U_\Gamma(p_0)\int_0^{\zeta_0}dze^{-(\zeta_0-z)u_\Gamma(p_0)}\Omega_{\Gamma,\Gamma'}^{1,\epsilon'}(z,p_0;0,p_0;p) \tag{C67}$$

$$+2U_\Gamma^2(p_0)\int_0^{\zeta_0}dze^{-(\zeta_0-z)u_\Gamma(p_0)}\Omega_{\Gamma,\Gamma'}^{0,\epsilon'}(z,p_0;p) \tag{C68}$$

$$+U_\Gamma^2(p_0)\zeta_0e^{-\zeta_0u_\Gamma(p_0)}\Omega_{\Gamma,\Gamma'}^{0,\epsilon'}(0,p_0;p), \tag{C69}$$

with the initial condition

$$\Omega_{\Gamma',\Gamma'}^{2n=0,\epsilon'}(\zeta_0,p_0;p)=\delta_{\epsilon',+1}\frac{2}{\Gamma'^2p^2}[U_{\Gamma'}(p_0+p)e^{-\zeta_0u_{\Gamma'}(p_0+p)}-U_{\Gamma'}(p_0)e^{-\zeta_0u_{\Gamma'}(p_0)} \tag{C70}$$

$$-p\partial_{p_0}(U_{\Gamma'}(p_0)e^{-\zeta_0u_{\Gamma'}(p_0)})]. \tag{C71}$$

In Laplace terms

$$P_{\Gamma,\Gamma'}^{\epsilon,\epsilon'}(q)=\int_0^\infty dxe^{-qx}P_{\Gamma,\Gamma'}^{\epsilon,\epsilon'}(x) \tag{C72}$$

relations (C51) become

$$P_{\Gamma,\Gamma'}^{+,+}(q)=\int_{\zeta_0}\Omega_{\Gamma,\Gamma'}^{0,+}(\zeta_0,0;q)+\frac{P_\Gamma(q)}{1-P_\Gamma^2(q)}\int_{\zeta_1,\zeta_2}E_{\Gamma,\Gamma'}^{++}(\zeta_1,q;\zeta_2,q;-q), \tag{C73}$$

$$P_{\Gamma,\Gamma'}^{+,-}(q)=\int_{\zeta_0}\Omega_{\Gamma,\Gamma'}^{0,-}(\zeta_0,0;q)+\frac{P_\Gamma(q)}{1-P_\Gamma^2(q)}\int_{\zeta_1,\zeta_2}E_{\Gamma,\Gamma'}^{+-}(\zeta_1,q;\zeta_2,q;-q), \tag{C74}$$

$$P_{\Gamma,\Gamma'}^{-,+}(q)=\frac{1}{1-P_\Gamma^2(q)}\int_{\zeta_1,\zeta_2}E_{\Gamma,\Gamma'}^{-+}(\zeta_1,q;\zeta_2,q;-q), \tag{C75}$$

$$P_{\Gamma,\Gamma'}^{-,-}(q)=\frac{1}{1-P_\Gamma^2(q)}\int_{\zeta_1,\zeta_2}E_{\Gamma,\Gamma'}^{--}(\zeta_1,q;\zeta_2,q;-q). \tag{C76}$$

Thus for the functions E, we need to solve only the case $p_1=p_2=q=-p$, i.e., with the notation $U_\Gamma(p=0)=u_\Gamma(p=0)=u_\Gamma$,

$$(\partial_\Gamma-\partial_{\zeta_1}-\partial_{\zeta_2})E_{\Gamma,\Gamma'}^{\pm,\epsilon'}(\zeta_1,q;\zeta_2,q;-q)=-2u_\Gamma E_{\Gamma,\Gamma'}^{\pm,\epsilon'}(\zeta_1,q;\zeta_2,q;-q) \tag{C77}$$

$$+U_\Gamma^2(q)\int_0^{\zeta_1}dze^{-(\zeta_1-z)u_\Gamma(q)}E_{\Gamma,\Gamma'}^{\pm,\epsilon'}(z,q;\zeta_2,q;-q) \tag{C78}$$

$$+U_\Gamma^2(q)\int_0^{\zeta_2}dze^{-(\zeta_2-z)u_\Gamma(q)}E_{\Gamma,\Gamma'}^{\pm,\epsilon'}(\zeta_1,q;z,q;-q) \tag{C79}$$

$$+U_\Gamma(q)u_\Gamma \frac{e^{-\zeta_1 u_\Gamma}-e^{-\zeta_1 u_\Gamma(q)}}{u_\Gamma(q)-u_\Gamma} E^{\mp,\epsilon'}_{\Gamma,\Gamma'}(0,q;\zeta_2,q;-q) \quad \text{(C80)}$$

$$+U_\Gamma(q)u_\Gamma \frac{e^{-\zeta_2 u_\Gamma}-e^{-\zeta_2 u_\Gamma(q)}}{u_\Gamma(q)-u_\Gamma} E^{\mp,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,q;0,q;-q) \quad \text{(C81)}$$

$$+u_\Gamma^2\int_0^{\zeta_1} dz e^{-(\zeta_1-z)u_\Gamma} E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(z,q;\zeta_2,q;-q) \quad \text{(C82)}$$

$$+u_\Gamma^2\int_0^{\zeta_2} dz e^{-(\zeta_2-z)u_\Gamma} E^{\pm,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,q;z,q;-q), \quad \text{(C83)}$$

and with the initial conditions

$$E^{\epsilon,\epsilon'}_{\Gamma',\Gamma'}(\zeta_1,q;\zeta_2,q;-q) = \delta_{\epsilon,\epsilon'}\frac{2}{\Gamma'^2 q^2}[U_{\Gamma'}(q)e^{-\zeta_1 u_{\Gamma'}(q)}-u_{\Gamma'}e^{-\zeta_1 u_{\Gamma'}}] \quad \text{(C84)}$$

$$[U_{\Gamma'}(q)e^{-\zeta_2 u_{\Gamma'}(q)}-u_{\Gamma'}e^{-\zeta_2 u_{\Gamma'}}]. \quad \text{(C85)}$$

For $n=0$ we need to solve only the cases $p_0=p_1=0$ and $p=q$, i.e., the equation

$$(\partial_\Gamma-\partial_{\zeta_0})\Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,0;q) = -2u_\Gamma \Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(\zeta_0,0;q) \quad \text{(C86)}$$

$$+U_\Gamma(q)\int_z E^{+,\epsilon'}_{\Gamma,\Gamma'}(z,q;\zeta_0-z,q;-q) \quad \text{(C87)}$$

$$+u_\Gamma\int_0^{\zeta_0} dz e^{-(\zeta_0-z)u_\Gamma}\Omega^{1,\epsilon'}_{\Gamma,\Gamma'}(0,0;z,0;q) \quad \text{(C88)}$$

$$+u_\Gamma\int_0^{\zeta_0} dz e^{-(\zeta_0-z)u_\Gamma}\Omega^{1,\epsilon'}_{\Gamma,\Gamma'}(z,0;0,0;q) \quad \text{(C89)}$$

$$+2u_\Gamma^2\int_0^{\zeta_0} dz e^{-(\zeta_0-z)u_\Gamma}\Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(z,0;q) \quad \text{(C90)}$$

$$+u_\Gamma^2\zeta_0 e^{-\zeta_0 u_\Gamma}\Omega^{0,\epsilon'}_{\Gamma,\Gamma'}(0,0;q), \quad \text{(C91)}$$

with the initial conditions

$$\Omega^{0,\epsilon'}_{\Gamma',\Gamma'}(\zeta_0,0;q) = \delta_{\epsilon',+1}\frac{2}{\Gamma'^2 q^2}[U_{\Gamma'}(q)e^{-\zeta_0 u_{\Gamma'}(q)}-U_{\Gamma'}e^{-\zeta_0 u_{\Gamma'}} \quad \text{(C92)}$$

$$-p\partial_{p_0}(U_{\Gamma'}(p_0)e^{-\zeta_0 u_{\Gamma'}(p_0)})|_{p_0=0}]. \quad \text{(C93)}$$

5. Solution for the functions *E*

Using the symmetry in (ζ_1,ζ_2), we set

$$E^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(\zeta_1,q;\zeta_2,q;-q) = \frac{2}{\Gamma^2}(A^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)e^{-\zeta_1 u_\Gamma(q)}e^{-\zeta_2 u_\Gamma(q)} + B^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)e^{-\zeta_1 u_\Gamma}e^{-\zeta_2 u_\Gamma(q)} \quad \text{(C94)}$$

$$+B^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)e^{-\zeta_1 u_\Gamma(q)}e^{-\zeta_2 u_\Gamma}+D^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)e^{-\zeta_1 u_\Gamma}e^{-\zeta_2 u_\Gamma}). \quad \text{(C95)}$$

It is in fact more convenient to use the following combinations:

$$I^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q) = \frac{\Gamma^2}{2}E^{\epsilon,\epsilon'}_{\Gamma',\Gamma'}(\zeta_1=0,q;\zeta_2=0,q;-q) \quad \text{(C96)}$$

$$=A^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)+2B^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)+D^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q), \quad \text{(C97)}$$

$$J^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q) = \frac{\Gamma^2}{2}\int_0^\infty d\zeta_2 E^{\epsilon,\epsilon'}_{\Gamma',\Gamma'}(\zeta_1=0,q;\zeta_2,q;-q) \quad \text{(C98)}$$

$$=\frac{A^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)}{u_\Gamma(q)}+B^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)\left(\frac{1}{u_\Gamma(q)}+\frac{1}{u_\Gamma}\right) + \frac{D^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)}{u_\Gamma}, \quad \text{(C99)}$$

$$K^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q) = \frac{\Gamma^2}{2}\int_0^\infty d\zeta_1\int_0^\infty d\zeta_2 E^{\epsilon,\epsilon'}_{\Gamma',\Gamma'}(\zeta_1,q;\zeta_2,q;-q) \quad \text{(C100)}$$

$$=\frac{A^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)}{u_\Gamma^2(q)}+2\frac{B^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)}{u_\Gamma(q)u_\Gamma}+\frac{D^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)}{u_\Gamma^2}. \quad \text{(C101)}$$

They satisfy the system of equations

$$\partial_\Gamma I^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q) = -2(u_\Gamma+u_\Gamma(q))I^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q)+2u_\Gamma u_\Gamma(q)J^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q), \quad \text{(C102)}$$

$$\partial_\Gamma J^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q) = \left(\frac{U_\Gamma^2(q)}{u_\Gamma(q)}-u_\Gamma(q)\right)J^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q)-I^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q) + u_\Gamma u_\Gamma(q)K^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q)+\frac{U_\Gamma(q)}{u_\Gamma(q)}I^{\mp,\epsilon'}_{\Gamma,\Gamma'}(q), \quad \text{(C103)}$$

$$\partial_\Gamma K^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q)=\left(2\frac{U^2_\Gamma(q)}{u_\Gamma(q)}+2u_\Gamma\right)K^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q)-2J^{\pm,\epsilon'}_{\Gamma,\Gamma'}(q)+2\frac{U_\Gamma(q)}{u_\Gamma(q)}J^{\mp,\epsilon'}_{\Gamma,\Gamma'}(q), \tag{C104}$$

and the initial conditions

$$I^{\epsilon,\epsilon'}_{\Gamma',\Gamma'}(q)=\delta_{\epsilon,\epsilon'}\frac{1}{q^2}[U_{\Gamma'}(q)-u_{\Gamma'}]^2, \tag{C105}$$

$$J^{\epsilon,\epsilon'}_{\Gamma',\Gamma'}(q)=\delta_{\epsilon,\epsilon'}\frac{1}{q^2}[U_{\Gamma'}(q)-u_{\Gamma'}]\left[\frac{U_{\Gamma'}(q)}{u_{\Gamma'}(q)}-1\right], \tag{C106}$$

$$K^{\epsilon,\epsilon'}_{\Gamma',\Gamma'}(q)=\delta_{\epsilon,\epsilon'}\frac{1}{q^2}\left[\frac{U_{\Gamma'}(q)}{u_{\Gamma'(q)}}-1\right]^2. \tag{C107}$$

It is convenient to introduce the following sums and differences:

$$I^{S,\epsilon'}_{\Gamma',\Gamma'}(q)=I^{+,\epsilon'}_{\Gamma',\Gamma'}(q)+I^{-,\epsilon'}_{\Gamma',\Gamma'}(q), \tag{C108}$$

$$I^{D,\epsilon'}_{\Gamma',\Gamma'}(q)=I^{+,\epsilon'}_{\Gamma',\Gamma'}(q)-I^{-,\epsilon'}_{\Gamma',\Gamma'}(q), \tag{C109}$$

$$J^{S,\epsilon'}_{\Gamma',\Gamma'}(q)=J^{+,\epsilon'}_{\Gamma',\Gamma'}(q)+J^{-,\epsilon'}_{\Gamma',\Gamma'}(q), \tag{C110}$$

$$J^{D,\epsilon'}_{\Gamma',\Gamma'}(q)=J^{+,\epsilon'}_{\Gamma',\Gamma'}(q)-J^{-,\epsilon'}_{\Gamma',\Gamma'}(q), \tag{C111}$$

$$K^{S,\epsilon'}_{\Gamma',\Gamma'}(q)=K^{+,\epsilon'}_{\Gamma',\Gamma'}(q)+K^{-,\epsilon'}_{\Gamma',\Gamma'}(q), \tag{C112}$$

$$K^{D,\epsilon'}_{\Gamma',\Gamma'}(q)=K^{+,\epsilon'}_{\Gamma',\Gamma'}(q)-K^{-,\epsilon'}_{\Gamma',\Gamma'}(q). \tag{C113}$$

Then, for ϵ' fixed, the three functions $I^{S,\epsilon'}$, $J^{S,\epsilon'}$, and $K^{S,\epsilon'}$ satisfy the system of equations

$$\partial_\Gamma I^{S,\epsilon'}_{\Gamma,\Gamma'}(q)=-2(u_\Gamma+u_\Gamma(q))I^{S,\epsilon'}_{\Gamma,\Gamma'}(q)+2u_\Gamma u_\Gamma(q)J^{S,\epsilon'}_{\Gamma,\Gamma'}(q), \tag{C114}$$

$$\partial_\Gamma J^{S,\epsilon'}_{\Gamma,\Gamma'}(q)=\left(\frac{U^2_\Gamma(q)}{u_\Gamma(q)}-u_\Gamma(q)\right)J^{S,\epsilon'}_{\Gamma,\Gamma'}(q)-I^{S,\epsilon'}_{\Gamma,\Gamma'}(q)+u_\Gamma u_\Gamma(q)K^{S,\epsilon'}_{\Gamma,\Gamma'}(q)+\frac{U_\Gamma(q)}{u_\Gamma(q)}I^{S,\epsilon'}_{\Gamma,\Gamma'}(q), \tag{C115}$$

$$\partial_\Gamma K^{S,\epsilon'}_{\Gamma,\Gamma'}(q)=\left(2\frac{U^2_\Gamma(q)}{u_\Gamma(q)}+2u_\Gamma\right)K^{S,\epsilon'}_{\Gamma,\Gamma'}(q)-2J^{S,\epsilon'}_{\Gamma,\Gamma'}(q)+2\frac{U_\Gamma(q)}{u_\Gamma(q)}J^{S,\epsilon'}_{\Gamma,\Gamma'}(q), \tag{C116}$$

with the initial conditions

$$I^{S,\epsilon'}_{\Gamma',\Gamma'}(q)=\frac{1}{q^2}[U_{\Gamma'}(q)-u_{\Gamma'}]^2, \tag{C117}$$

$$J^{S,\epsilon'}_{\Gamma',\Gamma'}(q)=\frac{1}{q^2}[U_{\Gamma'}(q)-u_{\Gamma'}]\left[\frac{U_{\Gamma'}(q)}{u_{\Gamma'}(q)}-1\right], \tag{C118}$$

$$K^{S,\epsilon'}_{\Gamma',\Gamma'}(q)=\frac{1}{q^2}\left[\frac{U_{\Gamma'}(q)}{u_{\Gamma'(q)}}-1\right]^2; \tag{C119}$$

thus the solutions are simply

$$I^{S,\epsilon'}_{\Gamma,\Gamma'}(q)=\frac{1}{q^2}[U_\Gamma(q)-u_\Gamma]^2, \tag{C120}$$

$$J^{S,\epsilon'}_{\Gamma,\Gamma'}(q)=\frac{1}{q^2}[U_\Gamma(q)-u_\Gamma]\left[\frac{U_\Gamma(q)}{u_\Gamma(q)}-1\right], \tag{C121}$$

$$K^{S,\epsilon'}_{\Gamma,\Gamma'}(q)=\frac{1}{q^2}\left[\frac{U_\Gamma(q)}{u_\Gamma(q)}-1\right]^2. \tag{C122}$$

For ϵ' fixed, the three functions $I^{D,\epsilon'}$, $J^{D,\epsilon'}$, and $K^{D,\epsilon'}$ satisfy the system of equations

$$\partial_\Gamma I^{D,\epsilon'}_{\Gamma,\Gamma'}(q)=-2(u_\Gamma+u_\Gamma(q))I^{D,\epsilon'}_{\Gamma,\Gamma'}(q)+2u_\Gamma u_\Gamma(q)J^{D,\epsilon'}_{\Gamma,\Gamma'}(q), \tag{C123}$$

$$\partial_\Gamma J^{D,\epsilon'}_{\Gamma,\Gamma'}(q)=\left(\frac{U^2_\Gamma(q)}{u_\Gamma(q)}-u_\Gamma(q)\right)J^{D,\epsilon'}_{\Gamma,\Gamma'}(q)-I^{D,\epsilon'}_{\Gamma,\Gamma'}(q)+u_\Gamma u_\Gamma(q)K^{D,\epsilon'}_{\Gamma,\Gamma'}(q)-\frac{U_\Gamma(q)}{u_\Gamma(q)}I^{D,\epsilon'}_{\Gamma,\Gamma'}(q), \tag{C124}$$

$$\partial_\Gamma K^{D,\epsilon'}_{\Gamma,\Gamma'}(q)=\left(2\frac{U^2_\Gamma(q)}{u_\Gamma(q)}+2u_\Gamma\right)K^{D,\epsilon'}_{\Gamma,\Gamma'}(q)-2J^{D,\epsilon'}_{\Gamma,\Gamma'}(q)-2\frac{U_\Gamma(q)}{u_\Gamma(q)}J^{D,\epsilon'}_{\Gamma,\Gamma'}(q), \tag{C125}$$

with the initial conditions

$$I^{D,\epsilon'}_{\Gamma',\Gamma'}(q)=(-1)^{\epsilon'}\frac{1}{q^2}[U_{\Gamma'}(q)-u_{\Gamma'}]^2, \tag{C126}$$

$$J^{D,\epsilon'}_{\Gamma',\Gamma'}(q)=(-1)^{\epsilon'}\frac{1}{q^2}[U_{\Gamma'}(q)-u_{\Gamma'}]\left[\frac{U_{\Gamma'}(q)}{u_{\Gamma'}(q)}-1\right], \tag{C127}$$

$$K^{D,\epsilon'}_{\Gamma',\Gamma'}(q)=(-1)^{\epsilon'}\frac{1}{q^2}\left[\frac{U_{\Gamma'}(q)}{u_{\Gamma'(q)}}-1\right]^2. \tag{C128}$$

Three linearly independent solutions of the system read

$$I_\Gamma^{D1}(q)=[U_\Gamma(q)+u_\Gamma]^2, \tag{C129}$$

$$J_\Gamma^{D1}(q)=[U_\Gamma(q)+u_\Gamma]\left[\frac{U_\Gamma(q)}{u_\Gamma(q)}+1\right], \tag{C130}$$

$$K_\Gamma^{D1}(q)=\left[\frac{U_\Gamma(q)}{u_{\Gamma(q)}}+1\right]^2, \tag{C131}$$

$$I_\Gamma^{D2}(q)=2u_\Gamma\frac{U_\Gamma(q)+u_\Gamma}{U_\Gamma(q)+u_\Gamma(q)}, \tag{C132}$$

$$J_\Gamma^{D2}(q)=\frac{(2u_\Gamma+U_\Gamma(q))\left[\frac{U_\Gamma(q)}{u_\Gamma(q)}+1\right]}{U_\Gamma(q)+u_\Gamma(q)}, \tag{C133}$$

$$K_\Gamma^{D2}(q)=2\frac{\left[\frac{U_\Gamma(q)}{u_{\Gamma(q)}}+1\right]}{u_\Gamma(q)}, \tag{C134}$$

$$I_\Gamma^{D3}(q)=\frac{U_\Gamma(q)[2u_\Gamma^2+(2u_\Gamma+U_\Gamma(q))(u_\Gamma(q)+U_\Gamma(q))]}{(u_\Gamma(q)+U_\Gamma(q))(u_\Gamma(q)^2-U_\Gamma(q)^2)}, \tag{C135}$$

$$J_\Gamma^{D3}(q)=\frac{U_\Gamma(q)[2u_\Gamma+u_\Gamma(q)+U_\Gamma(q)]}{u_\Gamma(q)(u_\Gamma(q)^2-U_\Gamma(q)^2))}, \tag{C136}$$

$$K_\Gamma^{D3}(q)=\frac{2U_\Gamma(q)}{u_\Gamma(q)^2(u_\Gamma(q)-U_\Gamma(q))}. \tag{C137}$$

It is useful to consider the matrix formed by these solutions,

$$N_\Gamma=\begin{pmatrix} I_\Gamma^{D1}(q) & I_\Gamma^{D2}(q) & I_\Gamma^{D3}(q) \\ J_\Gamma^{D1}(q) & J_\Gamma^{D2}(q) & J_\Gamma^{D3}(q) \\ K_\Gamma^{D1}(q) & K_\Gamma^{D2}(q) & K_\Gamma^{D3}(q) \end{pmatrix},$$

whose determinant gives the Wronskian W_Γ of the three linear independent solutions:

$$W_\Gamma=\det[N_\Gamma]=-\frac{U_\Gamma^3(q)}{u_\Gamma^3(q)}. \tag{C138}$$

The solution satisfying the initial conditions we are interested in will be obtained as a linear combination,

$$\begin{pmatrix} I_{\Gamma,\Gamma'}^{D,\epsilon'}(q) \\ J_{\Gamma,\Gamma'}^{D,\epsilon'}(q) \\ K_{\Gamma,\Gamma'}^{D,\epsilon'}(q) \end{pmatrix}=\sum_{i=1}^{3}\lambda_{\Gamma'}^{i\epsilon'}(q)\begin{pmatrix} I_\Gamma^{Di}(q) \\ J_\Gamma^{Di}(q) \\ K_\Gamma^{Di}(q) \end{pmatrix}\equiv N_\Gamma\begin{pmatrix} \lambda_{\Gamma'}^{1\epsilon'}(q) \\ \lambda_{\Gamma'}^{2\epsilon'}(q) \\ \lambda_{\Gamma'}^{3\epsilon'}(q) \end{pmatrix},$$

where the coefficients $\lambda_{\Gamma'}^{1\epsilon'}(q)$ are determined by the initial conditions

$$\begin{pmatrix} \lambda_{\Gamma'}^{1\epsilon'}(q) \\ \lambda_{\Gamma'}^{2\epsilon'}(q) \\ \lambda_{\Gamma'}^{3\epsilon'}(q) \end{pmatrix}=N_{\Gamma'}^{-1}\begin{pmatrix} I_{\Gamma',\Gamma'}^{D,\epsilon'}(q) \\ J_{\Gamma',\Gamma'}^{D,\epsilon'}(q) \\ K_{\Gamma',\Gamma'}^{D,\epsilon'}(q) \end{pmatrix}. \tag{C139}$$

The inverse of the matrix N_Γ is

$$N_\Gamma^{-1}=\begin{pmatrix} \frac{2u_\Gamma(q)}{U_\Gamma(q)^2[u_\Gamma(q)-U_\Gamma(q)]} & -\frac{2u_\Gamma(q)(2u_\Gamma u_\Gamma(q)+U_\Gamma(q)[u_\Gamma(q)+U_\Gamma(q)])}{U_\Gamma(q)^2(u_\Gamma(q)^2-U_\Gamma(q)^2)} & \frac{u_\Gamma(q)^2(2u_\Gamma^2u_\Gamma(q)+U_\Gamma(q)[2u_\Gamma+U_\Gamma(q)][u_\Gamma(q)+U_\Gamma(q)])}{U_\Gamma(q)^2[u_\Gamma(q)+U_\Gamma(q)][u_\Gamma(q)^2-U_\Gamma(q)^2]} \\ -\frac{(u_\Gamma(q)+U_\Gamma(q))}{U_\Gamma(q)^2} & \frac{u_\Gamma(q)(2u_\Gamma+U_\Gamma(q))}{U_\Gamma(q)^2} & -\frac{u_\Gamma u_\Gamma(q)^2[u_\Gamma+U_\Gamma(q)]}{U_\Gamma(q)^2[u_\Gamma(q)+U_\Gamma(q)]} \\ -\frac{[u_\Gamma(q)+U_\Gamma(q)]^2}{U_\Gamma(q)^2} & \frac{2u_\Gamma(q)[u_\Gamma+U_\Gamma(q)][u_\Gamma(q)+U_\Gamma(q)]}{U_\Gamma(q)} & -\frac{u_\Gamma(q)^2[u_\Gamma+U_\Gamma(q)]^2}{U_\Gamma(q)^2} \end{pmatrix}, \tag{C140}$$

and, finally, we obtain the coefficients

$$\lambda_{\Gamma'}^{1\epsilon'}(q)=-(-1)^{\epsilon'}\frac{\{-8u_{\Gamma'}^2u_{\Gamma'}(q)+[u_{\Gamma'}(q)+U_{\Gamma'}(q)][-5u_{\Gamma'}(q)^2+U_{\Gamma'}^2(q)+4u_{\Gamma'}(3u_{\Gamma'}(q)-U_{\Gamma'}(q))]\}}{q^2(u_{\Gamma'}(q)-U_{\Gamma'}(q))(u_{\Gamma'}(q)+U_{\Gamma'}(q))^2)} \tag{C141}$$

$$\lambda^{2\epsilon'}_{\Gamma'}(q) = -(-1)^{\epsilon'}\frac{2(u_{\Gamma'}-u_{\Gamma'}(q))(2u_{\Gamma'}-u_{\Gamma'}(q)-U_{\Gamma'}(q))}{q^2(u_{\Gamma'}(q)+U_{\Gamma'}(q))} \tag{C142}$$

$$\lambda^{3\epsilon'}_{\Gamma'}(q) = -(-1)^{\epsilon'}\frac{4(u_{\Gamma'}-u_{\Gamma'}(q))^2}{q^2}. \tag{C143}$$

In the end we are interested in

$$P^{+,+}_{\Gamma,\Gamma'}(q) = \int_{\zeta_0}\Omega^{0,+}_{\Gamma,\Gamma'}(\zeta_0,0;q) + \frac{2}{\Gamma^2 q^2}\frac{P_\Gamma(q)}{1-P^2_\Gamma(q)}K^{+,+}_{\Gamma,\Gamma'}(q), \tag{C144}$$

$$P^{+,-}_{\Gamma,\Gamma'}(q) = \int_{\zeta_0}\Omega^{0,-}_{\Gamma,\Gamma'}(\zeta_0,0;q) + \frac{2}{\Gamma^2 q^2}\frac{P_\Gamma(q)}{1-P^2_\Gamma(q)}K^{+,-}_{\Gamma,\Gamma'}(q), \tag{C145}$$

$$P^{-,+}_{\Gamma,\Gamma'}(q) = \frac{2}{\Gamma^2 q^2}\,\frac{1}{1-P^2_\Gamma(q)}K^{-,+}_{\Gamma,\Gamma'}(q), \tag{C146}$$

$$P^{-,-}_{\Gamma,\Gamma'}(q) = \frac{2}{\Gamma^2 q^2}\,\frac{1}{1-P^2_\Gamma(q)}K^{-,-}_{\Gamma,\Gamma'}(q). \tag{C147}$$

Since we have the relations

$$K^{S,+}_{\Gamma,\Gamma'}(q) = K^{S,-}_{\Gamma,\Gamma'}(q), \tag{C148}$$

$$K^{D,+}_{\Gamma,\Gamma'}(q) = -K^{D,-}_{\Gamma,\Gamma'}(q), \tag{C149}$$

we can express the four functions $K^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q)$ in terms of $K^{S,+}_{\Gamma,\Gamma'}(q)$ and $K^{D,+}_{\Gamma,\Gamma'}(q)$ only as

$$K^{\epsilon,\epsilon'}_{\Gamma,\Gamma'}(q) = \tfrac{1}{2}(K^{S,+}_{\Gamma,\Gamma'}(q) + (-1)^{\epsilon}(-1)^{\epsilon'}K^{D,+}_{\Gamma,\Gamma'}(q)) \tag{C150}$$

Since we have the constraints

$$P^{+,+}_{\Gamma,\Gamma'}(q) + P^{+,-}_{\Gamma,\Gamma'}(q) = P^{+}_{\Gamma}(q), \tag{C151}$$

$$P^{-,+}_{\Gamma,\Gamma'}(q) + P^{-,-}_{\Gamma,\Gamma'}(q) = P^{-}_{\Gamma}(q), \tag{C152}$$

$$P^{+,+}_{\Gamma,\Gamma'}(q) + P^{-,+}_{\Gamma,\Gamma'}(q) = P^{+}_{\Gamma'}(q), \tag{C153}$$

$$P^{+,-}_{\Gamma,\Gamma'}(q) + P^{-,-}_{\Gamma,\Gamma'}(q) = P^{-}_{\Gamma'}(q), \tag{C154}$$

where

$$P^{+}_{\Gamma}(q) = \frac{1}{q} - \frac{2}{\Gamma^2 q^2}\frac{1-P_\Gamma(q)}{1+P_\Gamma(q)}, \tag{C155}$$

$$P^{-}_{\Gamma}(q) = \frac{2}{\Gamma^2 q^2}\frac{1-P_\Gamma(q)}{1+P_\Gamma(q)}, \tag{C156}$$

we do not have to compute the functions $\Omega^{0,-}_{\Gamma,\Gamma'}(\zeta_0,0;q)$ separately. Indeed, we obtain

$$P^{+}_{\Gamma}(q) = \int_{\zeta_0}\Omega^{0,+}_{\Gamma,\Gamma'}(\zeta_0,0;q) + \int_{\zeta_0}\Omega^{0,-}_{\Gamma,\Gamma'}(\zeta_0,0;q) + \frac{2}{\Gamma^2}\frac{P_\Gamma(q)}{1-P^2_\Gamma(q)}K^{S,+}_{\Gamma,\Gamma'}(q), \tag{C157}$$

$$P^{-}_{\Gamma}(q) = \frac{2}{\Gamma^2}\,\frac{1}{1-P^2_\Gamma(q)}K^{S,+}_{\Gamma,\Gamma'}(q), \tag{C158}$$

$$P^{+}_{\Gamma'}(q) = \int_{\zeta_0}\Omega^{0,+}_{\Gamma,\Gamma'}(\zeta_0,0;q) + \frac{1}{\Gamma^2}\frac{1}{1-P^2_\Gamma(q)} \times[(1+P_\Gamma(q))K^{S,+}_{\Gamma,\Gamma'}(q) - (1-P_\Gamma(q))K^{D,+}_{\Gamma,\Gamma'}(q)], \tag{C159}$$

$$P^{-}_{\Gamma'}(q) = \int_{\zeta_0}\Omega^{0,-}_{\Gamma,\Gamma'}(\zeta_0,0;q) + \frac{1}{\Gamma^2}\frac{1}{1-P^2_\Gamma(q)} \times[(1+P_\Gamma(q))K^{S,+}_{\Gamma,\Gamma'}(q) + (1-P_\Gamma(q)) \times K^{D,+}_{\Gamma,\Gamma'}(q)]. \tag{C160}$$

The second equation is satisfied since $K^{S,+}_{\Gamma,\Gamma'}(q) = (1/q^2)[1-P_\Gamma(q)]^2$. The three other are compatible, and give

$$\int_{\zeta_0}\Omega^{0,+}_{\Gamma,\Gamma'}(\zeta_0,0;q) = \frac{1}{q} - \frac{2}{\Gamma'^2 q^2}\frac{1-P_{\Gamma'}(q)}{1+P_{\Gamma'}(q)} - \frac{1}{\Gamma^2 q^2} \times(1-P_\Gamma(q)) + \frac{1}{\Gamma^2}\frac{1}{1+P_\Gamma(q)} \times K^{D,+}_{\Gamma,\Gamma'}(q), \tag{C161}$$

$$\int_{\zeta_0}\Omega^{0,-}_{\Gamma,\Gamma'}(\zeta_0,0;q) = \frac{2}{\Gamma'^2 q^2}\frac{1-P_{\Gamma'}(q)}{1+P_{\Gamma'}(q)} - \frac{1}{\Gamma^2 q^2}(1-P_\Gamma(q)) - \frac{1}{\Gamma^2}\frac{1}{1+P_\Gamma(q)}K^{D,+}_{\Gamma,\Gamma'}(q). \tag{C162}$$

6. Final result

The Laplace transform of the correlation function can now be obtained as

$$\int_0^\infty dx e^{-qx}\overline{\langle S_0(t)S_x(t)\rangle\langle S_0(t')S_x(t')\rangle}$$
$$=P^{+,+}_{\Gamma,\Gamma'}(q)+P^{-,-}_{\Gamma,\Gamma'}(q)-P^{-,+}_{\Gamma,\Gamma'}(q)-P^{+,-}_{\Gamma,\Gamma'}(q) \quad \text{(C163)}$$

$$=\frac{1}{q}-\frac{4}{\Gamma'^2q^2}\frac{1-P_{\Gamma'}(q)}{1+P_{\Gamma'}(q)}+\frac{4}{\Gamma^2}\frac{1}{1-P^2_\Gamma(q)}K^{D,+}_{\Gamma,\Gamma'}(q), \quad \text{(C164)}$$

and thus the final explicit expression is

$$\int_0^\infty dx e^{-qx}\overline{\langle S_0(t)S_x(t)\rangle\langle S_0(t')S_x(t')\rangle}$$
$$=\frac{1}{q}-\frac{4}{\Gamma'^2q^2}\tanh^2\left(\frac{\Gamma'\sqrt{q}}{2}\right) \quad \text{(C165)}$$

$$+\frac{4}{\Gamma^2}\coth^2\left(\frac{\Gamma\sqrt{q}}{2}\right)\lambda^{1+}_{\Gamma'}(q)+\frac{8}{\Gamma^2}\frac{\coth\left(\frac{\Gamma\sqrt{q}}{2}\right)}{\sqrt{q}}$$
$$\times\lambda^{2+}_{\Gamma'}(q)+\frac{4}{\Gamma^2}\frac{1}{q\sinh^2\left(\frac{\Gamma\sqrt{q}}{2}\right)}\lambda^{3+}_{\Gamma'}(q), \quad \text{(C166)}$$

where

$$\lambda^{1+}_{\Gamma'}(q)=\frac{1}{2\Gamma'^2q^3\sinh^2(\Gamma'\sqrt{q})}[8+3\Gamma'^2q$$
$$-16\cosh(\Gamma'\sqrt{q})+8\Gamma'\sqrt{q}\sinh(\Gamma'\sqrt{q}) \quad \text{(C167)}$$

$$+(8+5\Gamma'^2q)\cosh(2\Gamma'\sqrt{q})$$
$$-12\Gamma'\sqrt{q}\sinh(2\Gamma'\sqrt{q})] \quad \text{(C168)}$$

$$\lambda^{2+}_{\Gamma'}(q)=-\frac{2}{q^2}\left(\frac{1}{\Gamma'}-\sqrt{q}\coth(\Gamma'\sqrt{q})\right)$$
$$\times\left(\frac{2}{\Gamma'\sqrt{q}}\tanh\left(\frac{\Gamma'\sqrt{q}}{2}\right)-1\right) \quad \text{(C169)}$$

$$\lambda^{3\epsilon'}_{\Gamma'}(q)=-\frac{4}{q^2}\left(\frac{1}{\Gamma'}-\sqrt{q}\coth(\Gamma'\sqrt{q})\right)^2. \quad \text{(C170)}$$

This leads to the scaling form given in the text in Eqs. (114) and (117).

APPENDIX D: FINITE SIZE PROPERTIES

In this appendix we sketch the derivation of the finite size results from the finite size measure for the RFIM. If spins at both ends are fixed to have the same (opposite) value, there are an odd (even) number of bonds in the finite size measure.

Assuming for definitness that spins at both extremities are fixed to the values $S_0=+1$ and $S_L=-1$. Then domains closest to the boundaries are domain walls of type A which as Sinai walkers see reflecting boundary conditions. In this case, as explained in Ref. [28], the probability $N^{2k+2}_{\Gamma,L}(l_1,l_2,\ldots,l_{2k+2})$ that the system at scale Γ has $(2k+2)$ bonds $(k=0,1,\ldots)$, with respective lengths $(l_1,\ldots,l_{2k+2})$, is

$$N^{2k+2}_{\Gamma,L}(l_1,l_2,\ldots,l_{2k+2})$$
$$=E^+_\Gamma(l_1)P^-_\Gamma(l_2)P^+_\Gamma(l_3)\ldots P^+_\Gamma(l_{2k+1})$$
$$\times E^-_\Gamma(l_{2k+2})\overline{l_\Gamma}\delta\left(L-\sum_{i=1}^{2k+2}l_i\right), \quad \text{(D1)}$$

where $P^\pm_\Gamma(l)$ are the bulk length distributions [Eq. (68)], $\overline{l_\Gamma}=\overline{l^+_\Gamma}+\overline{l^-_\Gamma}$, and $E^\pm_\Gamma(l)$ the length distribution of boundary bonds (see Ref. [28]). The normalization is

$$\sum_{k=0}^\infty\int_{l_1,\ldots l_{2k+2}}N^{2k+2}_{\Gamma,L}(l_1,l_2,\ldots,l_{2k+2})=1. \quad \text{(D2)}$$

The probability that the system has $(2k+2)$ domains at scale Γ is

$$I_L(k;\Gamma)=\int_{l_1,\ldots,l_{2k+2}}E^+_\Gamma(l_1)P^-_\Gamma(l_2)\cdots P^+_\Gamma(l_{2k+1})$$
$$\times E^-_\Gamma(l_{2k+2})\overline{l_\Gamma}\delta\left(L-\sum_{i=1}^{2k+2}l_i\right), \quad \text{(D3)}$$

so that the Laplace transform with respect to the length of the generating function is

$$\int_0^\infty dLe^{-qL}\left(\sum_{k=0}^\infty z^kI_L(k;\Gamma)\right)=\overline{l_\Gamma}\frac{E^+_\Gamma(q/2g)E^-_\Gamma(q/2g)}{1-zP^+_\Gamma(q/2g)P^-_\Gamma(q/2g)}$$
$$=\frac{2g}{q}\frac{1-P^+_\Gamma(q/2g)P^-_\Gamma(q/2g)}{1-zP^+_\Gamma(q/2g)P^-_\Gamma(q/2g)}, \quad \text{(D4)}$$

where $p=q/2g$. This leads to the results given in the text.

The magnetization can also be obtained. In the ''full'' renormalized landscape, the magnetization is given by $M_L=\sum_{i=1}^{i=2k+2}(-1)^{i+1}l_i$. The probability that it has value M simply is

$$F_L(M;\Gamma)=\sum_{k=0}^{k=+\infty}\int_{l_1,\ldots,l_{2k+2}} E^+(l_1)P^-(l_2)P^+(l_3)\cdots \times P^+(l_{2k-1})E^-(l_{2k+2}) \tag{D5}$$

$$\overline{l_\Gamma}\delta\left(L-\sum_{i=1}^{2k+2} l_i\right)\delta\left(M-\sum_{i=1}^{i=2k+2}(-1)^{i+1}l_i\right). \tag{D6}$$

This leads to the result given in the text.

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